

Non-standard analysis and singular perturbations of ordinary differential equations

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Non-standard analysis and singular perturbations of ordinary differential equations

A.K. Zvonkin and M.A. Shubin

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Introduction

The subject of this paper is known in the mathematical literature under the picturesque name of “duck-shooting”. Ducks are certain singular solutions of equations with a small parameter, which are studied in the theory of relaxation oscillations. These solutions were first found for the van der Pol equation, and their form resembled that of a flying duck. Duck theory is, in the authors’ opinion, the most striking application of the techniques of non-standard analysis. Some results concerning ducks were mentioned in the report by P. Cartier [1]. This paper is a supplement to his article. Apart from presenting the theory of ducks we wanted to show how the results of this theory can be interpreted in the language of classical (“standard”) mathematics.

We should mention that all our results can be stated and proved without the use of non-standard analysis (this is altogether true for applications of non-standard analysis). However, this would have made all the statements more cumbersome and all the proofs longer and intuitively less clear. It was not by chance that ducks were discovered with the help of non-standard analysis and in connection with it. We think that the language of non-standard analysis will make it easy for a wide circle of mathematicians to become acquainted with the theory of ducks and the theory of relaxation oscillations in general.

Advertising infinitesimal calculus, Leibniz wrote: “The difference from the style of Archimedes lies only in expressions which in our method are

more direct and more suitable for the art of invention". Quoting this passage Lutz and Goze [3] noted that the same could be said about non-standard analysis (which is, properly speaking, the present-day infinitesimal calculus), replacing Archimedes by Bourbaki.

Of course, non-standard analysis does not offer as much economy of thought as the differential and integral calculus did in their time. But even the economy that it does provide can be essential in difficult problems of the theory of singular perturbations of non-linear equations (as it turned out in the problems discussed in this paper).

In what follows we sometimes use notions from [1] without saying so specifically.

We start by constructing in §1 the simplest version of non-standard analysis. It is based on the construction described in §3 of [1] and also in [2], but we do not introduce a scale of sets, extending the construction at once to the entire category of sets. Here we have to sacrifice the general principle of idealization (the direction theorem). However, the use of a countable set Ω allows us to prove easily the existence of infinitely large and infinitely small numbers, which is quite sufficient for the applications considered in this paper.

In §1 we give non-standard formulations of a number of important theorems from the theory of ordinary differential equations (the existence and uniqueness theorem, theorems about extension of solutions, the Poincaré-Bendixson theorem). We also give a standard interpretation of the theorem on the existence of a large cycle for the van der Pol equation on the Liénard plane.

In §2 we give a definition of solution-ducks, that is, trajectories of a rapid-slow field depending on an additional parameter a (apart from the small parameter ϵ), which at first move along the attracting part of the slow curve and then move to the repelling part and continue along it; along both the attracting and repelling parts appreciable (that is, not infinitesimal) intervals are covered. By means of a magnifying glass, representing an infinitely large extension of the scale, we prove here that there are solution-ducks in the case of a non-degenerate minimum point of the slow curve. Here the simplest version of the "law of preservation of ducks" is established: the fact that a solution-duck on the Liénard plane remains a duck under the application of a magnifying glass, which corresponds in this case to a transition to the phase plane. We also give a standard interpretation of our results concerning ducks.

§3 contains various results specifying the behaviour of solution-ducks. We prove that both the ducks themselves and the values of the parameter a corresponding to them have complete asymptotic expansions in powers of the small parameter ϵ . The coefficients of the expansion do not depend on the choice of a duck. We obtain recurrence formulae for the coefficients of the expansion and prove the existence of the expansion by means of

a countable sequence of magnifying glasses. In addition, by considering separately the case when all the coefficients in the asymptotic expansion of the duck differ only by a numerical factor, we construct some examples of ducks defined by explicit formulae.

In §3 we also describe an exponential microscope: a special change of variables, which gives an exponentially large expansion (in relation to $1/\epsilon$) of the scale in the neighbourhood of a curve. Surprisingly well-adapted to the study of ducks, this change of variables makes it possible to distinguish between different ducks and the values of the parameter corresponding to them. (For values of the parameter a this gives an exponentially small interval with respect to ϵ , which is vividly expressed by the words “the life of ducks is short”.) By means of the exponential microscope we construct an entry-exit function, which describes the correspondence between the point at which a trajectory enters the halo of a solution-duck and the point at which it leaves it. It is interesting that this function can be found explicitly. Also, by means of an exponential microscope we give a second proof of the existence of asymptotic expansions of ducks and values of the parameter a . Next, in §3 we describe certain specific objects connected with ducks, such as tunnels, funnels, and showers.

§4 is devoted mainly to the study of duck-cycles, that is, ducks that are periodic trajectories. First we prove a general theorem on the existence of duck-cycles for the same configuration of the slow curve as in the van der Pol equation (the minimum point may be degenerate). From this it follows also that there are solution-ducks in the case of an arbitrary isolated minimum or maximum point of the slow curve. The proof of the general theorem on the existence of duck-cycles is based on the analysis of the successor function. In addition we give another proof, based on the use of the entry-exit function and suitable for the case of a non-degenerate minimum point. We also describe the behaviour of the period of a large cycle and cycle-ducks, discuss the question of the stability of cycle-ducks, and describe their evolution in the case of the van der Pol equation. At the end of §4 we give an outline of the description of the behaviour of duck-cycles for one equation studied by M. Diener.

Lest we interrupt the exposition, in §§1–4 we do not name authors of individual results and we give hardly any references to the literature. All this is done in §5, where we also give a short survey of the literature available to us. We only point out that most of the results described here are due to young mathematicians from Strasbourg, who later worked in Mulhouse (France), and also in Tlemcen and Oran (Algeria). Their work was inspired by one question formulated by G. Reeb in 1977. The enthusiasm of the authors of the theory described here has influenced, in particular, their terminology. We have tried to preserve their vivid style, since it undoubtedly helps the comprehension of the results.

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§1. Non-standard analysis and ordinary differential equations

1.1. The simplest non-standard extension of the real line.

Let Ω be an infinite countable set. Fixing a numbering of its points, we can identify it with the set \mathbf{N} of all natural numbers. We consider on Ω the filter \mathcal{Q}_0 consisting of all sets $A \subset \Omega$ whose complement is finite. We extend this to an ultrafilter \mathcal{Q} and define on Ω a "probability measure" (in the sense of [1]), associated with it, setting $P(A) = 1$ for $A \in \mathcal{Q}$ and $P(A) = 0$ for $A \notin \mathcal{Q}$. We remark that if a subset $A \subset \Omega$ is finite, then $\Omega \setminus A \in \mathcal{Q}$, hence, $A \notin \mathcal{Q}$ so that $P(A) = 0$. In what follows the measure P is fixed and the words "almost everywhere" (or briefly a.e.) mean "almost everywhere in measure P ".

Let us introduce the set ${}^*\mathbf{R}$ of "random real numbers",⁽¹⁾ which consists of equivalence classes of maps $\Omega \rightarrow \mathbf{R}$, where two maps $x', x'' : \Omega \rightarrow \mathbf{R}$ are regarded as equivalent (we write $x' \sim x''$) if $x'(\omega) = x''(\omega)$ a.e. on Ω . If $x \in {}^*\mathbf{R}$, then any of the maps $x' : \Omega \rightarrow \mathbf{R}$ defining x is called a *version* of the element x .

There is a natural embedding $\mathbf{R} \subset {}^*\mathbf{R}$ under which to every number $a \in \mathbf{R}$ there corresponds the class of the constant function on Ω (everywhere equal to a).

It is easy to see that ${}^*\mathbf{R}$ has the natural structure of an ordered field. Namely, the algebraic operations (addition and multiplication) are introduced by means of the corresponding operations over versions, carried out for each $\omega \in \Omega$. The order is introduced as follows: if $x, y \in {}^*\mathbf{R}$, and x', y' are versions of x and y , then Ω splits into the union of disjoint subsets:

$$\begin{aligned} \Omega &= \Omega_+ \cup \Omega_0 \cup \Omega_-, \text{ where } \Omega_+ = \{\omega: x'(\omega) > y'(\omega)\}, \\ \Omega_0 &= \{\omega: x'(\omega) = y'(\omega)\}, \Omega_- = \{\omega: x'(\omega) < y'(\omega)\}. \end{aligned}$$

Owing to the properties of P one of the three subsets Ω_+ , Ω_0 , and Ω_- has measure 1, and the other two have measure 0. In accordance with this we see that either a.e. $x'(\omega) > y'(\omega)$ (then we must put $x > y$), or a.e. $x'(\omega) = y'(\omega)$ (which means that $x = y$), or a.e. $x'(\omega) < y'(\omega)$ (in this case we must put $x < y$), which gives an order relation in ${}^*\mathbf{R}$. Similarly, we can verify that if $x \in {}^*\mathbf{R} \setminus \{0\}$, then there is an inverse element $1/x$.

Let us verify that ${}^*\mathbf{R} \setminus \mathbf{R}$ is non-empty, that is, ${}^*\mathbf{R}$ is actually a non-trivial extension of \mathbf{R} . Identifying Ω with \mathbf{N} , we put $\varepsilon(n) = 1/n$ ($n = 1, 2, \dots$) and let ε be the corresponding equivalence class, that is, an element of ${}^*\mathbf{R}$.

⁽¹⁾Cartier [1] puts the asterisk on the right. We keep to the notation adopted by Davis [2].

Since finite sets have measure 0 in Ω , it is clear that $0 < \varepsilon < \varepsilon_0$ for any $\varepsilon_0 > 0$, $\varepsilon_0 \in \mathbf{R}$. Therefore, $\varepsilon \in {}^*\mathbf{R} \setminus \mathbf{R}$, which proves that ${}^*\mathbf{R} \setminus \mathbf{R}$ is non-empty.

The number $\varepsilon \in {}^*\mathbf{R}$ we have just constructed is *infinitely small*. Having one such number, by means of algebraic operations we can construct arbitrarily many other numbers belonging to ${}^*\mathbf{R} \setminus \mathbf{R}$. For instance, if $a \in \mathbf{R} \setminus \{0\}$, then $a\varepsilon$ is also infinitely small and differs from ε for $a \neq 1$. The number $1/\varepsilon$ is infinitely large. Adding to $a \in \mathbf{R}$ all infinitesimals we obtain the *halo* of a . The haloes of any two distinct numbers $a_1, a_2 \in \mathbf{R}$ do not intersect.

The ‘‘asterisk’’ procedure of constructing a set of random elements can be applied not only to \mathbf{R} , but to any set X , and we obtain from it a set *X in the same way as ${}^*\mathbf{R}$ from \mathbf{R} . There is a natural embedding $X \subset {}^*X$, and if X is infinite, then ${}^*X \setminus X$ is non-empty. In particular, there is a set ${}^*\mathbf{N}$, a subset of ${}^*\mathbf{R}$. If a map $f: X \rightarrow Y$ is given, then it generates a natural map ${}^*f: {}^*X \rightarrow {}^*Y$. Namely, if $x \in {}^*X$ and $x' = x'(\omega)$ is a version of x , then ${}^*f(x)$ must be put equal to the class of the map $\omega \mapsto f(x'(\omega))$ from Ω to Y (that is, on versions the map *f is defined by applying f to each ω). We obtain a functor from the category of sets to itself. It induces functors on many natural subcategories, for example, the categories of groups, rings, fields, partially ordered sets, and so on.

If $A \subset X$, then ${}^*A \subset {}^*X$. However, subsets of the form *A (which in [1] are called *standard*) are not the only subsets of *X . An interesting class of subsets can be obtained by applying the functor $*$ to the set $\mathcal{P}(X)$ of all subsets of X . Elements of ${}^*\mathcal{P}(X)$ can be interpreted as random subsets of *X (such a subset can be thought of as a fibration over Ω , whose fibre over each point ω is an ordinary subset of X). If a version $\omega \mapsto A'(\omega)$ of such a random set A is given and x is a random element of X with a version $x' = x'(\omega)$, then $x \in A$ means that $x'(\omega) \in A'(\omega)$ a.e. Thus, elements of ${}^*\mathcal{P}(X)$ are interpreted as subsets of *X . (It is easy to verify that two random subsets coincide if and only if they contain the same random elements.) Thus, ${}^*\mathcal{P}(X) \subset \mathcal{P}({}^*X)$. Elements of ${}^*\mathcal{P}(X)$ are called *internal* subsets of *X . All subsets of *X that are not internal are called *external*.

If $Z = X \times Y$, where X and Y are sets, then there is a canonical isomorphism ${}^*Z \cong {}^*X \times {}^*Y$ under which the canonical projections of *Z to *X and *Y are the maps *p_1 and *p_2 obtained by applying the functor $*$ to the canonical projections p_1 and p_2 of the direct product $Z = X \times Y$ to X and Y . Now if $f: X \rightarrow Y$ is a map and Γ_f its graph, $\Gamma_f \subset X \times Y$, then ${}^*\Gamma_f$ turns out to be the same as the graph of the map ${}^*f: {}^*X \rightarrow {}^*Y$. Thus, to study the functor $*$ it is sufficient to confine its application to sets, identifying maps with their graphs (bearing into mind, of course, that graphs are defined on sets with a fixed structure of a direct product). In addition, the operation $*$ can be applied not only to maps, but also to relations between X and Y (representing subsets of the direct product $X \times Y$) and turns them into relations between *X and *Y .

Developing these remarks, we see that any mathematical structure described in the framework of set theory allows us to apply the operation $*$. Any statement \mathcal{Y} in the language of set theory, after applying the operation $*$, turns into another statement $*\mathcal{Y}$, also in the language of set theory. Of course, the notion of *statement* here needs more precision. This is achieved by a standard formalization of the language in the spirit of textbooks on mathematical logic. Each mathematician intuitively knows what a mathematical statement means. However, negligence can lead to misunderstanding. It is worth mentioning that in a mathematical statement (in its strict sense, as discussed here) all letters in it must be described in the spirit of modern programming languages (for example, Pascal or Ada) with the use of the membership sign \in , though often such descriptions are only implied. For example, the standard definition of the continuity of a given function $f: \mathbf{R} \rightarrow \mathbf{R}$ at a given point $x_0 \in \mathbf{R}$, written in the form

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x: |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon,$$

can lead to a misunderstanding if it is not made more precise in the following way:

$$\forall \varepsilon \in \mathbf{R}, \varepsilon > 0 \Rightarrow \{\exists \delta \in \mathbf{R}, \delta > 0 \ \& \ \forall x \in \mathbf{R}: |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon\}.$$

In the general statements discussed here we are allowed to use any sets, functions, quantifiers, the implication sign \Rightarrow , parentheses, the signs $=$, \in , substitution of an argument in a function, and so on. Without going into details we refer the reader to the book by Davis [2], where the necessary formalization is made, apparently with the least possible purism, though observing the usual requirements of mathematical rigour.

We point out that since all the sets and functions arising in the statements we use must also be described (for a set A such a description must, as a rule, have the form $A \in \mathcal{P}(X)$, where X is some set already given), applying the operation $*$ we arrive at statements referring only to internal sets and functions. In this context it is important to be able to distinguish between internal sets and functions and external ones. We have something to say about this later.

Now we can state the transfer principle.

Transfer principle. *Any statement \mathcal{Y} in the language of set theory is equivalent to the corresponding statement $*\mathcal{Y}$, that is, they are true or false simultaneously.*

Here an expert in logic may ask: what does it mean to say “the statement is true”? A mathematician can understand this intuitively (and will not come to a contradiction!) or assume that truth means the existence of a proof. (A proof of a statement \mathcal{Y} can be transformed into a proof of $*\mathcal{Y}$ and vice versa.)

The idea of the proof of the transfer principle is given in [1]. The essence of the matter consists in the fact that any statement can be written in the form of an assertion about coincidence or non-coincidence of two sets (for example, the statement $\forall x: x \in X \Rightarrow x \in Y$ can be written in the form $X \setminus Y = \emptyset$). The statement $X = Y \leftrightarrow *X = *Y$ (here X and Y are sets), as well as the possibility of inserting $*$ under the sign of any standard operations on sets, can easily be verified directly.

Let us dwell on the question of internal sets. Let X and Y be sets, $R \subset X \times Y$. We consider the standard set $*R \subset *X \times *Y$. Then for each fixed $\bar{y} \in *Y$ the set

$$A_{\bar{y}} = \{x: (x, \bar{y}) \in *R\} \subset *X$$

is an internal subset of $*X$. This follows easily from the transfer principle, applied to the statement

$$\forall y \in Y, \{x: x \in X, (x, y) \in R\} \in \mathcal{F}(X),$$

and can also be obtained by a direct construction of the corresponding random set. (If \bar{y} is given by a random element $\{y_\omega\}$, then $A_{\bar{y}}$ is determined by the random set $A_\omega = \{x: x \in X, (x, y_\omega) \in R\}$.)

It turns out that the above construction of an internal set is general: any internal subset of $*X$ can be obtained in the same way. To prove this it suffices to take $Y = \mathcal{F}(X)$ and to define R as the inclusion relation \in on $X \times \mathcal{F}(X)$. Of course, this proof does not have any practical value, but internal sets actually arising are always obtained by fixing parameters (in general, non-standard) in standard sets. Similarly, internal functions are obtained by fixing parameters (generally speaking, non-standard) in standard functions. A typical example: $f(x) = \sin(\omega x)$, where $\omega \in *R$.

The most important examples of external sets are the sets of infinitesimal numbers, of finite numbers, of positive infinitely large numbers, and of standard numbers. (Indeed, none of them has a least upper or greatest lower bound, whereas by the transfer principle an internal set bounded above or below ought to have a least upper or greatest lower bound, respectively.) Additional examples are given by one-to-one images of the above sets under internal mappings (see [8]).

1.2. Axioms.

From this point on we may entirely forget about the construction of the functor $*$. Only the following properties of the operation $*$, which can be considered as axioms, are important:

- a) if X is a set, then $*X$ is a set, and $X \subset *X$;
- b) the transfer principle is valid;
- c) $*R \setminus R \neq \emptyset$.

Let us give an example of the use of these axioms. From the transfer principle it follows that $*R$ is an ordered field. From this and from c) it is easy to deduce that $*R$ contains infinitely large and infinitely small numbers.

We can do this by introducing some notions that are incidentally important for what follows. An element $x \in {}^*\mathbf{R}$ is called *finite* if there is a number $c \in \mathbf{R}$ such that $-c < x < c$. Each finite $x \in {}^*\mathbf{R}$ has a *shadow* (or a standard part, see [2]), that is, a number ${}^\circ x \in \mathbf{R}$ such that $(x - {}^\circ x)$ is infinitely small. (Henceforth we write $x \sim y$ if x and y are infinitely close, that is, $x - y$ is infinitely small; this will not produce any misunderstanding with respect to the corresponding notation from § 1.1, since the equivalence in § 1.1 will no longer be used.) This can be proved, for example, by successively bisecting the interval $[-c, c]$ (in \mathbf{R} !) and applying the standard theorem of analysis about nested intervals (after each division, x falls into one of the resulting intervals), or by defining ${}^\circ x$ directly by the formula ${}^\circ x = \inf\{a: a \in \mathbf{R}, a > x\}$. Further, let $x \in {}^*\mathbf{R} \setminus \mathbf{R}$. If x is finite, then $\varepsilon = (x - {}^\circ x)$ is an infinitesimal, and $1/\varepsilon$ is infinitely large. If x is not finite, then $|x| > c$ for any standard $c > 0$, from which it follows that x is infinitely large and $1/x$ is infinitely small.

The given axioms are sufficient for those applications of non-standard analysis to differential equations that are discussed in this paper. However, in a number of problems (for example, in applications to general topology) it is necessary to replace axiom c) by a general idealization principle, which guarantees the existence of ideal elements not only in \mathbf{R} , but in much more general objects. The general idealization principle can be made to hold by choosing a set Ω coarser than \mathbf{N} with a suitable ultrafilter (see [1] or [2]).

1.3. The simplest theorems on ordinary differential equations.

Let us apply the transfer principle to the existence and uniqueness theorem for ordinary differential equations. We consider the equation

$$(1.1) \quad \dot{x} = V(x)$$

with the initial conditions

$$(1.2) \quad x(t_0) = x_0.$$

Here $x_0 \in G$, where G is an open subset of \mathbf{R}^n , $x = x(t)$ is a function of t defined on the interval $(a, b) \subset \mathbf{R}$ and taking values in G , $t_0 \in (a, b)$, and $V(x)$ is a vector field on G . A non-standard approach to the problem (1.1)–(1.2) assumes the replacement of all objects t_0, x_0, a, b, G, V , and $x(t)$ by their non-standard analogues. We indicate the relevant refinements. First, we must take $t_0, a, b \in {}^*\mathbf{R}$, $x_0 \in {}^*\mathbf{R}^n$. Next, we denote by $\mathcal{F}(\mathbf{R}^n)$ the set of all open subsets of \mathbf{R}^n . Then we assume that $G \in {}^*[\mathcal{F}(\mathbf{R}^n)]$. It is natural to call such G internal open subsets of ${}^*\mathbf{R}^n$. By the transfer principle, in fact, $G \subset {}^*\mathbf{R}^n$ for any $G \in {}^*[\mathcal{F}(\mathbf{R}^n)]$. Examples of internal open subsets of ${}^*\mathbf{R}^n$ are any open balls (including those of infinitely small or infinitely large radius). Such balls need not contain any standard points.

The interval $(a, b) \subset {}^*\mathbf{R}$, for any $a, b \in {}^*\mathbf{R}$, $a < b$, consists of $c \in {}^*\mathbf{R}$ such that $a < c < b$. By the transfer principle, it is an internal open subset of ${}^*\mathbf{R}$.

Let $a, b \in \mathbf{R}$, $a < b$, and $G \in \mathcal{F}(\mathbf{R}^n)$. We denote by $C((a, b), G)$ the set of continuous functions on (a, b) with values in G . What is the non-standard analogue of this space? The easiest way to conceive this is by means of the transfer principle. We introduce the set

$$\Pi = \{C((a, b), G): a, b \in \mathbf{R}, a < b, G \in \mathcal{F}(\mathbf{R}^n)\},$$

whose elements are the usual function spaces $C((a, b), G)$ for all possible standard a, b ($a < b$) and G . Let I be the set of all non-empty intervals $(a, b) \subset \mathbf{R}$, where $-\infty < a < b < +\infty$. Then there is an obvious map

$$\begin{aligned} C: I \times \mathcal{F}(\mathbf{R}^n) &\rightarrow \Pi, \\ ((a, b), G) &\mapsto C((a, b), G). \end{aligned}$$

We now introduce the set ${}^*\Pi$ and canonically extend the map C just constructed to a map

$${}^*C: {}^*I \times {}^*[\mathcal{F}(\mathbf{R}^n)] \rightarrow {}^*\Pi.$$

We denote the image of a pair $((a, b), G)$ under this map by ${}^*C((a, b), G)$. Here $a, b \in {}^*\mathbf{R}$, $a < b$, and G is any internal open set. By the transfer principle, any element $x \in {}^*C((a, b), G)$ is a function $x: (a, b) \rightarrow G$. By the same principle, a necessary and sufficient condition for the internal function $x: (a, b) \rightarrow G$ to belong to ${}^*C((a, b), G)$ is obtained by repeating the definition of continuity. (We have to use the metric on G obtained by the canonical extension of the standard metric $\rho: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}_+$ to ${}^*\rho: {}^*\mathbf{R}^n \times {}^*\mathbf{R}^n \rightarrow {}^*\mathbf{R}_+$ and repeat the usual (ε, δ) -definition of continuity, assuming that ε, δ , and all the points occurring are arbitrary and non-standard.) Elements of the space ${}^*C((a, b), G)$ will be called internal continuous functions on (a, b) with values in G .

By analogy with ${}^*C((a, b), G)$ we define the space ${}^*C^1((a, b), G)$ of internal continuously differentiable functions on (a, b) with values in G .

The derivative $\frac{d}{dt}$ for any standard $(a, b) \in I$ and $G \in \mathcal{F}(\mathbf{R}^n)$ defines a map

$$\frac{d}{dt}: C^1((a, b), G) \rightarrow C((a, b), \mathbf{R}^n).$$

By the transfer principle, for any $(a, b) \in {}^*I$ and $G \in {}^*[\mathcal{F}(\mathbf{R}^n)]$ we obtain a map

$${}^*\left(\frac{d}{dt}\right): {}^*C^1((a, b), G) \rightarrow {}^*C((a, b), {}^*\mathbf{R}^n),$$

which for brevity we denote simply by $\frac{d}{dt}$. (This cannot lead to a contradiction, since for standard (a, b) and G the map ${}^*\left(\frac{d}{dt}\right)$ is an extension of $\frac{d}{dt}$.) Instead of ${}^*\left(\frac{d}{dt}\right)x = \frac{d}{dt}x$ we write \dot{x} , which also cannot lead to a misunderstanding.

The space ${}^*C(G, {}^*\mathbf{R}^n)$ of internal continuous vector-valued functions on an internal open set G (with values in ${}^*\mathbf{R}^n$) is defined in the same way as the space ${}^*C((a, b), G)$ introduced above. We assume that $\mathbf{V} \in {}^*C(G, {}^*\mathbf{R}^n)$. Now (1.1) makes sense for such \mathbf{V} and for $x \in {}^*C^1((a, b), G)$. A narrower space of vector fields is the naturally defined space ${}^*C^1(G, {}^*\mathbf{R}^n)$.

We can now state an existence and uniqueness theorem for the solution of the problem (1.1)–(1.2).

Theorem 1.1. *Let G be an internal open subset of ${}^*\mathbf{R}^n$, $\mathbf{V} \in {}^*C^1(G, {}^*\mathbf{R}^n)$, $t_0 \in {}^*\mathbf{R}$, and $x_0 \in G$. Then there is an $\varepsilon > 0$, $\varepsilon \in {}^*\mathbf{R}$, such that the solution $x \in {}^*C^1((t_0 - \varepsilon, t_0 + \varepsilon), G)$ of (1.1) with the initial condition (1.2) exists and is unique.*

By an application of the transfer principle the proof follows immediately from the standard existence and uniqueness theorem.

The number $\varepsilon > 0$ in Theorem 1.1 may be infinitely small, even if $G = {}^*\mathbf{R}^n$. This happens, for instance, for the equation $\dot{x} = x^2$ with the initial condition $x(0) = B$, where $B > 0$ is infinitely large (here the solution has the form $x(t) = -(t - 1/B)^{-1}$ and is defined only for $t < 1/B$), or for the equation $\dot{x} = Ax^2$ with infinitely large $A > 0$ and the initial condition $x(0) = 1$ (the solution has the form $x(t) = (1 - At)^{-1}$). However, all the usual extension theorems remain valid. For example, if a solution $x(t)$ is not defined for all $t \geq t_0$, then for any (including infinitely small) $\varepsilon > 0$, $\varepsilon \in {}^*\mathbf{R}$, it has to be shifted for some t from the maximal interval on which it is defined either into the set $\{x: \rho(x, {}^*\mathbf{R}^n \setminus G) < \varepsilon\}$ or the set $\{x: |x| > 1/\varepsilon\}$. Another example: let $G = {}^*\mathbf{R}^n$, and suppose that for any $T > t_0$, $T \in {}^*\mathbf{R}$, there is an $R > 0$, $R \in {}^*\mathbf{R}$, such that if the solution is defined on $[t_0, t_1]$, where $t_1 \leq T$, then $|x(t)| \leq R$ for⁽¹⁾ $t \in [t_0, t_1]$; then the solution $x(t)$ is defined for all $t \geq t_0$, $t \in {}^*\mathbf{R}$. Proofs are easily obtained by applying the transfer principle to a known standard fact. Similarly, a linear equation (with continuous variable coefficients) has a solution on the whole interval on which the coefficients are defined.

Let us give a non-standard statement of the Poincaré-Bendixson theorem, which we need in the following form: suppose that K is a compact set in \mathbf{R}^2 that does not contain positions of equilibrium of a field $\mathbf{V} \in C^1(\mathbf{R}^2, \mathbf{R}^2)$, and that the solution $x(t)$ of the problem (1.1)–(1.2) (for some $t_0 \in \mathbf{R}$ and $x_0 \in K$) belongs to K for all $t \geq t_0$; then K contains a periodic trajectory of \mathbf{V} . This makes it possible to apply the transfer principle along the same lines as above and leads to the following result.

Theorem 1.2. *Let \mathcal{K} be the set of all compact sets in \mathbf{R}^2 , $K \in {}^*\mathcal{K}$. If K does not contain positions of equilibrium of a field $\mathbf{V} \in C^1({}^*\mathbf{R}^2, {}^*\mathbf{R}^2)$ and if there is a trajectory $x(t)$ (a solution of (1.1)) defined for all $t \geq t_0$, $t \in {}^*\mathbf{R}$, such that $x(t) \in K$ for all $t \geq t_0$, then some periodic trajectory of \mathbf{V} is contained in K .*

⁽¹⁾Here $R > 0$ may be infinitely large!

For K we can take, for example, the closed ball $\{x: x \in {}^*\mathbf{R}^2, |x| \leq R\}$ of any radius $R \in {}^*\mathbf{R}$ (in particular, R may be infinitely large).

1.4. The large cycle for the van der Pol equation.

The preceding arguments gave an almost tautological transfer of standard theorems to the non-standard case. However, ultimately we should like to be able to obtain standard theorems by non-standard methods. Here is the simplest example of this situation in differential equations.

Consider the van der Pol equation

$$(1.3) \quad \varepsilon \ddot{x} + (x^2 - 1)\dot{x} + x = 0$$

on the Liénard plane (x, y) , where $y = \varepsilon \dot{x} + f(x)$, $f(x) = x^3/3 - x$, so that (1.3) is equivalent to the system

$$(1.4) \quad \begin{cases} \dot{x} = \varepsilon^{-1}(y - f(x)), \\ \dot{y} = -x. \end{cases}$$

We consider the closed curve $\Gamma \subset \mathbf{R}^2$, described in §22 of [1], and having the property that if $\varepsilon > 0$ is infinitely small, then any semi-trajectory $\{(x(t), y(t)), t \geq t_0\}$ starting at any finite point of the non-standard plane ${}^*\mathbf{R}^2$ other than the origin enters the halo of Γ , beginning at some instant. We fix a standard number $\delta > 0$, $\delta \in \mathbf{R}$, and denote by $\Gamma^{(\delta)}$ the δ -neighbourhood of Γ in \mathbf{R}^2 . If $\varepsilon > 0$ is infinitely small, then the unique periodic solution of (1.4) is contained in ${}^*\Gamma^{(\delta)}$. Conversely, Γ is also contained in the δ -neighbourhood of this periodic solution. In general, it is convenient to introduce the usual distance between two compact sets in \mathbf{R}^2 :

$$(1.5) \quad \rho(K_1, K_2) = \max_{z \in K_1} \{\max \rho(z, K_2), \max_{z \in K_2} \rho(z, K_1)\}.$$

By the transfer principle, this distance is defined for any pair of closed curves in ${}^*\mathbf{R}^2$ determined by continuous periodic functions (that is, belonging to ${}^*C^1(\mathbf{R}, \mathbf{R}^2)$). We have proved that the unique periodic solution γ_ε of (1.4) has the property that

$$(1.6) \quad \rho(\gamma({}^*\mathbf{R}), \Gamma) < \delta \quad \text{for any } \delta > 0, \delta \in \mathbf{R}.$$

In particular, if we fix $\delta > 0$, $\delta \in \mathbf{R}$, and take $\varepsilon_0 > 0$ as infinitely small, then we find that (1.6) holds for any $\varepsilon \in {}^*\mathbf{R}$ for which $0 < \varepsilon < \varepsilon_0$. By the transfer principle we find that for the system (1.4) on the standard plane \mathbf{R}^2 the following theorem is true.

Theorem 1.3. *There is an $\varepsilon_0 > 0$, $\varepsilon_0 \in \mathbf{R}$, such that for $0 < \varepsilon < \varepsilon_0$ the system (1.4) has a unique periodic solution $\gamma_\varepsilon: \mathbf{R} \rightarrow \mathbf{R}^2$. For any $\delta > 0$ there is an $\varepsilon_\delta = \varepsilon_0(\delta) > 0$ such that if $0 < \varepsilon < \varepsilon_\delta$, then $\rho(\gamma_\varepsilon(\mathbf{R}), \Gamma) < \delta$.*

Thus, as $\varepsilon \rightarrow +0$, the periodic solution γ_ε of (1.4) converges to Γ in the metric (1.5). By similar arguments it is easy to see that the smallest periods

T_ϵ of periodic solutions converge as $\epsilon \rightarrow +0$ to a number T_0 , which is the shadow of the smallest period of such a solution for infinitely small $\epsilon > 0$. (This shadow does not depend on ϵ : it is easy to verify that $T_0 = 3 - \log 4$.)

§2. Existence of solution-ducks

2.1. Definition of solution-ducks.

We consider the system of differential equations

$$(2.1) \quad \begin{cases} \dot{x} = \epsilon^{-1}(y - f(x)), \\ \dot{y} = a - x. \end{cases}$$

Here (and almost everywhere in what follows) we treat it on the plane ${}^*\mathbf{R}^2$ and assume that $a \in {}^*\mathbf{R}$, $\epsilon \in {}^*\mathbf{R}$, $\epsilon > 0$, ϵ is infinitely small, and f is a standard function of class C^∞ . (More precisely, it is a function of class $C^\infty(\mathbf{R}, \mathbf{R})$, extended to a map ${}^*\mathbf{R} \rightarrow {}^*\mathbf{R}$). However, at the end of this section we give a standard interpretation of the results obtained, and then the system (2.1) is treated on \mathbf{R}^2 , and we assume that $a \in \mathbf{R}$ and $\epsilon \in \mathbf{R}$, where $\epsilon > 0$ is a small parameter.

For $a = 0$ and $f(x) = x^3/3 - x$, the system (2.1) goes over to the van der Pol equation on the Liénard plane. This is why (2.1) is often called the Liénard system, and the (x, y) -plane the Liénard plane. The class of equations of the form (2.1) is convenient, because on the one hand, it contains many interesting examples and on the other hand, it is simple enough from the technical point of view.

In (2.1) the variable y is slow; this means that \dot{y} takes finite values at all finite points of the plane. The variable x is rapid, that is, \dot{x} takes infinitely large values at some finite points of the plane. For systems on the plane with one slow and one rapid variable, we define the slow curve as the set of points at which the time-derivative of the rapid variable vanishes: for the system (2.1) this gives $\dot{x} = 0$, that is, $y = f(x)$. Thus, in this case the slow curve is the graph of $f(x)$. (For many systems occurring later the slow curve is not a standard, but an internal set; in this situation some authors use a different terminology, calling the slow curve the shadow of the given set.)

We define the ϵ -galaxy of a plane set A as the set of points whose distance d from A is such that d/ϵ is finite. It is evident that the ϵ -galaxy of the slow curve is the set of points (x, y) such that $\epsilon^{-1}(y - f(x))$ is finite. Consequently, outside this ϵ -galaxy \dot{x} is infinitely large, that is, the vector field corresponding to the system (2.1) is almost horizontal; above the graph $f(x)$ it is directed to the right, and below the graph to the left (see Fig. 2.1; the double arrows here and in subsequent figures show *rapid motion*, that is, motion with infinitely large velocity). It is obvious that the parts of the graph where $f(x)$ increases are attracting (that is, locally in the neighbourhood of these parts the rapid movement is directed towards them), and the parts of the graph where $f(x)$ is decreasing are repelling (that is, in their neighbourhood the rapid motion is directed away from them).

In this paper we are mainly interested in trajectories that move from the attracting part of the slow curve to the repelling part. We consider the extremum point x_0 that separates these parts; for definiteness, let x_0 be a minimum (Fig. 2.1).

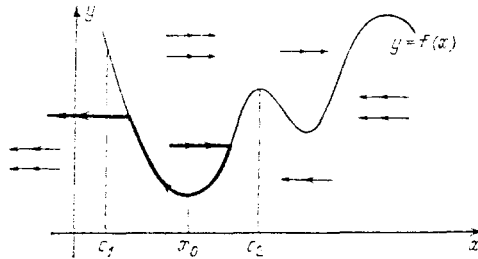


Fig. 2.1.

We consider the whole motion in the neighbourhood of x_0 , an isolated minimum point of f . Then for certain standard $c_1 < x_0$ and $c_2 > x_0$ the function $f(x)$ is non-increasing in (c_1, x_0) and non-decreasing in (x_0, c_2) (Fig. 2.1). Throughout the rest of this section, without saying so each time, we consider the system (2.1) and the corresponding motions only in the strip $\{(x, y): c_1 < x < c_2\}$. Moreover, we assume that $f'(x) \neq 0$ for $x \in (c_1, c_2)$, $x \neq x_0$, so that $f'(x) < 0$ for $x \in (c_1, x_0)$ and $f'(x) > 0$ for $x \in (x_0, c_2)$.

We now define solution-ducks for a system of the type (2.1).

Definition 2.1. A solution $(x(t), y(t))$ of (2.1) is called a *solution-duck*, or simply a *duck*, if there are standard $t_1 < t_0 < t_2$ such that ${}^{\circ}[x(t_0)] = x_0$, for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t), y(t))$ is infinitely close to the attracting part and for $t \in (t_0, t_2)$ to the repelling part of the slow curve. (The attracting and repelling pieces which the trajectory covers are not infinitely small.)

With *abus de language* we sometimes define a duck as not the whole trajectory $(x(t), y(t))$, but only the segment of it that is infinitely close to the slow curve. In Cartier's paper [1] ducks are only those solution-ducks that are at the same time cycles, that is, solutions periodic with respect to t . We study duck-cycles later, in §4. As we shall see, many properties of duck-cycles can be deduced from the fact that they are simply solution-ducks.

Our definition depends on the choice of the point x_0 ; however, we do not explicitly mention this choice each time, when it is clear from the context. In particular, for the time being we consider ducks in the strip $\{(x, y): c_1 < x < c_2\}$, where (c_1, c_2) is the interval described above in which f has exactly one (isolated) extremum point, the minimum point x_0 . Instead of a minimum point we can, of course, consider a maximum point. In this case, the definition of a solution-duck is the same (though now, according to the definition, the shadow of this solution moves through the maximum point from left to right, whereas at a minimum point its shadow moves from

right to left). However, this case is easily reduced to the case of a minimum point by changing y to $-y$. On the other hand, “false ducks”, which move from the unstable branch of the slow curve to the stable one, cannot be obtained from ducks by reversing time, as one might think at first glance, since in this case the attracting parts of the slow curve become repelling, and vice versa.

We give a simple necessary condition for the existence of a duck close to the minimum point x_0 of $f(x)$.

Proposition 2.1. *If there is a duck of the system (2.1) close to the minimum point x_0 , then $a \sim x_0$.*

Proof. Suppose, on the contrary, that a is not infinitely close to x_0 , for example, that $a \ll x_0$. But from the second equation (2.1) it follows that $\dot{y} < 0$ for $x > a$, that is, for $x > a$ the motion is such that y decreases as t increases. Thus, no solution $(x(t), y(t))$ can arise to the left of the point x_0 along⁽¹⁾ the slow curve. (This is intuitively obvious from the direction of the field on the slow curve itself, as shown in Fig. 2.2.) In particular, in this situation a solution-duck is not possible. The case $a \gg x_0$ is similar. ■

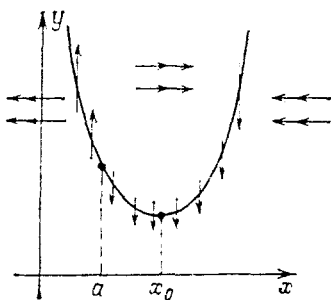


Fig. 2.2.

2.2. Existence of ducks in the case of a non-degenerate extremum.

To avoid misunderstanding, we remark that the existence of solutions of (2.1) that move for some time along the repelling part of the slow curve is a trivial fact, since if we change from t to $-t$ the repelling part becomes the attracting one. However, what is not trivial is the existence of solutions that first move along the attracting part and then along the repelling part, that is, ducks. We prove here the following theorem, which is due to M. Diener [34].

Theorem 2.1. *Suppose that a function f has a non-degenerate (Morse) minimum at x_0 , that is, $f'(x_0) = 0$ and $f''(x_0) > 0$. Then there are values of the parameter a for which in the neighbourhood of x_0 there are solution-ducks for (2.1). What is more, if $[c_1, c_2]$ is a standard finite interval of the axis \mathbf{R} such that $x_0 \in (c_1, c_2)$ and $f'(x) \neq 0$ for all $x \in [c_1, c_2]$, $x \neq x_0$, then there is a value a and a duck corresponding to it whose shadow is the same as the graph of $f(x)$ for all $x \in [c_1, c_2]$.*

⁽¹⁾“Along” means infinitely close to.

The existence of ducks for degenerate extremum points will be proved in §4.

Proof of Theorem 2.1. Let γ be a trajectory starting from a finite point (\bar{x}, \bar{y}) in the “zone of attraction” of the attracting branch of the slow curve (Fig. 2.3, a and b). This means that, firstly, the point (\bar{x}, \bar{y}) lies outside the ε -galaxy of the repelling branch of the slow curve, and secondly, either the point (\bar{x}, \bar{y}) already lies in the ε -galaxy of the attracting branch of the slow curve, or it lies in the zone of almost horizontal motion, which by changing it to a horizontal one leads to some point of the slow curve whose abscissa is denoted by x_2 , so that this point has the coordinates (x_2, \bar{y}) .

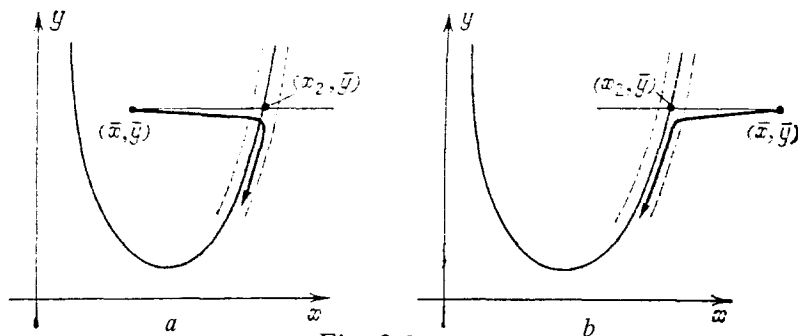


Fig. 2.3.

We claim that γ enters the ε -galaxy of the slow curve in the halo of the point (x_2, \bar{y}) . This means that, firstly, there are points of the trajectory γ that belong to the intersection of the halo of (x_2, \bar{y}) with the ε -galaxy of the slow curve, and secondly, for any point of the trajectory γ in the ε -galaxy of the slow curve there is a point of this trajectory preceding it in the halo of (x_2, \bar{y}) . The terminology just introduced must be used with care, since the ε -galaxy is not an internal set and there is no “first” point where the trajectory enters the ε -galaxy. However, for simplicity and brevity we use this terminology later, having in mind the precise sense just mentioned (or a similar sense, which the reader can easily make precise himself).

To prove the proposition stated above we need a transition to the phase plane, which in the given case coincides with an important device, which plays an essential role in what follows, namely, a magnifying glass or a stretch of the scale with an infinitely large coefficient. We go over to the plane of the variables (x, v) , where $v = \dot{x} = \varepsilon^{-1}(y - f(x))$. Then along a trajectory of the system (2.1) we have

$$\dot{v} = \frac{d}{dt} \dot{x} = \varepsilon^{-1} \frac{d}{dt} (y - f(x)) = \varepsilon^{-1} (\dot{y} - f'(x) \cdot \dot{x}) = \varepsilon^{-1} [a - x - f'(x)v],$$

so that (2.1) on the phase plane (x, v) takes the form

$$(2.2) \quad \begin{cases} \dot{x} = v, \\ \dot{v} = \varepsilon^{-1} [a - x - f'(x)v]. \end{cases}$$

Finite points of the phase plane correspond to points of the Liénard plane that belong to the ε -galaxy of the curve $y = f(x)$. Now (2.2) is again an equation of slow-rapid type (but this time the rapid motion is vertical, since the rapid variable is v). The slow curve on the phase plane is given by the equation

$$(2.3) \quad a - x - f'(x)v = 0;$$

for $a \neq x_0$ it is the graph of the function

$$(2.4) \quad v = \frac{a-x}{f'(x)},$$

and for $a = x_0$ it is the union of the vertical line $x = x_0$ and the smooth curve $y = g(x)$, where $g(x) = -(x - x_0)/f'(x)$, and at x_0 the function $g(x)$ is defined by continuity as $-1/f''(x_0)$ (the smoothness of $g(x)$ follows from the non-singularity of the minimum point x_0). The vector field corresponding to the system (2.2) is illustrated in Fig. 2.4 on the Liénard plane, the attracting branch of the slow curve is given by $x > x_0$, and the repelling branch by $x < x_0$.

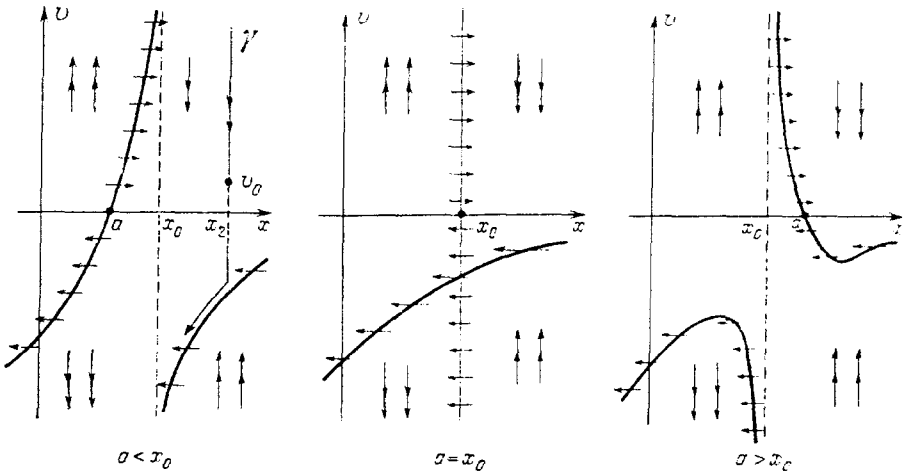


Fig. 2.4.

As we have already mentioned, the trajectory γ falls into the ε -galaxy of the curve $y = f(x)$ on the Liénard plane; this means that we can observe it in the finite part (or the so-called “principal galaxy”, see [1], §14) of the phase plane. A segment on the phase plane coming “from infinity” appears to us as an almost vertical ray (see the first graph in Fig. 2.4). The statement that on the Liénard plane γ enters the ε -galaxy of $f(x)$ in the halo of (x_2, \bar{y}) is equivalent to the following lemma.

Lemma 2.1. *Points of an almost vertical segment of the trajectory γ on the phase plane have an abscissa infinitely close to x_2 , where x_2 is the abscissa of the point of intersection of the line $y = \bar{y}$ and the attracting branch of the curve $y = f(x)$ on the Liénard plane.*

Proof. For definiteness we assume that the almost vertical portion of the trajectory is situated above the slow curve on the phase plane. (The opposite case is considered in exactly the same way.) Then it is the graph of some function $x = \varphi(v)$, which is automatically defined for all finite $v \geq v_0$ for some v_0 . We extend the domain of definition of this function so that it becomes internal. Namely, the trajectory $\gamma = (x(t), v(t))$ can be written in the form $x = \varphi(v)$ wherever $\dot{v}(t) \neq 0$. The set $\{t: \dot{v}(t) \neq 0\}$ is an internal open set. We take its maximal interval I containing some t corresponding to the vertical part of γ . Then the set of points $\{(x(t), v(t)), t \in I\}$ has the form of the graph $\{(x, v): x = \varphi(v)\}$, where φ is an internal function. In particular, its domain of definition (which consists of all points of the form $v(t)$, where $t \in I$) is also an internal set (and even an interval). Since this domain of definition contains all finite $v \geq v_0$ for some v_0 , it contains a whole interval $[v_0, \omega_0]$ for some infinitely large ω_0 . The internal function $\psi(v) = \varphi(v) - \varphi(v_0)$ is defined on the same interval, and $\psi(v)$ takes infinitely small values for all finite $v \geq v_0$. According to Robinson's lemma ([1], §11) there is an infinitely large ω_1 such that $\psi(v) \sim 0$ for all $v \in [v_0, \omega_1]$, that is, $\varphi(v) \sim \varphi(v_0)$ for these v . The point of γ with the coordinate $v = \omega_1$ on the Liénard plane lies outside the ε -galaxy of $f(x)$, hence on the almost horizontal part of γ , that is, at this point $y \sim \bar{y}$. As v varies from ω_1 to v_0 the coordinate x varies infinitesimally. Since the slow curve on the Liénard plane has a finite non-zero slope, the coordinate y also varies infinitesimally. Thus, the point of γ corresponding to $v = v_0$ on the Liénard plane is also infinitely close to the line $y = \bar{y}$. ■

Remark. One might be surprised that we have made such an effort to prove such an obvious statement. However, let us consider the system

$$(2.5) \quad \begin{cases} \dot{x} = \varepsilon^{-1}x^2, \\ \dot{y} = -1. \end{cases}$$

Its slow curve is the line $x = 0$ and its trajectories are given by the equations $y = C + \varepsilon/x$. Outside the ε -galaxy of the slow curve these trajectories are almost horizontal, that is, infinitely close to the line $y = C$. However, any point of the trajectory that lies in the ε -galaxy of the slow curve is at an appreciable distance from this line. The difference between the systems (2.1) and (2.5) is that if we stretch the ε -galaxy of the slow curve by a factor ε^{-1} , we do not obtain on the new plane a field of slow-rapid type.

Repeating arguments similar to the proof of Lemma 2.1 for the phase plane, we can verify that *the trajectory γ enters the ε -galaxy of the slow curve on the phase plane in the halo of the point with the abscissa x_2 .*

All we have said up to now is true not only for ducks, but for any trajectory that enters the halo of the slow curve $y = f(x)$ not in the halo of the extremum point. However, a duck can not only enter but also leave the halo of the slow curve, and not even in the halo of the extremum. It is

easy to see that the propositions stated earlier for the points of entry are also true for the coordinates of the points of “leaving” the ε -galaxy of the slow curve: it is sufficient to replace t by $-t$, then the repelling branch of $f(x)$ becomes the attracting one, and the exit part becomes the entry part, which allows us to repeat all the preceding arguments. We have thus proved the following proposition.

Proposition 2.2. *The trajectory enters the ε -galaxy of the slow curve on the Liénard plane in the halo of the point with the abscissa $x_2 \in \mathbf{R}$, $x_2 \neq x_0$, if and only if it enters the ε -galaxy of the slow curve on the phase plane in the halo of the point with the same abscissa x_2 . The trajectory leaves the ε -galaxy of the slow curve on the Liénard plane in the halo of the point with the abscissa $x_1 \in \mathbf{R}$, $x_1 \neq x_0$, if and only if it leaves the ε -galaxy of the slow curve on the phase plane in the halo of the point with the same abscissa x_1 .*

Let γ be a trajectory-duck with abscissas of the points of entry and exit x_2 and x_1 , respectively. What does this trajectory look like on the phase plane? To answer this question we must first of all recall that $a \sim x_0$ (Proposition 2.1). From this it follows that the slow curve on the phase plane is infinitely close to the union of the line $x = x_0$ and the curve $v = g(x)$.

Lemma 2.2 (about the absence of excursions). *In the interval between the points x_2 and x_1 the trajectory γ lies entirely in the halo of the curve $v = g(x)$.*

Proof. We know that at the beginning γ enters the halo of the graph $g(x)$ and at the end it leaves it. We assume that on the way γ deviates appreciably from $g(x)$, for example, upwards. However, this contradicts the topology of solutions of the differential equation in question.

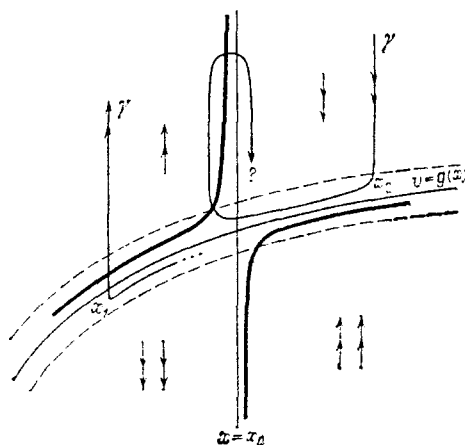


Fig. 2.5.

For, upward motion is possible only to the left of the slow curve, whereas the returning downward motion is possible only to the right (see Fig. 2.5, in which the halo of $g(x)$ is shown by a dotted line). Thus, for γ to be able to leave the halo of $g(x)$ near the point x_1 after it has already left it once (for example, near the point x_0) it is necessary that γ has a loop with a self-intersection (a double point). The remaining cases of the impossible behaviour of γ are treated similarly. ■

Corollary 2.1. *The segment of the trajectory-duck γ on the Liénard plane situated between the entry point x_2 and the exit point x_1 lies entirely in the ε -galaxy of the curve $y = f(x)$.*

Proof. This follows from the fact that on the phase plane the segment of γ between x_2 and x_1 contains only finite points. ■

For degenerate extremum points the analogous statement is false: a trajectory-duck that stays in the halo of $f(x)$ nevertheless leaves the ε -galaxy of $f(x)$ near the point x_0 (and on the phase plane it leaves the principal galaxy).

Corollary 2.2. *The segment of the trajectory-duck γ on the Liénard plane and on the phase plane between the entry point x_2 and the exit point x_1 is in the form of the graph of a smooth function of x .*

Proof. On the phase plane this statement is a direct consequence of Lemma 2.2, since by this lemma the whole relevant part of the trajectory-duck lies in the domain $v < 0$, hence, $\dot{x} < 0$ on it. But then $\dot{x} < 0$ also on the Liénard plane. ■

The trajectory γ in question first moves on the phase plane along the attracting part and then along the repelling part of the slow curve, that is, it has the same property as a duck on the Liénard plane. Although, formally speaking, Definition 2.1 of a duck refers only to equations of the form (2.1), it can be applied without any modification to (2.2) on the phase plane. Then Proposition 2.2 combined with Lemma 2.2 has a simpler formulation.

Proposition 2.3 (law of preservation of ducks). *The trajectory γ is a duck on the Liénard plane if and only if it is one on the phase plane. The shadow of γ on the Liénard plane contains a segment of the graph of $f(x)$ for $x \in [x_1, x_2]$ if and only if its shadow on the phase plane contains a segment of the graph of $g(x)$ for $x \in [x_1, x_2]$.*

In accordance with this proposition we can study the question of the existence or non-existence of ducks directly on the phase plane. We need the following important lemma, which we first state in the standard version.

Lemma 2.3 (about shooting). *Suppose we are given:*

1) *a connected and simply-connected domain D in \mathbf{R}^2 with boundary ∂D of class C^1 , that is, having a continuously varying tangent;*

2) a family of vector fields $v_a(y)$, defined in a fixed neighbourhood \mathcal{U} of the closure \bar{D} of D , which are smooth (of class C^∞) and depend smoothly on the parameter⁽¹⁾ $a \in [a_1, a_2]$. Suppose also that

3) v_a does not have singular points in \mathcal{U} ;

4) v_a is transversal to ∂D everywhere except for two points $T_1(a)$ and $T_2(a)$.

We denote by $E(a)$ the interval of entry of a field v_a , the subset⁽²⁾ of ∂D consisting of points $y \in \partial D$ such that $v_a(y)$ is directed inside D , and by $S(a)$ the interval of exit (the complement to $E(a)$ in $\partial D \setminus \{T_1(a), T_2(a)\}$).

Suppose that a point $y_0 \in \partial D$ is such that $y_0 \in \bigcap_{a_1 \leq a \leq a_2} E(a)$, that is, y_0 belongs to all intervals of entry. We denote by $z(a)$ the point of exit on ∂D of the trajectory of v_a starting at y_0 . Then $z(a) \in S(a)$ and the function $a \mapsto z(a)$ is continuous; in particular, it takes all intermediate values, that is, its range is a closed subinterval of ∂D .

Proof. It is easy to see that from the assumptions we have made it follows that the trajectory of v_a starting at $T_1(a)$ or $T_2(a)$ (at the instant $t = 0$) for $t \in (-\delta, \delta)$ and for sufficiently small $\delta > 0$ has only one point in common with \bar{D} (Fig. 2.6).

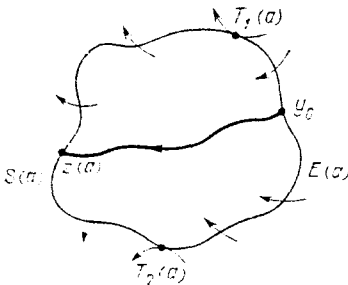


Fig. 2.6.

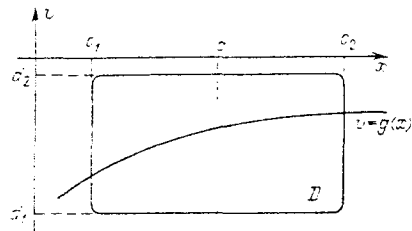


Fig. 2.7.

Therefore, $T_1(a)$ and $T_2(a)$ cannot be exit points for the trajectory starting at y_0 , hence, $z(a) \in S(a)$. Now continuity of $z(a)$ follows immediately from the theorem on the smooth dependence of the solution on parameters and from the implicit function theorem. ■

We need a non-standard version of Lemma 2.3, which is obtained from it by applying the transfer principle. We do not state it explicitly, but only indicate the necessary modifications.

1°. We have to take $D \in {}^*\mathcal{Z}$, where \mathcal{Z} is the set of all bounded domains with boundary of class C^1 in \mathbf{R}^2 . In particular, $D \subset {}^*\mathbf{R}^2$ (and D is an internal subset of ${}^*\mathbf{R}^2$). The shadow of $D \in {}^*\mathcal{Z}$ does not have to be a bounded domain with smooth boundary: for example, this shadow can

⁽¹⁾This means that $(x, a) \mapsto v_a(x)$ is infinitely differentiable as a function from $\mathcal{U} \times [a_1, a_2]$ to \mathbf{R}^2 .

⁽²⁾This set is diffeomorphic to an open interval of \mathbf{R} .

coincide with the whole plane \mathbf{R}^2 (for D we can take a disc of infinitely large radius) or with a rectangle in \mathbf{R}^2 (if D itself is obtained by an infinitely small rounding of the corners of this rectangle).

2°. The maps of going over to the closure and taking the boundary are extended from \mathcal{Z} to $*D$, so that \bar{D} and ∂D are defined for $D \in *\mathcal{Z}$. Here, $\partial D \in *\Gamma$, where Γ is the set of all closed curves without self-intersection of class C^1 in \mathbf{R}^2 ; $\bar{D} = D \cup \partial D$.

3°. The neighbourhood \mathcal{U} of \bar{D} can be defined, for example, as an element of $*\mathcal{Z}$, representing a set containing \bar{D} .

4°. Vector fields must be taken from a function space that is an element of the set obtained by application of the functor $*$ to the corresponding set of standard function spaces (compare §1.3). In particular, the parameters a_1 and a_2 may be non-standard.

5°. By the transfer principle, at each point $y \in \partial D$ (for $D \in *\mathcal{Z}$) there is a tangent and a vector normal to the boundary; this allows us to define transversality and intervals of entry and exit just as in the standard case. Transversality in the non-standard sense means that the vector field makes a non-zero angle with a tangent to the boundary, but this angle may be infinitely small!

6°. The conclusion of the lemma remains the same, but continuity of $z(a)$ must also be understood in the sense of belonging to a non-standard function space obtained by analogy with the arguments of §1.3. However, by the transfer principle all non-standard intermediate values of $z(a)$ are assumed in the usual sense.

Now we are ready to turn to the proof of the existence of ducks.

Proof of Theorem 2.1. We want to apply Lemma 2.1 (in the non-standard version) and for this purpose we consider in the phase plane a domain $D \in *\mathcal{Z}$ that has the form of a rectangle $(c_1, c_2) \times (d_1, d_2)$ with corners rounded by means of arcs of infinitely small radius (Fig. 2.7).

We take c_1 and c_2 as in the condition of the theorem, and choose d_1 and d_2 so that the following conditions are satisfied:

a) d_1, d_2 are standard and $d_1 < d_2 < 0$;

b) the numbers $|d_1|$ and $1/|d_2|$ are sufficiently large (this condition will be made more precise later).

In particular, we assume that $d_1 < \min_{x \in [c_1, c_2]} g(x)$ and $d_2 > \max_{x \in [c_1, c_2]} g(x)$, so that the part of the graph of $g(x)$ above $[c_1, c_2]$ belongs entirely to the rectangle $[c_1, c_2] \times [d_1, d_2]$ and intersects its boundary in the vertical sides $x = c_1$ and $x = c_2$.

Let us verify that D satisfies the conditions of the non-standard version of Lemma 2.3 with respect to the vector field (2.2). The unique position of equilibrium $(a, 0)$ of (2.2) does not belong to D . Transversality of (2.2) with respect to the vertical parts of ∂D is evident from the fact that $v < 0$ in D . (In this case the shadow of the field in many places is vertical and is

not transversal to the shadow of ∂D !) An elementary consideration of the field shows that (2.2) is transversal to ∂D also on the rounded parts. Points of non-transversality, therefore, lie on the horizontal parts, namely, where $a - x - f'(x)v = 0$, that is, on the slow curve.

Now we want to choose d_1 and d_2 so that the slow curve has precisely one point of intersection with each of the segments $I_j = \{(x, d_j): c_1 \leq x \leq c_2\}$ ($j = 1$ or 2). Let $F_j(x) = x - x_0 + f'(x)d_j$ ($j = 1$ or 2). Then the points of intersection of the slow curve with I_j have the form (x, d_j) , where x is a solution of the equation $F_j(x) = a - x_0$ in the segment $[c_1, c_2]$. We recall that it is sufficient to consider the case when a is sufficiently close⁽¹⁾ to x_0 . Since $F'_j(x) = 1 + f''(x)d_j$, we see that if a standard interval (\bar{c}_1, \bar{c}_2) is chosen so that $f''(x) > 0$ for $x \in [\bar{c}_1, \bar{c}_2]$, then for sufficiently large $|d_1|$ and $1/|d_2|$

$$F'_1(x) < 0 \quad \text{and} \quad F'_2(x) > 0 \quad \text{for} \quad x \in [\bar{c}_1, \bar{c}_2].$$

Since $f''(x_0) > 0$, the interval (\bar{c}_1, \bar{c}_2) can be chosen so that $x_0 \in (\bar{c}_1, \bar{c}_2)$ and $[\bar{c}_1, \bar{c}_2] \subset [c_1, c_2]$. Now we can arrange that F_1 and F_2 do not vanish on $[c_1, \bar{c}_1]$ and $[\bar{c}_2, c_2]$ by a further increase of $|d_1|$ and $1/|d_2|$. (This is possible, because by our assumption $f'(x) < 0$ on $[c_1, \bar{c}_1]$ and $f'(x) > 0$ on $[\bar{c}_2, c_2]$.) But then for sufficiently small $a - x_0$ the equation $F_j(x) = a - x_0$ does not have solutions on $[c_1, \bar{c}_1]$ and $[\bar{c}_2, c_2]$ (even in ${}^*\mathbf{R}$), since $F_j(x)$, as a standard continuous function, is non-zero on these intervals. At the same time, since $F_j(x_0) = 0$ and $F'_j(x) \neq 0$ on (\bar{c}_1, \bar{c}_2) , the equation $F_j(x) = a - x_0$ has one and only one solution (by the implicit function theorem) in (\bar{c}_1, \bar{c}_2) , again on condition that $a - x_0$ is sufficiently small.

Thus, the domain D described above satisfies the conditions of the non-standard version of Lemma 2.3 if we take $a \in [x_0 - \delta, x_0 + \delta]$, where δ is a sufficiently small standard positive number. We consider the disposition of the intervals of entry and exit. It is clear that the whole left-hand side of the boundary of the "rectangle" D (that is, the left vertical part with the roundings adjacent to it) belongs to the interval of exit $S(a)$ of the field (2.2) for any a sufficiently close to x_0 . At the same time, the whole right-hand side of the boundary of D belongs entirely to the interval of entry $E(a)$.

Suppose now that $y_0 = (c_2, g(c_2)) \in \partial D$, so that y_0 belongs to the intersection of all intervals of entry $E(a)$ for $a \in [x_0 - \delta, x_0 + \delta]$. Applying the non-standard version of Lemma 2.3 we see that the exit point $z(a)$ of the trajectory of (2.2) starting at y_0 depends continuously on $a \in [x_0 - \delta, x_0 + \delta]$. But from an examination of the directions of the field (Fig. 2.8) it is clear that if $a \ll x_0$ (for example, $a = x_0 - \delta$), then the exit point $z(a)$ must be situated on the lower side of the boundary of D (its shadow must coincide with the point of intersection of the shadow of the

⁽¹⁾These words and their synonym " $a - x_0$ is sufficiently small" here and later are understood in the following way: $a \in {}^*\mathbf{R}$ and $|a - x_0| < \delta$ for some standard sufficiently small $\delta > 0$.

slow curve and the horizontal line $v = d_1$), and if $a \gg x_0$ (for example, $a = x_0 + \delta$), then $z(a)$ is situated on the upper side of ∂D (Fig. 2.8).

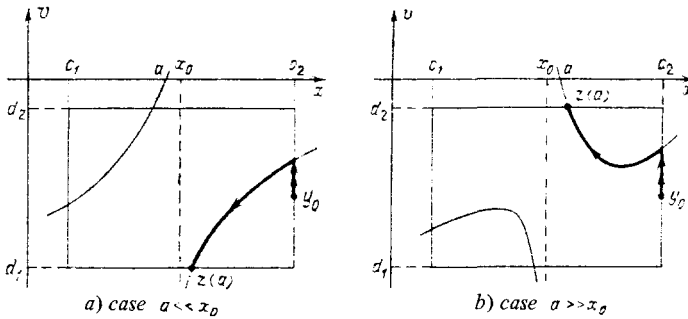


Fig. 2.8.

By the non-standard version of Lemma 2.3 we find that there is a parameter value $a \in [x_0 - \delta, x_0 + \delta]$ such that $z(a) = (c_1, g(c_1))$. (Instead of this we can, of course, take any other point on the left-hand side of ∂D .) But then it is clear that the corresponding solution is a duck, whose shadow contains all points $(x, g(x))$ for $x \in [c_1, c_2]$, as required. This completes the proof of Theorem 2.1. ■

Remark. It stands to reason that solution-ducks are far from being unique. This follows clearly and simply from the theorem about the continuous dependence of a solution on the parameters and initial conditions: if we change the parameter a and the initial condition of the solution-duck by a sufficiently small amount (even in the non-standard sense), then it remains a duck. Quite different is the case of duck-cycles, which may be unique (for a fixed value of a) and stable (as is the case, for example, in the van der Pol equation; see §4 later).

2.3. Standard interpretation.

We now give a standard interpretation of the results just proved (in particular, Theorem 2.1 about the existence of ducks). This is done by analogy with the standard interpretation of the theorem on the existence of a large cycle for the van der Pol equation (see Theorem 1.3).

We regard (2.1) as a standard equation in \mathbf{R}^2 with a small parameter $\varepsilon > 0$, $\varepsilon \in \mathbf{R}$. We also assume that the parameter a is a standard real number. We denote by $(2.1)_{\varepsilon, a}$ the equation (2.1) with the indicated values of the parameters ε and a . As above, let x_0 be a non-degenerate minimum point of $f(x)$.

Definition 2.2. We define a *standard duck* of the system (2.1) as a sequence $(x_n(t), y_n(t))$ ($n = 1, 2, \dots$) consisting of solutions of the equations $(2.1)_{\varepsilon_n, a_n}$ such that:

- a) $\varepsilon_n > 0$ for all n ; $\lim_{n \rightarrow \infty} \varepsilon_n = 0$;
- b) $\lim_{n \rightarrow \infty} a_n = a_0$ exists;

c) the solution $(x_n(t), y_n(t))$ is defined for $t \in (c_n, d_n)$ and there are two closed disjoint subintervals $[c'_n, d'_n]$ and $[c''_n, d''_n]$ of (c_n, d_n) , of which the first lies to the left of the second and the restriction of this solution to $[c'_n, d'_n]$ (respectively, $[c''_n, d''_n]$) defines a curve on the Liénard plane that converges⁽¹⁾ as $n \rightarrow \infty$ to the portion of the slow curve $y = f(x)$ above a non-trivial interval of the semi-axis $\{x: x > x_0\}$ (respectively, $\{x: x < x_0\}$).

Let us establish a correspondence between ducks in the sense of Definition 2.1 and standard ducks in the sense of Definition 2.2. If $(x_n(t), y_n(t))$ is a standard duck, then taking an infinitely large natural number n_0 and putting $(x(t), y(t)) = (x_{n_0}(t), y_{n_0}(t))$ we obtain the duck of (2.1) corresponding to the parameter values $\varepsilon = \varepsilon_{n_0}$ and $a = a_{n_0}$. Since $\circ(a_{n_0}) = a_0$, from Proposition 2.1 it is clear that a necessary condition for the existence of a standard duck is $a_0 = x_0$.

Conversely, let $(x(t), y(t))$ be a duck of (2.1). We denote by Γ_1 and Γ_2 the two compact connected pieces of the attracting and repelling parts of the slow curve, respectively, where $\Gamma = \Gamma_1 \cup \Gamma_2$ belongs to the shadow of the duck in question. Choosing an arbitrary standard $\delta > 0$ we can make the following assertion:

There are numbers $\varepsilon > 0, a \in \mathbf{R}$, and a solution $(x(t), y(t))$ of the system (2.1) _{ε, a} such that $\varepsilon < \delta, |a - x_0| < \delta$, and the compact set Γ lies in the δ -neighbourhood of the curve determined by the solution $(x(t), y(t))$ on the Liénard plane.

This is true in the non-standard sense, that is, for equations in ${}^*\mathbf{R}^2$ (for $(x(t), y(t))$ we can take the duck itself). But then by the transfer principle it is true also in the standard sense. Now taking $\delta = 1/n$ and denoting the resulting solutions by $(x_n(t), y_n(t))$ we obtain a standard duck in the sense of Definition 2.2.

Thus, the existence of standard and non-standard ducks occurs simultaneously. An obvious refinement of these arguments shows that the following result is true.

Theorem 2.2. *Under the conditions of Theorem 2.1 the family of standard equations (2.1) has standard ducks. If (c_1, c_2) is a standard finite interval of \mathbf{R} such that $x_0 \in (c_1, c_2)$ and $f'(x) \neq 0$ for $x \in [c_1, c_2]$, $x \neq x_0$, then there is a standard duck such that for any piece of the slow curve above a closed subinterval of $[c_1, c_2]$ that does not contain the point x_0 there are parts of the curves $(x_n(t), y_n(t))$ forming the relevant standard duck that converge uniformly to this piece.*

⁽¹⁾Here we can understand convergence, for example, in the sense of the metric (1.5). However, as is easy to verify, we may assume that the restrictions of the solution $(x_n(t), y_n(t))$ to the subintervals $[c'_n, d'_n]$ and $[c''_n, d''_n]$ for large n have the form of graphs $y = g_n(x)$, and then convergence can be understood as uniform convergence of $g_n(x)$ to $f(x)$ as $n \rightarrow \infty$.

Although in essence Definition 2.1 and Theorem 2.1 are equivalent to Definition 2.2 and Theorem 2.2, the non-standard definition and theorem look considerably more simple, clear, and elegant than their standard analogues. It is not surprising that the very formulation of the duck problem comes from a paper of Reeb in connection with non-standard analysis (see his letter, reproduced in [19]). A comparison of the standard and non-standard versions of the statements shows that the language of non-standard analysis is natural in problems of the type under consideration.

Remark. The reader may have noticed that certain statements of non-standard analysis allow for an “automatic” translation into the standard language with the help of the transfer principle, while others need some additional effort to work out a “standard interpretation”. Statements of the first type are called internal and those of the second type external. The formal difference between them is described in a paper by Nelson [4]; there a purely syntactic algorithm for constructing a standard interpretation of any external statement is given. Note that the majority of theorems about ducks are external.

§3. Approximations of ducks

The solution-ducks constructed in §2 have remarkable qualities from the point of view of the construction of approximate solutions. We can obtain in explicit form not only asymptotic expansions of solution-ducks and the corresponding parameter values a in powers of ϵ (§3.1), but even exponential terms of the expansion (§§3.3, 3.4). However, the role of the results obtained in this section is far from being exhausted by constructing approximate solutions. Thus, the explicit form of the asymptotic expansion in powers of ϵ allows us to construct a number of examples in which the solution-ducks themselves are also in the explicit form (§3.2). The construction of exponential terms of the expansion allows us to construct an entry-exit function (§3.3), which is the most important instrument for the studying the qualitative behaviour of solutions (actually used in §4). Also, the discovery of exponential terms is very rare in the theory of asymptotic expansions and therefore deserves special attention.

3.1. Asymptotic expansion of solution-ducks and the parameter a in powers of ϵ .

To begin with we indicate the meaning of asymptotic expansion in powers of an infinitely small $\epsilon \in {}^*\mathbf{R}$, which appears later. We say that a number $a \in {}^*\mathbf{R}$ admits an *asymptotic expansion in powers of ϵ*

$$(3.1) \quad a \sim a_0 + a_1\epsilon + a_2\epsilon^2 + \dots + a_n\epsilon^n + \dots,$$

if $a_k \in \mathbf{R}$ ($k = 0, 1, \dots$) (that is, the coefficients a_k are standard) and for any standard integer $n \geq 0$ the number a can be written in the form

$$(3.2) \quad a = a_0 + a_1\epsilon + a_2\epsilon^2 + \dots + a_n\epsilon^n + \eta_n\epsilon^n,$$

where $\eta_n \sim 0$, that is, η_n is infinitely small.⁽¹⁾ It can be proved in the usual way that all the coefficients a_k in (3.1) are uniquely determined. For example, it is clear that $a_0 = {}^o a$, $a_1 = {}^o[\varepsilon^{-1}(a - a_0)]$ and so on.

We say that an internal function $y(x)$, defined on the closed interval $[c_1, c_2] \subset {}^* \mathbf{R}$ with standard ends c_1 and c_2 , admits an asymptotic expansion in powers of ε on this interval:

$$(3.3) \quad y \sim f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots + \varepsilon^n f_n + \dots,$$

if f_1, f_2, \dots are standard functions on $[c_1, c_2]$ and for any standard integer $n \geq 0$ the function y can be written in the form

$$(3.4) \quad y(x) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \dots + \varepsilon^n f_n(x) + \varepsilon^n \eta_n(x),$$

where $\eta_n(x) \sim 0$ for any $x \in [c_1, c_2]$. Clearly, all the functions $f_k(x)$ are uniquely determined by this expansion. Note that $\eta_n(x)$ is an internal function, so that by Robinson's principle (see [1], §10) there is an infinitely small $\bar{\eta} > 0$ such that $|\eta_n(x)| \leq \bar{\eta}$ for all $x \in [c_1, c_2]$.

We also mention that if y is a function of class ${}^* C^1([c_1, c_2], {}^* \mathbf{R})$ (see §1) and its derivative y' admits an asymptotic expansion

$$(3.5) \quad y' \sim g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \dots + \varepsilon^n g_n + \dots,$$

where $g_k \in C([c_1, c_2], \mathbf{R})$ ($k = 0, 1, 2, \dots$), and y itself has the asymptotic expansion written down above with coefficients f_k ($k = 0, 1, 2, \dots$), then $f_k \in C^1([c_1, c_2], \mathbf{R})$ and the expansion (3.5) for y' can be obtained by term-by-term differentiation of (3.3), that is, $g_k = f'_k$ ($k = 0, 1, 2, \dots$). To prove this we must integrate the expansion for y' from c_1 to x and use the uniqueness of the coefficients of the asymptotic expansion.

We now proceed to the study of asymptotic expansions of ducks and the corresponding values of the parameter a .

Suppose that the trajectory γ of (2.1) on the Liénard plane enters the ε -galaxy of the slow curve in the halo of the point with the abscissa $x_2 \gg x_0$ and leaves it in the halo of the point with the abscissa $x_1 \ll x_0$. In the interval between these two points it is defined by a function $y = y(x)$ infinitely close to $y = f(x)$. Let a be the parameter value corresponding to the solution in question.

Theorem 3.1. *The parameter a , and also the function $y(x)$, considered on any standard interval $[x'_1, x'_2]$, $x_1 \ll x'_1 \ll x'_2 \ll x_2$, admit an asymptotic expansion in powers of ε :*

$$(3.6) \quad \begin{cases} a \sim a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots + a_n \varepsilon^n + \dots \\ y(x) \sim f(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \dots + \varepsilon^n f_n(x) + \dots, \end{cases}$$

⁽¹⁾The symbol \sim in (3.1) and in the relation $\eta_n \sim 0$ is used in different senses. Unfortunately, this cannot be avoided, though the difference between these two uses of the sign \sim is always clear from the context.

Corollary 3.1. *The shadow on Π_n of the slow curve (that is, the curve on which $\dot{y}_n = 0$) is the union of the line $x = a_0$ and the curve $y_n = f_n(x)$. Moreover, outside the halo of $(a_0, f_n(a_0))$ the slow curve lies in the ε -galaxy (and not only in the halo) of its shadow.*

This follows from the fact that $a^{(n)}$ and P_n in the second equation (3.9) are finite.

(B_n). *The trajectory γ on Π_n enters the ε -galaxy of the slow curve (hence, by the corollary, also the ε -galaxy of $y_n = f_n(x)$) in the halo of the point with the abscissa $x_2 \gg a_0$ and leaves it in the halo of the point with the abscissa $x_1 \ll a_0$.*

Π_1 is the phase plane. The validity of the corresponding assertions (A_1) and (B_1), except of finiteness of $a^{(1)}$, was proved in § 2 where $a^{(0)} = a$. As we see below by induction on n , the finiteness of $a^{(1)}$ follows from the remaining assertions in (A_1) and (B_1). (More generally: the finiteness of $a^{(n)}$ follows from the remaining assertions in (A_n) and (B_n).)

Lemma 3.1 (about the absence of excursions). *If (A_n) and (B_n) hold, then in the interval between x_2 and x_1 the trajectory γ on Π_n lies entirely in the halo of $y_n = f_n(x)$.*

The proof is similar to that of Lemma 2.2 (the only difference being that the curve $v = g(x)$ in Fig. 2.5 must be replaced by $y_n = f_n(x)$). ■

Now we take the inductive step.

1) We make the change $y_{n+1} = \varepsilon^{-1}(y_n - f_n(x))$ and obtain the following system on Π_{n+1} :

$$(3.10) \quad \begin{cases} \dot{x} = f_1 + \varepsilon f_2 + \dots + \varepsilon^{n-1} f_n + \varepsilon^n y_{n+1}, \\ \dot{y}_{n+1} = \varepsilon^{-1} (\dot{y}_n - f'_n(x) \cdot \dot{x}) = \\ = \varepsilon^{-1} [y_{n+1} (x - a_0) h(x) + a^{(n)} + Q_n - f'_n \cdot (f_1 + \dots + \varepsilon^n y_{n+1})], \end{cases}$$

where the polynomial Q_n is obtained from P_n by the substitution $y_n = f_n + \varepsilon y_{n+1}$. In the expression $Q_n - f'_n \cdot (f_1 + \dots + \varepsilon^n y_{n+1})$ certain terms contain the factor ε (among them all those in which y_{n+1} occurs); we put these terms outside the square brackets, having first cancelled ε , and obtain

$$(3.11) \quad \dot{y}_{n+1} = \varepsilon^{-1} [y_{n+1} (x - a_0) h(x) + a^{(n)} - F_n(x)] + P_{n+1},$$

where P_{n+1} is some standard polynomial in the variables $\varepsilon, f_1, \dots, f_n, f'_1, \dots, f'_n, y_{n+1}$, and $F_n(x)$ is the sum of the terms that do not contain the factor ε . Finally, by writing $F_n(a_0) = a_n$ and $F_n(x) - a_n = (x - a_0)h(x)f_{n+1}(x)$ (which is the last equation in (3.7)), we obtain

$$(3.12) \quad \dot{y}_{n+1} = \varepsilon^{-1} (y_{n+1} - f_{n+1}(x)) (x - a_0) h(x) + a^{(n+1)} + P_{n+1},$$

where $a^{(n+1)} = (a^{(n)} - a_n)/\varepsilon$. Now (3.12) together with the first equation of the system (3.10) forms a system of exactly the same type as (3.9); only the finiteness of $a^{(n+1)}$ is not yet proved.

2) We construct the shadow of the slow curve of (3.12) on Π_{n+1} . Here it is convenient to use (3.11): the finiteness of $a^{(n)}$ guarantees that the branches of the curve lie at a finite distance from one another (Fig. 3.1).

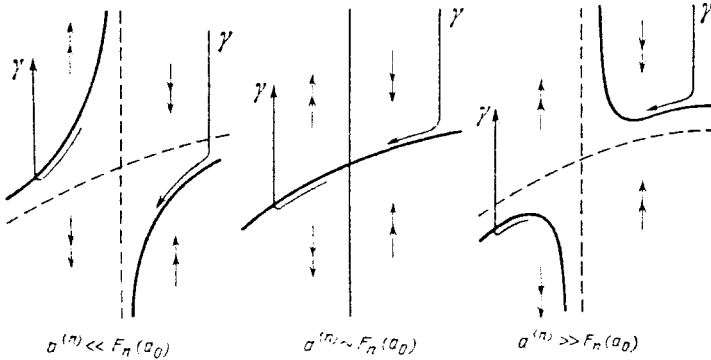


Fig. 3.1.

By (B_n) , the trajectory γ on Π_n , has a point whose abscissa is infinitely close to x_2 in the ϵ -galaxy of $y_n = f_n(x)$. The corresponding point on Π_{n+1} has the same abscissa and a *finite* ordinate y_{n+1} . Hence, by virtue of the slow-rapid character of the field on Π_{n+1} , the trajectory γ enters the ϵ -galaxy of the slow curve on Π_{n+1} in the halo of the point with the abscissa x_2 (see the argument in the proof of Theorem 2.1). Similar arguments on the “exit” of the trajectory show that (B_{n+1}) holds.

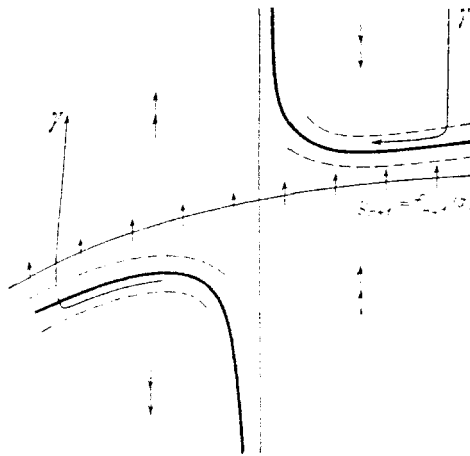


Fig. 3.2.

3) Next we prove that the finiteness of $a^{(n+1)}$ in (3.12) follows from the remaining assertions in (A_{n+1}) and (B_{n+1}) . Let $a^{(n+1)}$ be infinitely large and, to be definite, positive. Then on $y_{n+1} = f_{n+1}(x)$ the field is directed vertically upwards (Fig. 3.2). At the same time the character of the configuration of the branches of the slow curve is such that the right branch

lies above the curve $y_{n+1} = f_{n+1}(x)$ and the left branch below. (This follows from the fact that $h(x) < 0$ and $a^{(n+1)} > 0$.) What is more, the entire ε -galaxy of the right branch of the slow curve lies above the curve $y_{n+1} = f_{n+1}(x)$ and that of the left branch below. (This follows from the fact that $a^{(n+1)}$ is infinitely large.) Consequently, in order to satisfy (B_{n+1}) , γ must intersect the curve $y_{n+1} = f_{n+1}(x)$ from top to bottom, which contradicts the direction of the field on it.

The case of infinitely large negative $a^{(n+1)}$ is treated similarly: in this case the field on the curve $y_{n+1} = f_{n+1}(x)$ is directed vertically downwards, and the trajectory must intersect this curve from bottom to top. Thus, $a^{(n+1)}$ is finite, and (A_{n+1}) is true.

Let us sum up. We have proved by induction the validity of (A_n) and (B_n) for any n , and thus the existence of the expansions (3.6). In addition, we have proved (3.7), except those formulae that give an explicit expression for the functions $F_n(x)$. This expression can be obtained by writing out the polynomials P_n explicitly, which is, however, rather cumbersome. Instead, we use the fact that the *existence* of the expansions (3.6) has already been proved and write out the following equalities:

$$\begin{aligned} \dot{x} &= \varepsilon^{-1} (y(x) - f(x)) = f_1 + \varepsilon f_2 + \dots + \varepsilon^{n-1} f_n + \dots, \\ \dot{y} &= a - x = (a_0 - x) + a_1 \varepsilon + \dots + a_n \varepsilon^n + \dots, \\ y'(x) &= f' + \varepsilon f' + \dots + \varepsilon^n f'_n + \dots \end{aligned}$$

(The existence of an expansion for y' can be obtained by dividing the expansion of \dot{y} by that of \dot{x} .) Substituting the resulting expressions in $\dot{y} = y'(x)\dot{x}$ and collecting similar terms, we obtain

$$(3.13) \quad (f'f_1 + x - a_0) + (f'f_2 + f'_1f_1 - a_1) \varepsilon + (f'f_3 + f'_1f_2 + f'_2f_1 - a_2) \varepsilon^2 + \dots + (f'f_{n+1} + f'_1f_n + f'_2f_{n-1} + \dots + f'_nf_1 - a_n) \varepsilon^n + \dots \equiv 0.$$

Equating each bracket to zero for $x = a_0$, we find that $a_n = F_n(a_0)$; for arbitrary x , by equating the first bracket to zero we find f_1 , then f_2 , and so on. As a result we obtain (3.7).

Theorem 3.1 is now proved. ■

Corollary 3.2. *If $a_0 = 0$ and $f(x)$ is even, then in the expansion (3.6) for a all the coefficients $a_n = 0$ ($n = 0, 1, 2, \dots$).*

Proof. From the fact that $f(x)$ is even it follows, by (3.7), that all the $f_n(x)$ are even, all the $f'_n(x)$ are odd, and all the $F_n(x)$ ($n = 1, 2, \dots$) are odd, hence, $a_n = F_n(0) = 0$. ■

Remark. The corollary, strictly speaking, does not yet mean that (2.1) with even $f(x)$ for $a = 0$ has solution-ducks: it only means that the values of a corresponding to the ducks are less than any finite power of ε . Nevertheless, ducks for $a = 0$ actually exist, which can be proved by symmetry arguments.

For if $a = 0$ and $f(x)$ is even, then the vector field corresponding to (2.1) is such that if (x, y) is mapped to $(-x, y)$, then the horizontal component of the vector is unchanged and the vertical component changes sign. Hence, all the trajectories that intersect $0y$ are symmetrical about this axis (Fig. 3.3).

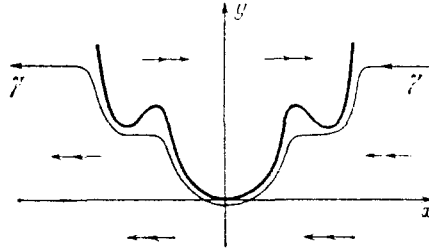


Fig. 3.3.

In the same way we can prove the existence of ducks for functions $f(x)$ such as, for example, $f(x) = x^4$, which does not satisfy the condition $f''(a_0) \neq 0$ assumed in Theorems 2.1 and 3.1. However, here the result of Theorem 3.1 is false: we can prove that a trajectory-duck on the Liénard plane deviates from the slow curve $f(x) = x^4$ by a quantity of order $\epsilon^{2/3}$ and not of order ϵ .

The qualitative behaviour of the solutions, in particular the evolution of cycles, depends principally on whether the values of a corresponding to the ducks lie to the left or to the right of a_0 , in other words, on the sign of a_1 . Therefore it is convenient that we can determine a_1 directly from $f(x)$.

Corollary 3.3. *The following formula holds: $a_1 = -\frac{1}{2} \frac{f'''}{(f'')^3}(a_0)$. In particular, if a_0 is a minimum point of $f(x)$, then the sign of a_1 is opposite to that of $f'''(a_0)$.*

Proof. We recall that $f_1(x) = 1/h(x)$, and so $F_1(x) = f_1'(x)f_1(x) = -h'(x)/h^3(x)$, where the function $h(x)$ is determined by the relation $f'(x) = -h(x)(x - a_0)$. By Taylor's formula, $h(a_0) = -f''(a_0)$ and $h'(a_0) = -f'''(a_0)/2$, as required. ■

In conclusion of this section we give the first three terms in the expansion (3.6) for the van der Pol equation. Substituting $f(x) = x^3/3 - x$, $a_0 = 1$ in (3.7), we obtain

$$f_1 = -\frac{1}{x+1}; \quad f_2 = -\frac{x^2+4x+7}{8(x+1)^3};$$

$$f_3 = -\frac{3x^5+21x^4+66x^3+126x^2+159x+121}{32(x+1)^7};$$

$$a_1 = -\frac{1}{8}; \quad a_2 = -\frac{3}{32}; \quad a_3 = -\frac{173}{1024}.$$

3.2. Live ducks.

Here we construct a number of examples, in which solution-ducks can be found explicitly.

The relatively simple form of (3.7) suggests the following question: is it possible to choose a function $\varphi(x)$ such that all the $f_n(x)$ ($n = 1, 2, \dots$) have the form

$$(3.14) \quad f_n(x) = k_n \cdot \varphi(x),$$

that is, they differ only by a factor (something like “eigenfunctions” for the non-linear map (3.7)). Then the trajectory-duck takes the form

$$(3.15) \quad y(x) = f(x) + (k_1\varepsilon + k_2\varepsilon^2 + \dots + k_n\varepsilon^n + \dots)\varphi(x) = \\ = f(x) + K(\varepsilon) \cdot \varphi(x).$$

We can substitute this solution in the initial equation (2.1) and obtain explicit values for the constants $K = K(\varepsilon)$ and $a = a(\varepsilon)$.

To find a suitable function $\varphi(x)$ we substitute f_n from (3.14) in the recursion formulae (3.7) and obtain

$$F_n(x) = k_1 k_n \varphi' \varphi + k_2 k_{n-1} \varphi' \varphi + \dots + k_n k_1 \varphi' \varphi = C_n \varphi'(x) \cdot \varphi(x), \\ f_{n+1}(x) = k_{n+1} \varphi(x) = \frac{F_n(x) - a_n}{x - a_0} f_1(x) = \frac{C_n \varphi' \varphi - a_n}{x - a_0} k_1 \varphi(x).$$

Cancelling $\varphi(x)$ we obtain

$$\varphi'(x) \cdot \varphi(x) = px + q,$$

where p and q are constants. From this we deduce that

$$(\varphi^2(x))' = 2(px + q), \quad \varphi^2(x) = Ax^2 + Bx + C,$$

and, finally,

$$(3.16) \quad \varphi(x) = \sqrt{Ax^2 + Bx + C},$$

where A, B, C are constants. Since the choice of k_1 is arbitrary, it is convenient to take $k_1 = -1$; then from $f_1(x) = -\varphi(x) = -(x - a_0)/f'(x)$ we obtain $f'(x) = (x - a_0)/\varphi(x)$, which allows us, knowing $\varphi(x)$, to find the function $f(x)$:

$$(3.17) \quad f(x) = \int \frac{x - a_0}{\varphi(x)} dx.$$

Various kinds of examples of ducks can be obtained for various choices of A, B, C , and a_0 in (3.16) and (3.17).

Example 1. $f(x) = x^2, a_0 = 0, \varphi(x) = \text{const}$. We look for the solution in the form $y(x) = x^2 + K$; substituting this in

$$dy/dx = 2x = \dot{y}/\dot{x} = (a - x)/\varepsilon^{-1}K,$$

we obtain $2Kx = \varepsilon a - \varepsilon x$, hence, $a = 0$ and $K = -\varepsilon/2$. Thus, for $a = 0$ we obtain the solution-duck $y = x^2 - \varepsilon/2$.

Example 2. $f(x) = \sqrt{1+x^2}$, $a_0 = 0$, $\varphi(x) = \sqrt{1+x^2}$. We look for a solution in the form $y(x) = (1+K)\sqrt{1+x^2}$; substituting it in

$$dy/dx = (1+K)x/\sqrt{1+x^2} = \dot{y}/\dot{x} = (a-x)/\varepsilon^{-1}K\sqrt{1+x^2},$$

we obtain $(1+K)Kx = \varepsilon(a-x)$, and so $a = 0$ and $K^2 + K + \varepsilon = 0$. Of the two solutions to this quadratic equation only $K = \frac{-1+\sqrt{1-4\varepsilon}}{2}$ is infinitely small, and $1+K = \frac{1+\sqrt{1-4\varepsilon}}{2}$. Hence, for $a = 0$ we obtain the solution-duck $y = \frac{1+\sqrt{1-4\varepsilon}}{2} \cdot \sqrt{1+x^2}$.

Example 3. $f(x) = (x-1) - \log x$ ($x > 0$), $a_0 = 1$, $\varphi(x) = x$. As above, substituting $y(x) = f(x) + Kx$ in $dy/dx = \dot{y}/\dot{x}$ we obtain $K^2 + K + \varepsilon = 0$ and $\varepsilon a = -K$, and so $K = \frac{-1+\sqrt{1-4\varepsilon}}{2}$, $a = \frac{1-\sqrt{1-4\varepsilon}}{2\varepsilon} =$

$= 1 + \varepsilon + 2\varepsilon^2 + 5\varepsilon^3 + 14\varepsilon^4 + \dots$. For this value of a we obtain the solution duck $y = \frac{1+\sqrt{1-4\varepsilon}}{2} x - 1 - \log x$. It is of interest that in this

example the coefficients a_n ($n = 0, 1, \dots$) form the sequence of Catalan numbers: they satisfy the recursion relation $a_0 = 1$, $a_n = a_0 a_{n-1} + a_1 a_{n-2} + \dots + a_{n-1} a_0$ and are also determined by the formula $a_n = \frac{1}{n+1} \binom{2n}{n}$, and $a(\varepsilon) = \frac{1-\sqrt{1-4\varepsilon}}{2\varepsilon}$ is their generating function (see [83]).

Examples 2 and 3 are particular cases of a more cumbersome formula:

$f(x) = \sqrt{x^2 + C} - a_0 \log(x + \sqrt{x^2 + C})$, $\varphi(x) = \sqrt{x^2 + C}$. Here for a solution-duck $K = \frac{-1+\sqrt{1-4\varepsilon}}{2}$, and $a = -(K/\varepsilon)a_0 = \frac{1-\sqrt{1-4\varepsilon}}{2\varepsilon} a_0 =$

$= a_0(1 + \varepsilon + 2\varepsilon^2 + \dots)$. Geometrically different pictures are obtained for $C < 0$, $C = 0$, and $C > 0$, and also for $a_0 < 0$, $a_0 = 0$, and $a_0 > 0$.

Example 4. $f(x) = (\sqrt{x})^3/3 - \sqrt{x}$ ($x \geq 0$), $a_0 = 1$, $\varphi(x) = 2\sqrt{x}$. We look for a solution in the form $y(x) = (\sqrt{x})^3/3 - \sqrt{x} + K\sqrt{x}$ and obtain

$$dy/dx = (x+K-1)/2\sqrt{x} = \dot{y}/\dot{x} = (a-x)/\varepsilon^{-1}K\sqrt{x},$$

and so $K = -2\varepsilon$ and $a = 1 + 2\varepsilon$. Hence, for $a = 1 + 2\varepsilon$ we obtain the solution-duck $y = (\sqrt{x})^3/3 - (1 + 2\varepsilon)\sqrt{x}$.

The form of the function $f(x)$ in the last example resembles the function $f(x) = x^3/3 - x$ in the van der Pol equation, so that the substitution $x' = x^{1/2}$ is quite natural. It leads to an equation that makes sense not only for positive values of x' ; however, we do not obtain the van der Pol system, since the second equation of the system also changes. As a result we obtain the next example.

Example 5. The equation

$$\begin{cases} \dot{x} = \varepsilon^{-1}(y - f_\varepsilon(x)), & f(x) = x^3/3 - x, \\ \dot{y} = 2x(a - x^2) \end{cases}$$

with $a = 1 + 2\varepsilon$ has the solution-duck $y = x^3/3 - (1 + 2\varepsilon)x$.

Of course, substitutions of a different kind could have been made in the preceding examples. In all five examples the resulting explicit solution gives the longest duck.

In conclusion we mention one interesting fact: in all our examples the series (3.6) turned out to be convergent. The function $f(x) = e^{x^2/2}$ gives an example of the situation when the series for $y(x)$ is only asymptotic. For from (3.7) it is easy to see that in this case $f_n(x) = (-1)^n k_n e^{-(2n-1)x^2/2}$, where the numbers k_n satisfy the relation

$$(3.18) \quad k_1 = 1, \quad k_{n+1} = n(k_1 k_n + \dots + k_n k_1).$$

Thus, the series (3.6) for $y(x)$ takes the form

$$y(x) = e^{x^2/2} \left(1 + \sum_{n=1}^{\infty} k_n \cdot (-\varepsilon e^{-x^2})^n \right) = e^{x^2/2} \left(1 + \sum_{n=1}^{\infty} k_n z^n \right),$$

where $z = -\varepsilon e^{-x^2}$, that is, it is actually a power series. Its coefficients k_n satisfy $k_n \geq (n-1)!$ (This is proved by a trivial induction from (3.18).) Consequently, by the Cauchy-Hadamard formula, the series has radius of convergence 0.

The authors do not know of any examples in which the series (3.6) for a is convergent.

3.3. The exponential microscope. Tunnels, funnels, showers. The entry-exit function.

By the transfer principle, on the non-standard plane ${}^*\mathbf{R}$ the theorem that solutions of differential equations depend continuously on the initial values remains in force. One of the consequences of this theorem is that if an equation has one solution-duck, then near this solution there must be other solution-ducks. From the results of §3.1 it follows that any two ducks (of course, living near one and the same extremum of $f(x)$) coincide to within any finite power of ε . However, on ${}^*\mathbf{R}^2$ the usual topology of solutions of differential equations is valid: no two phase curves can have common points. The only conclusion from this is that the distance between two trajectory-ducks is infinitely small of higher order, that is, less than ε^n for any finite n . Thus, the sequence of substitutions (“linear magnifying glasses”) $y_{n+1} = \varepsilon^{-1}(y_n - f_n(x))$ ($n = 1, 2, \dots$) does not allow us to distinguish between different ducks, separating them from one another by an appreciable distance: on each plane Π_n the ducks are infinitely close to one another. To obtain such a separation we must use a more powerful magnification—the so-called “exponential microscope”, which we now describe.

For any $\alpha > 0$ we introduce the odd function

$$x \mapsto x^{[\alpha]} = \begin{cases} x^\alpha, & x \geq 0, \\ -(-x)^\alpha, & x < 0. \end{cases}$$

This determines a one-to-one map of ${}^*\mathbf{R}$ onto ${}^*\mathbf{R}$. Its graph for an infinitely small α is conventionally shown in Fig. 3.4.

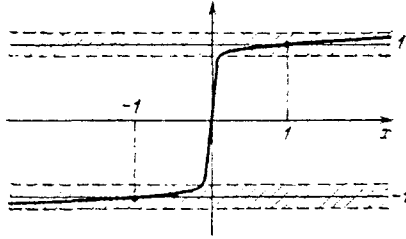


Fig. 3.4.

In this case this function maps all perceptible points into the halo of ± 1 ; many infinitely small numbers also fall into this halo, for example, all numbers of the form α^n for finite n . On the other hand, numbers of the form $e^{-1/k\alpha}$ for all finite positive k are mapped to numbers of the interval $[0, 1]$ at appreciable distances from 1. The function in question is infinitely differentiable everywhere except at $x = 0$. In what follows we need the relations

$$(3.19) \quad \begin{cases} (x^{[\alpha]})' = \alpha \cdot |x|^{\alpha-1}, \\ y = x^{[\alpha]} \iff x = y^{[1/\alpha]}, \\ |x|^{\alpha-1} \cdot x = x^{[\alpha]}. \end{cases}$$

Let $v = \beta(x)$ be the phase curve of (2.2) on the phase plane, which is a duck on some closed interval $[x_1, x_2]$, $x_1 \ll a_0 \ll x_2$, that is, it lies in the halo of the curve $v = f_1(x)$. On the phase plane we consider a domain \mathcal{T} bounded by the lines $x = x_1$ and $x = x_2$ and the curves $v = \beta(x) + 1$ and $v = \beta(x) - 1$. We substitute

$$(3.20) \quad z = (v - \beta(x))^{[\varepsilon]}.$$

On the (x, z) -plane the image of \mathcal{T} is precisely the rectangle $P = [x_1, x_2] \times [-1, 1]$. The graph of $v = \beta(x)$ goes into the line $z = 0$. We still have to rewrite (2.2) in the new coordinates. It is convenient to do this in two steps: putting $w = v - \beta(x)$ we obtain (after suitable transformations) the system

$$\begin{cases} \dot{x} = w + \beta(x), \\ \dot{w} = \varepsilon^{-1} (x - a) w / \beta(x); \end{cases}$$

writing $z = u^{1/\epsilon}$ and taking (3.19) into account we obtain the system on the (x, z) -plane

$$(3.21) \quad \begin{cases} \dot{x} = z^{1/\epsilon} + \beta(x), \\ \dot{z} = (x - a) z / \beta(x). \end{cases}$$

We find that the right-hand sides of both the equations in (3.21) are finite in the entire rectangle P . Outside the halo of the lines $z = \pm 1$ the value of $z^{1/\epsilon}$ is infinitely small, so that we can write down a standard system infinitely close to (3.21):

$$(3.22) \quad \begin{cases} \dot{x} = f_1(x) = 1/h(x), \\ \dot{z} = (x - a_0) z / f_1(x) = (x - a_0) h(x) z; \end{cases}$$

we recall that $f_1(x)$ is the shadow of the trajectory-duck on the phase plane, so that the function $h(x) = 1/f_1(x)$ can be found from $f'(x) = -h(x)(x - a_0)$. Now (3.22) is remarkable in that as an equation with separable variables it can be solved explicitly:

$$(3.23) \quad z = \bar{z}(x) = C \cdot \exp \left\{ \int h^2(x)(x - a_0) dx \right\}.$$

For example, for the van der Pol equation $h(x) = -(x + 1)$, $a_0 = 1$, hence,

$$z = C \cdot \exp \left\{ \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} - x \right\}.$$

We write⁽¹⁾

$$(3.24) \quad H(x) = \int h^2(x)(x - a_0) dx = \int \frac{(f'(x))^2}{x - a_0} dx = - \int f'(x) h(x) dx.$$

Lemma 3.2. *On the interval $[x_1, x_2]$ the function $H(x)$ has a unique minimum at $x = a_0$; $z = \bar{z}(x)$ has a unique extremum at the same point.*

The proof is straightforward if we take into account the fact that $h(x)$ does not vanish on $[x_1, x_2]$. ■

This lemma allows us to draw a picture of the solutions of (3.22) (Fig. 3.5).

We must establish the relationship between the trajectories of the systems (3.21) and (3.22). Outside the halo of the lines $z = \pm 1$ the right-hand sides of these systems are infinitely close, consequently the trajectories are also infinitely close (see [1], §16). It turns out that they stay infinitely close right up to the lines $z = \pm 1$. For suppose that a trajectory of (3.21) is in the halo of some standard point $(x_0, 1)$, $x_0 \in [x_1, x_2]$, $x_0 \neq a_0$. Then its

⁽¹⁾It is useful to note that this integral, multiplied by ϵ^{-1} , becomes infinitely close to $\int \text{div } \mathcal{Y} dt$, where \mathcal{Y} is a vector field on the Liénard plane and γ is a trajectory-duck. For, $\text{div } \mathcal{Y} = -\epsilon^{-1}f'(x)$ and $dt = dx/\dot{x} \sim h(x)dx$, since for ducks $\dot{x} \sim 1/h(x)$.

velocity along the vertical \dot{z} is not infinitely small; therefore, the trajectory covers an infinitely small distance along the vertical (up to the line $z = 1$) in infinitely small time. Since the velocity along the horizontal \dot{x} stays finite, in this same time the trajectory covers only an infinitely small distance in the horizontal direction. (Outside P the analogous assertion becomes false: being still in the halo of $z = \pm 1$, the value of $z^{[1/\varepsilon]}$ becomes infinitely large, so that trajectories of (3.21) become almost horizontal.) We have obtained the following theorem.

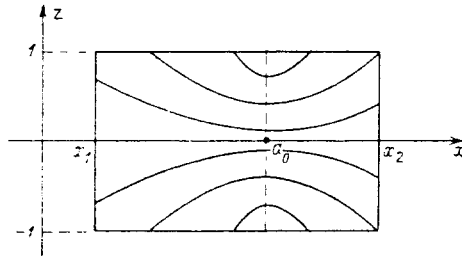


Fig. 3.5.

Theorem 3.2. *The functions $\bar{z}(x)$ defined by (3.23) within the boundaries of the rectangle $P = [x_1, x_2] \times [-1, 1]$ (and for standard C) are shadows of trajectories of (3.21).*

We still have to establish the relationship between the behaviour of a trajectory on the (x, z) -plane and that of the corresponding trajectory on the (x, v) -plane. Suppose that some trajectory $(x(t), v(t))$ of (2.2) on the (x, v) -plane enters the halo of the graph of $\beta(x)$ (or, what is equivalent, the halo of the graph of $f_1(x)$) in the halo of a point with the abscissa p , where p is standard, $a_0 < p \leq x_2$. In this case the “almost vertical” part of this trajectory intersects the curve $v = \beta(x) + 1$ (the upper boundary of V) at a point whose abscissa is equivalent to p . Hence, the corresponding trajectory $(x(t), z(t))$ on the (x, z) -plane intersects the line $z = 1$ in the halo of the point $(p, 1)$. In this case the shadow of the trajectory $(x(t), z(t))$ in P is the graph of one of the functions $\bar{z}(x)$ of the family (3.23), namely, the one that passes through $(p, 1)$. Two cases are possible:

1) The graph of $\bar{z}(x)$ intersects the line $z = 1$ for the second time at a point $(q, 1)$, where $x_1 \leq q < a$ (Fig. 3.6). In this case the trajectory $(x(t), z(t))$ also intersects the line $z = 1$ at an infinitely close point. Hence, returning to the (x, v) -plane, we can conclude that the trajectory $(x(t), v(t))$ leaves the halo of the graph of $\beta(x)$ (and $f_1(x)$) in the halo of a point with the abscissa q . The number q can be obtained from the conditions

$$(3.25) \quad H(q) = H(p), \quad q \neq p,$$

where H is defined by (3.24); the uniqueness of q follows from Lemma 3.2.

2) The graph of $\bar{z}(x)$ leaves the rectangle P on its lateral side, that is, at the point (x_1, z_1) , where $z_1 \ll 1$. (In Fig. 3.6 it is the graph passing through $(p', 1)$.) In this case the trajectory $(x(t), v(t))$ on the (x, v) -plane stays infinitely close to the graph of $\beta(x)$ up to the point x_1 .

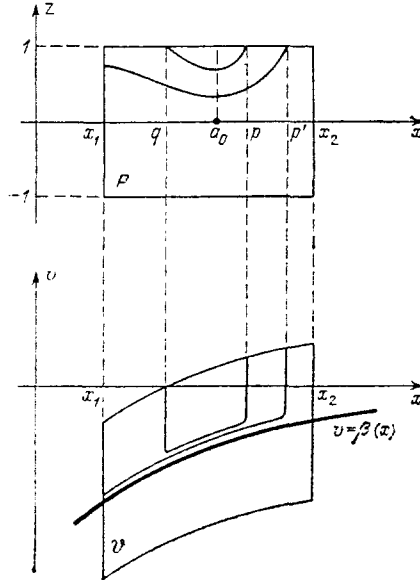


Fig. 3.6.

Similar statements can be made for the Liénard plane: if the trajectory $(x(t), y(t))$ enters the halo of $y = f(x)$ in the halo of a point $(p, f(p))$ (respectively, $(p', f(p'))$), then it leaves it in the halo of $(q, f(q))$ (respectively, it stays in the halo of the curve $y = f(x)$ up to $(x_1, f(x_1))$). But all this is true only under the condition that there is a solution-duck on the whole interval $[x_1, x_2]$.

The results we have obtained allow us to construct an *entry-exit function* $\Phi : p \mapsto q$, mapping “entry points” p into the corresponding “exit points” q . The function Φ is standard and is defined by (3.24) and (3.25) or by the equivalent relations

$$\int_q^p h^2(x) (x - a_0) dx = - \int_q^p f'(x) h(x) dx = 0.$$

A remarkable feature of Φ is that it does not depend on the parameter a . More precisely, its dependence on a is indirect: it influences only the domain of definition and range of the function Φ , but the functional dependence of q on p is the same for different a . The domain and the range of Φ are constructed in the following way. Suppose that for a given value of a there are solution-ducks of (2.1). From them we choose that which is the “longest duck” for the given value of a , that is, whose shadow includes the longest segment of the graph of $f(x)$. Suppose that this

segment of the graph is projected into the interval $[x_1, x_2]$ of the x -axis (one or both endpoints of $[x_1, x_2]$ may be infinite). We consider intervals $[q, p]$ such that $H(q) = H(p) = \alpha$, $\alpha > H(a)$, $q < p$. By Lemma 3.2, all these intervals contain the point a_0 , and for different α they are nested in each other. Therefore, among them there is a longest interval $[q_0, p_0]$ nested in $[x_1, x_2]$. In this case the domain of Φ for the given value of a is $[a_0, p_0]$, and its range is $[q_0, a_0]$; here $\Phi(a_0) = q_0$, $\Phi(p_0) = a_0$, and Φ decreases monotonically. (If we study ducks in the neighbourhood of a maximum point of $f(x)$, then the motion on all three planes—the Liénard plane, the phase plane, and the (x, z) -plane—is from left to right, so that $[q_0, a_0]$ becomes the domain for the entry-exit function and $[a_0, p_0]$ its range.)

The motion of the trajectories of (2.1) (and all equations connected with it) on the non-standard plane ${}^*\mathbf{R}^2$ can be imagined as a sort of *theatre of shadows*. What we can “observe” (or, if you prefer, “draw” on the graph) is not the trajectory itself but its shadow. In particular, we “do not see” how the trajectory moves over the set of infinitely large numbers; nor can we “distinguish” between infinitely close trajectories: they merge into one for us.

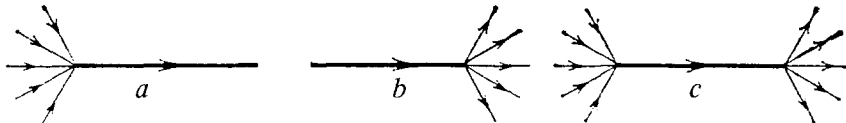


Fig. 3.7.

This colourful point of view helps us to notice three similar phenomena, which are represented in Fig. 3.7, *a*, *b*, and *c* and are called, respectively, a *funnel*, a *shower*, and a *tunnel*. A *funnel* is a bundle of trajectories that at the beginning are at an appreciable distance from one another and then become infinitely close. A *shower* is a bundle of trajectories that at the beginning are infinitely close and then deviate considerably from one another. A *tunnel* is a bundle of trajectories that forms a funnel at the beginning and then a shower. All three notions have their standard analogues, of course: these are bundles of trajectories such that the distance between them somewhere tends to zero as $\epsilon \rightarrow 0$, and somewhere does not. However, these phenomena were discovered by non-standard methods, namely in connection with ducks: we have seen that each solution-duck generates a whole tunnel of close trajectories; entering a tunnel and leaving it is regulated by the entry-exit function. (To discover funnels and showers we need to know the global picture of the behaviour of the trajectories of an equation; therefore, suitable examples will be given below in §4.) In the study of the qualitative behaviour of solutions of equations with a small parameter, funnels, tunnels, and showers play a role no less important than that of singular points in the qualitative study of ordinary equations.

For instance, the presence of a tunnel allows us to establish the existence of a duck-cycle (see §4.3), from which it follows in turn that there are funnels and showers.

We mention in conclusion that tunnels can be observed by means of a computer, if we draw the trajectories by means of a graph plotter. For example, let us choose $\epsilon = 1/20$ and a value of a corresponding to a duck. In this case the distance between two trajectory-ducks (in the domain where they are close) is of the order $e^{-1/\epsilon} = e^{-20} \approx 10^{-9}$. Such precision is still quite attainable for a numerical solution, so that two trajectories stay numerically distinct and can diverge later by an appreciable distance. At the same time, a graph plotter under any reasonable choice of scale illustrates precisely the lines that at first coincide and then diverge (that is, something like Fig. 3.7).

3.4. The life of ducks is short.

In §3.1 we established that any two values of a for which there are solution-ducks have the same asymptotic expansions in powers of ϵ (see (3.6)) hence differ from each other by a quantity less than any finite power of ϵ . In this section we establish that actually the difference between the two values of the parameter is a number of the form $e^{-1/h\epsilon}$, where k is some finite positive number. Thus, the range of the parameter corresponding to ducks is incredibly narrow: if (as is said about the van der Pol equation at the beginning of Cartier’s paper) we assume that ϵ is of the order 10^{-5} , then to “catch” a duck we need to make computations with an accuracy approximately $e^{-1/\epsilon} = e^{-10^5} \approx 10^{-43\ 000}$, that is, we must take into account more than 40000 decimal places.⁽¹⁾ The precise examples of §3.2 are of great interest.

Theorem 3.3. *Let \bar{a} be a parameter value for which there is a solution-duck. The parameter value $a \sim \bar{a}$ also corresponds to a solution-duck if and only if there is a finite positive number k such that*

$$(3.26) \quad |a - \bar{a}| = e^{-1/h\epsilon}.$$

Proof. Let $\beta(x)$ denote the solution-duck on the phase plane corresponding to the parameter value \bar{a} . As in §3.3, we construct on the phase plane the domain \mathcal{T} bounded by the lines $x = x_1$ and $x = x_2$ and the curves $v = \beta(x) + 1$ and $v = \beta(x) - 1$. (Here $\beta(x)$ is a duck on the whole interval $[x_1, x_2]$.) We carry out the same change of variables (3.20) as in §3.3: $z = (v - \beta(x))^{1/\epsilon}$, but now we apply it to the equation

$$(3.27) \quad \begin{cases} \dot{x} = v, \\ \dot{v} = \epsilon^{-1}(a - x - f'(x)v), \end{cases}$$

corresponding to the parameter value a and not \bar{a} .

⁽¹⁾At the same time, ducks and all phenomena connected with them can be effectively discovered by numerical computations for such “moderate infinitely small” values of ϵ as $1/10$ or $1/20$.

Let us outline briefly the corresponding computations. We put $w = v - \beta(x)$; from (3.27) we have $v'_x = \varepsilon^{-1}((a - x)/v - f'(x))$; from the analogous equation for \bar{a} we have $\beta'(x) = \varepsilon^{-1}((\bar{a} - x)/\beta(x) - f'(x))$ hence, $w'_x = \varepsilon^{-1}((a-x)/v - (\bar{a} - x)/\beta(x))$, and $\dot{w} = \varepsilon^{-1}(a - x - (a - x)v/\beta(x))$. The last equality is transformed by the substitution $v = w + \beta(x)$ to the form

$$(3.28) \quad \dot{w} = \varepsilon^{-1}w((x - \bar{a})/\beta(x) + (a - \bar{a})/w).$$

Next, by substituting $z = w^{[\varepsilon]}$ and bearing (3.19) in mind we obtain the equation

$$\dot{z} = z((x - \bar{a})/\beta(x) + (a - \bar{a})/z^{[1/\varepsilon]}).$$

At this moment it is convenient to write $b = (a - \bar{a})^{[\varepsilon]}$, and so $a - \bar{a} = b^{[1/\varepsilon]}$. Substituting this expression in the last equation we finally obtain the system

$$(3.29) \quad \begin{cases} \dot{x} = z^{[1/\varepsilon]} + \beta(x), \\ \dot{z} = z \left(\frac{x - \bar{a}}{\beta(x)} + \left(\frac{b}{z} \right)^{[1/\varepsilon]} \right). \end{cases}$$

We have to analyze the behaviour of the solutions of (3.29) in the rectangle $P = [x_1, x_2] \times [-1, 1]$ on the (x, z) -plane. First of all, $|a - \bar{a}| < 1$ and so $|b| < 1$. It is useful to consider two cases: (1) $|b| \ll 1$; (2) $|b| \sim 1$.

In the first case the lines $z = \pm b$ divide P into three strips, in each of which the trajectories behave differently. In the domain $|z| \ll |b|$ the term $(b/z)^{[1/\varepsilon]}$ in the second equation of (3.29) is infinitely large, so that the field (3.29) is almost vertical⁽¹⁾ (Fig. 3.8).

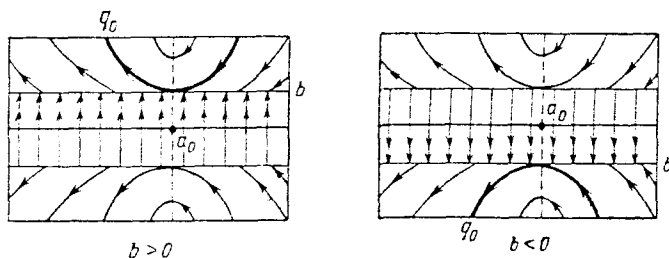


Fig. 3.8.

(Owing to the factor z the direction of the field is preserved on crossing the line $z = 0$, thus, it depends only on the sign of b .) Conversely, in the domains $|z| \gg |b|$ the term $(b/z)^{[1/\varepsilon]}$ is infinitely small, so that as in §3.3, (3.29) can be replaced by the infinitely close system (3.22) (see also Fig. 3.5). It is evident that a trajectory entering P at the point with the abscissa $p \gg a_0$ leaves at the point with the abscissa $q \ll a_0$, hence, is a duck. Thus, we can make our first deduction: if $|b| = |a - \bar{a}|^\varepsilon = e^{-1/k} \ll 1$, that is, if $k > 0$ is finite, then (3.27) has solution-ducks.

⁽¹⁾For $z = 0$ the second equation of (3.29) is undefined. Nevertheless, the direction field can be extended at these points by continuity, making it vertical.

Now we consider the case $|b| \sim 1$ and show that there are no ducks. Suppose, to be definite, that $b < 0$, so that we have a picture similar to the second one in Fig. 3.8, with the difference that this time the strip of vertical (rapid) motion occupies almost the whole rectangle P , keeping below and above only infinitely narrow strips of slow motion. We consider, again to be definite, a trajectory that enters P from above at the point with the abscissa $p \gg a_0$. (The other cases, including that of $b > 0$, are treated similarly.) At first this trajectory passes through the strip $z \sim -b$; to do this it requires an infinitely small amount of time, since in this domain $\dot{z} \ll 0$ (both terms $z(x - \bar{a})/\beta(x)$ and $z(b/z)^{[1/\varepsilon]}$ are negative, and the first is appreciable). It then passes through the strip $|z| \ll |b|$, which also needs an infinitely small amount of time (since in this domain \dot{z} is negative and infinitely large), and goes into the strip $z \sim b$. In this domain the negative term $z(b/z)^{[1/\varepsilon]}$, being finite, can be compensated by the positive term $z(x - \bar{a})/\beta(x)$. Thus, the trajectory, staying in the strip $z \sim b$, can reach the point with the abscissa $x = \bar{a}$. But later, for $x < \bar{a}$, both terms in the expression for \dot{z} become negative again. The assumption that the trajectory leaves P at the point with the abscissa $q \ll \bar{a}$ leads to a contradiction: to cover the path from $x = \bar{a}$ to $x = q$ needs an appreciable amount of time (we recall that \dot{x} is finite throughout P), and at those points of the trajectory where $x \ll \bar{a}$ the term $z(x - \bar{a})/\beta(x)$ in \dot{z} is appreciable; this contradicts the fact that the trajectory describes only an infinitely small path in the vertical direction. This proves the theorem. ■

Remark. A more careful examination of Fig. 3.8 allows us to obtain more information on trajectories of (3.29) than we needed to prove the theorem. In particular, it is easy to show that no trajectory stays in the halo of the lines $z = \pm b$ in the domain $x \leq a_0$ for more than an infinitely small amount of time. The same applies to the line $z = -b$ in the domain $x \geq a_0$. This allows us to define uniquely the shadow of any trajectory of (3.29) in P . In particular, *the longest duck corresponding to the parameter a leaves the halo of the slow curve in the halo of the point with the abscissa q_0 , where $q_0 \ll a_0$ is the abscissa of the points of intersection of the lines $z = \pm 1$ with the trajectories of (3.22) tangent to the lines $z = \pm b$ at the points with the abscissa a_0 .* (In Fig. 3.8 these trajectories are shown in boldface; of course, for other (smaller) values of b these trajectories may leave P through the lateral side, and then the longest duck corresponding to a reaches at least the point x_1 .)

This phenomenon has been called a *curtain*: for $a \ll a_0$ the curtain is closed; when a enters the domain of ducks, the curtain opens a little—the left fold moves up to $x = q$ and the right one up to $x = p$, where $\Phi(p) = q$, and the duck above the interval $[q, p]$ is the longest for the given value of a . In the opened vertical strip $q \leq x \leq p$ we can observe ducks and all phenomena connected with them. As a increases further, the curtain at first opens as wide as possible (“the longest duck”) and then starts to close again. For $a \gg a_0$ it is closed.

3.5. Second proof of the existence of asymptotic expansions of ducks and the parameter a in powers of ϵ .

We show here how we can use an exponential microscope for a certain modification of the proof of the existence of asymptotic expansions (3.6) for ducks and the corresponding values of a in powers of ϵ . For this purpose we use the comparatively simple algebraic part of the proof of Theorem 3.1 in §3.1 (the existence of a sequence of changes of variables $y_1 = \epsilon^{-1}(y - f(x)), \dots, y_{n+1} = \epsilon^{-1}(y_n - f_n(x)), \dots$, transforming the equations to the form (3.9) or (3.11)), but we do not use its rather complicated logical part.

Thus, we consider first of all for $n \geq 0$ on the (x, y_{n+1}) -plane Π_{n+1} the system (see (3.9) and (3.11))

$$(3.30) \quad \begin{cases} \dot{x} = f_1 + \epsilon f_2 + \dots + \epsilon^{n-1} f_n + \epsilon^n y_{n+1}, \\ \dot{y}_{n+1} = \epsilon^{-1} [y_{n+1} (x - a_0) h(x) + a^{(n)} - F_n(x)] + P_{n+1}. \end{cases}$$

Let $[c_1, c_2]$ be a standard interval such that $a_0 \in (c_1, c_2), f'(x) < 0$ for $x \in [c_1, a_0], f'(x) > 0$ for $x \in (a_0, c_2]$. We prove first the following lemma.

Lemma 3.3. *There is a finite value of the parameter $a^{(n)}$ such that (3.30) has a trajectory passing above the whole interval $[c_1, c_2]$ in the finite part of Π_{n+1} .*

Proof. In the proof we assume for simplicity that $n \geq 1$. Choosing $d_n > 0$ to be standard and sufficiently large, we consider on Π_{n+1} an internal domain D_n with boundary of class C^1 (the element $\ast\mathcal{D}$ in the notation of §2), whose shadow is the rectangle $[c_1, c_2] \times [-d_n, d_n]$. We assume that D_n is obtained from this rectangle by infinitely small rounding of its corners by means of arcs of circles of infinitely small radius. In the finite plane Π_{n+1} above $[c_1, c_2]$ there are no positions of equilibrium of (3.30), since $f_1(x) < 0$ for $x \in [c_1, c_2]$. In particular, D_n does not contain positions of equilibrium for any standard $d_n > 0$. The slow curve of (3.30) for $x \neq a_0$ can be written in the form

$$y_{n+1} = \frac{F_n(x) - a^{(n)} - \epsilon P_{n+1}}{(x - a_0) h(x)}.$$

Though this equation is implicit, y_{n+1} is involved in the right-hand side only in the infinitely small term ϵP_{n+1} ; therefore, by the implicit function theorem the shadow of the slow curve outside the halo of the line $x = a_0$ is the same as that of the curve

$$y_{n+1} = \frac{F_n(x) - a^{(n)}}{(x - a_0) h(x)}.$$

Next, as in the corresponding argument in the proof of Theorem 2.1, we can verify that if we fix an interval $[a_1^{(n)}, a_2^{(n)}]$ with standard end-points such that $F_n(a_0) \in (a_1^{(n)}, a_2^{(n)})$, then for sufficiently large $d_n > 0$ the equations

$$\pm d_n = \frac{F_n(x) - a^{(n)} - \epsilon P_{n+1}}{(x - a_0) h(x)}$$

(in which y_{n+1} in the polynomial P_{n+1} must be also replaced by $\pm d_n$) have for any $a^{(n)} \in [a_1^{(n)}, a_2^{(n)}]$ exactly one solution $x_{\pm}^{(n)} \in [c_1, c_2]$. This means that the slow curve of (3.30) has exactly one point of intersection with each of the upper and lower horizontal parts of the boundary of D_n . But then we can apply Lemma 2.3 (about shooting) to D_n and to the vector field (3.30) with the parameter $a^{(n)} \in [a_1^{(n)}, a_2^{(n)}]$. The whole left-hand side of the boundary ∂D_n of D_n belongs to the interval of exit $S(a^{(n)})$, and the whole right-hand side to that of entry $E(a^{(n)})$. Therefore, the point $(c_2, 0) \in \partial D_n$ belongs to the intersection of all intervals of entry $E(a^{(n)})$ for $a^{(n)} \in [a_1^{(n)}, a_2^{(n)}]$. But from an inspection of the field directions (Fig. 3.9) it is clear that for $a^{(n)} \ll F_n(a_0)$ the trajectory that starts at $(c_2, 0)$ leaves ∂D_n through its lower part and for $a^{(n)} \gg F_n(a_0)$ through its upper part. By Lemma 2.3 there is a parameter value $a^{(n)} \in [a_1^{(n)}, a_2^{(n)}]$ such that the point of exit lies on the left vertical part of ∂D_n . But this immediately gives the assertion of the lemma. ■

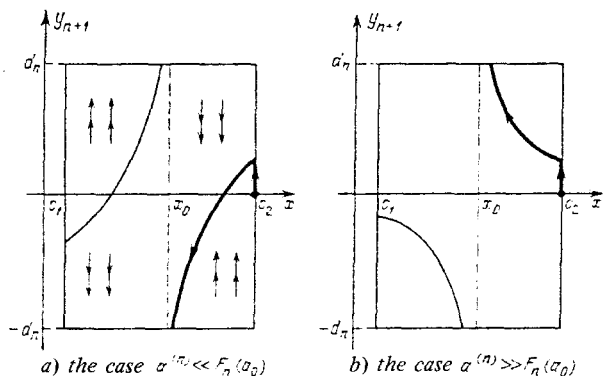


Fig. 3.9.

We now discuss what conclusions can be drawn from Lemma 3.3. First of all, recalling that

$$a = a_0 + a_1 \varepsilon + \dots + a_{n-1} \varepsilon^{n-1} + a^{(n)} \varepsilon^n,$$

we see that there is a parameter value a with a finite expansion (3.2) to which there corresponds a solution-duck. But then, by Theorem 3.3, any value of a corresponding to a duck also has a finite expansion (3.2). Since n is arbitrary here, we see that any value of a corresponding to a duck has a complete expansion of the form (3.1). (Incidentally, from this argument it is clear that if $a_n = \circ(a^{(n)})$, then $a_n = F_n(a_0)$.)

Next, the fact that $y_{n+1}(x)$ is finite means that the function $y(x)$ corresponding to the solution-duck has a finite expansion (3.4). But then, applying an exponential microscope $z = (y_1 - \beta(x))^{[e]}$ in the neighbourhood of the duck, as in §3.3, we find that any duck has such an expansion (on the part on which it is a duck). From this it follows that all ducks have a complete expansion (3.3) with coefficients that do not depend on the choice of the duck. This completes the second proof of the existence of asymptotic expansions (3.6).

§4. Qualitative behaviour of solutions of rapid-slow equations

The main aim of this section is the study of *solution-ducks*. Incidentally, however, we discover many additional special features of the qualitative behaviour of solutions of the equations in question. We begin, as usual, with an investigation of singular points.

4.1. Singular points of (2.1).

The only singular point of (2.1) is $(a, f(a))$. Writing down at this point the matrix of the linearized system

$$\begin{pmatrix} -\epsilon^{-1}f'(a) & \epsilon^{-1} \\ -1 & 0 \end{pmatrix}$$

and finding its eigenvalues

$$\lambda_{1,2} = \frac{-f'(a) \pm \sqrt{(f'(a))^2 - 4\epsilon}}{2\epsilon},$$

we conclude that:

- 1) if $f'(a) \geq 2\sqrt{\epsilon}$ (in particular, if $f'(a)$ is an appreciable positive number), then the singular point is a *stable node* (degenerate if $f'(a) = 2\sqrt{\epsilon}$);
- 2) if $f'(a) \leq -2\sqrt{\epsilon}$ (in particular, if $f'(a)$ is an appreciable negative number), then the singular point is an *unstable node* (degenerate if $f'(a) = -2\sqrt{\epsilon}$);
- 3) if $0 < f'(a) < 2\sqrt{\epsilon}$, then the singular point is a *stable focus*;
- 4) if $-2\sqrt{\epsilon} < f'(a) < 0$, then the singular point is an *unstable focus*.

If $f'(a) = 0$, in particular, at the extremum points of interest to us, both eigenvalues are purely imaginary, and the linear part does not give a precise answer to the question of the character of the singular point, which may turn out to be a centre, a stable or unstable focus, or a centre-focus. In this case we can proceed as follows: we go over to the phase plane,⁽¹⁾ obtain on it a singular point $(a, 0)$, and consider the Lyapunov function

$$\begin{aligned} U(x, v) &= (k/2)(x - a)^2 + \epsilon(v - k^{-1} \log(1 + kv)) = \\ &= (k/2)(x - a)^2 + \epsilon v^2(k/2 - (k^2/3)v + \dots), \end{aligned}$$

where $k = f''(a)$. Computing $dU/dt = k(x - a)v + \frac{kv}{1 + kv}(a - x - f'(x)v)$ and replacing $f'(x)$ by its quadratic part

$$\begin{aligned} f'(x) &\approx f'(a) + f''(a)(x - a) + f'''(a) \times (x - a)^2/2 = \\ &= k(x - a) + f'''(a)(x - a)^2/2, \end{aligned}$$

we obtain an approximation

$$dU/dt \approx -kf'''(a)v^2(x - a)^2/2(1 + kv).$$

Thus, if a is a minimum point of $f(x)$, then $k > 0$, $U(x, v) \geq 0$ in the neighbourhood of $(a, 0)$, and the sign of dU/dt is opposite to that of $f'''(a)$.

⁽¹⁾Transition to the phase plane, in spite of stretching by an infinitely large factor, is a diffeomorphism, therefore, preserves the character of the singular point.

Hence, the inequality $f'''(a) > 0$ implies that the singular point is stable, and then it is a stable focus, while $f'''(a) < 0$ implies that it is unstable, and then it is an unstable focus. Similarly, if a is a maximum point of $f(x)$, then $k < 0$, $U(x, v) \leq 0$, and the sign of dU/dt is the same as that of $f'''(a)$. Therefore, the conditions of stability or instability of the singular point remain the same.

Example. In the van der Pol equation $f(x) = x^3/3 - x$, hence $f'''(x) = 2$. This means that the singular points $(1, -2/3)$ (for $a = 1$) and $(-1, 2/3)$ (for $a = -1$) are both stable foci.

Stability or instability of the singular point $(a_0, f(a_0))$ for $f'(a_0) = 0$ and $f'''(a_0) \neq 0$ determines the character of the Hopf bifurcation, which occurs when a passes through the critical point a_0 . Suppose, to be definite, that a_0 is a non-degenerate minimum point of $f(x)$ (so that $f'(a_0) = 0$, $f''(a_0) > 0$), and the singular point $(a, f(a))$, as a increases, changes from unstable to stable. Then if $f'''(a_0) > 0$ (and $(a_0, f(a_0))$ is stable) as a passes through a_0 (in the direction of increasing a), absorption of the stable cycle by a singular point occurs, and for $f'''(a_0) < 0$ (and instability of the singular point $(a_0, f(a_0))$) an unstable cycle is generated from the singular point (see [63]).

4.2. Existence of duck-cycles and duck-solutions.

Among numerous solution-ducks of special interest are those that are cycles, that is, periodic trajectories.

The existence of duck-cycles, and also their form and evolution, depend on the specific form of the function f . To begin with we describe a general class of functions f for which there exist duck-cycles. It turns out that for this it is sufficient that the graph of f has the same form as that of the cubic curve $f(x) = x^3/3 - x$ for the van der Pol equation.

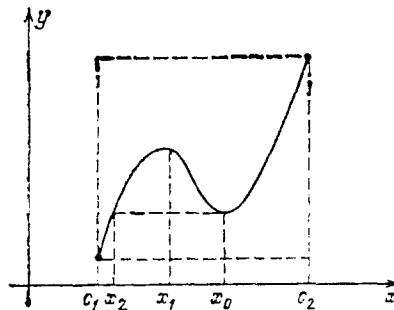


Fig. 4.1.

We say that f has an *admissible form* on $[c_1, c_2]$ if:

- a) f is a standard function of class C^2 on $[c_1, c_2]$;
- b) f has exactly two extremum points on $[c_1, c_2]$: an isolated minimum point x_0 and an isolated maximum point x_1 , where $c_1 < x_1 < x_0 < c_2$, so that $f'(x) > 0$ on $[c_1, x_1)$ and $(x_0, c_2]$ and $f'(x) < 0$ on (x_1, x_0) ;
- c) $f(c_1) < f(x_0)$ and $f(c_2) > f(x_1)$, so that the absolute minimum of f on $[c_1, c_2]$ is attained at c_1 (and only there), and the absolute maximum at c_2 .

Thus, the graph of f has the form shown in Fig. 4.1. Let $x_2 \in (c_1, x_1)$ be such that $f(x_2) = f(x_0)$ (it is evident that such an x_2 is unique).

Theorem 4.1. *Suppose that f has an admissible form on $[c_1, c_2]$. If $x_b \in \mathbf{R}$, $x_2 < x_b < x_0$, then there is a value of a such that the system (2.1) has a duck-cycle with the abscissa of the beak⁽¹⁾ x_b .*

Proof. 1) We consider the part of the slow curve in ${}^*\mathbf{R}^2$ strictly to the left of the singular point and denote it by Γ_a , that is,

$$\Gamma_a = \{(x, f(x)): x \in [c_1, a]\}.$$

We always assume that $a \in {}^*[a_1, a_2]$, where $a_1, a_2 \in \mathbf{R}$, $a_1 \in (x_1, x_0)$, $a_2 \in (x_0, c_2)$.

We consider on Γ_a the Poincaré succession map P_a with the domain of definition D_a , so that $P_a : D_a \rightarrow \Gamma_a$. We assume that $(x, f(x)) \in D_a$ if $x \in [c_1, a)$ and the trajectory of (2.1) that starts at $(x, f(x))$ once again intersects Γ_a , and then $P_a((x, f(x)))$ denotes the first point of intersection after $(x, f(x))$. Let $p_a(x)$ be the abscissa of $P_a((x, f(x)))$, so that

$$P_a((x, f(x))) = (p_a(x), f(p_a(x))).$$

It is easy to see that the set D_a and the map P_a are internal. The phase trajectories of (2.1) intersect Γ_a vertically upwards (and hence transversally). Therefore, the set

$$U = \{(a, x): a \in {}^*[a_1, a_2], (x, f(x)) \in D_a\},$$

is an internal open subset of the trapezium

$$\{(a, x): a \in {}^*[a_1, a_2], x \in [c_1, a]\},$$

and the function

$$U \rightarrow {}^*[c_1, c_2], \quad (a, x) \mapsto p_a(x)$$

is internal and continuous (in the sense of belonging to the corresponding space $C(U, {}^*[c_1, c_2])$) (see §1.3). Now if $(a, x_b) \in U$ is such that $p_a(x_b) = x_b$, and $x_2 \ll x_b \ll x_0$, then the trajectory of (2.1) that starts at $(x_b, f(x_b))$ is a duck-cycle with the abscissa of the beak x_b . Thus, for the given x_b the matter reduces to the proof of the existence of an a such that $p_a(x_b) = x_b$.

2) We fix a standard x_b such that $x_2 < x_b < x_0$. We assume that standard a_1 and a_2 are chosen so that $x_b < a_1, f(a_2) < f(x_1)$. The set

$$A = \{a: a \in {}^*[a_1, a_2], p_a(x_b) \text{ determinate}\} = \{a: (a, x_b) \in U\}$$

is an internal open subset of ${}^*[a_1, a_2]$. We put

$$a_3 = \inf({}^*[a_1, a_2] \setminus A) \text{ for } A \neq {}^*[a_1, a_2],$$

$$a_3 = a_2 \text{ for } A = {}^*[a_1, a_2].$$

⁽¹⁾The beak and other elements of the morphology of ducks are described in [1], §24.

It is easy to see that $p_a(x_b)$ can be indeterminate only when the singular point $(a, f(a))$ is a stable node (thus, $a > x_0$), and the trajectory γ_a of (2.1) that starts at $(x_b, f(x_b))$ goes into this singular point. The set of values a corresponding to stable nodes is closed; therefore, in any case we find that $a_3 > x_0$ and $(a_3, f(a_3))$ is a stable node for (2.1) with $a = a_3$.

3) We follow the change in $p_a(x_b)$ as a varies on $^*[a_1, a_3]$. Firstly, if $a = a_1$, then the shadow of γ_a breaks away from the slow curve at the minimum point $(x_0, f(x_0))$, since in the neighbourhood of this point there is a negative and not infinitely small vertical component of the velocity (Fig. 4.2). Therefore, $p_{a_1}(x_b) \sim x_2$, in particular, $p_{a_1}(x_b) \ll x_b$.

We now claim that $p_{a_3-\delta}(x_b) \sim a_3 - \delta$ for sufficiently small $\delta > 0$. We consider the two cases separately.

Case 1. $p_{a_3}(x_b)$ is determinate. In this case $a_3 = a_2 \gg x_0$. But then $p_{a_3}(x_b) \sim a_3$, since γ_{a_3} cannot descend appreciably below the point $(a_2, f(a_2))$ along the right-hand branch of the slow curve, since for $x < a_2$ the field has a positive vertical component (Fig. 4.3). For the same reason, in this case $p_{a_3-\delta}(x_b) \sim a_3 - \delta$ if $\delta > 0$ is chosen so that $a_3 - \delta \gg x_0$.

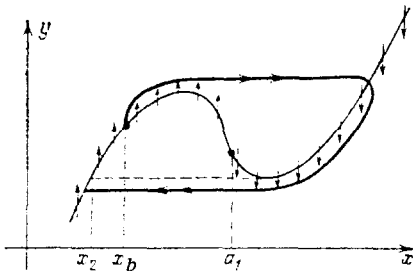


Fig. 4.2.

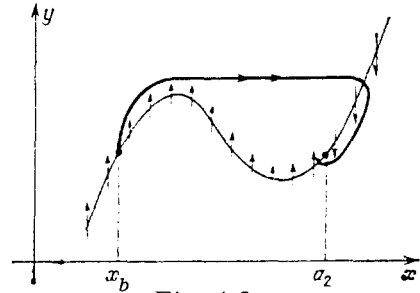


Fig. 4.3.

Case 2. $p_{a_3}(x_b)$ is indeterminate. In this case the singular point $(a_3, f(a_3))$ of (2.1) with $a = a_3$ is a stable node, and the trajectory γ_{a_3} goes into this point. By means of the usual construction of the Lyapunov function it is easy to construct an ellipse V with centre at $(a_3, f(a_3))$ and infinitely small semi-axes, which is a trap for the trajectories of (2.1) with $a = a_3$, that is, such that the vector field defined by the right-hand sides of this system is directed strictly inside V on the boundary of V (Fig. 4.4).

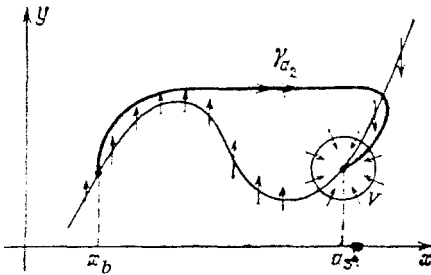


Fig. 4.4.

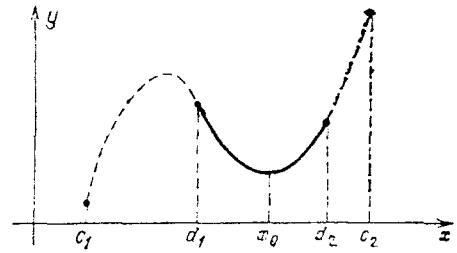


Fig. 4.5.

But then V remains a trap for trajectories of (2.1) for $a = a_3 - \delta$ if $\delta > 0$ is sufficiently small, that is, $\delta < \delta_0$ for some $\delta_0 \in \mathbf{R}^*$, $\delta_0 > 0$. In particular, the trajectory γ_a enters V and stays there for ever, and for sufficiently small $\delta > 0$ it does not intersect Γ_a before entering V . From this it follows that γ_a and Γ_a intersect inside V , that is, in the halo of $(a, f(a))$ (which is the same as that of $(a_3, f(a_3))$, since it is clear that in this case $\delta \sim 0$). Thus $p_{a_3-\delta}(x_b) \sim a_3 - \delta$, hence, in particular $p_{a_3-\delta}(x_b) \gg x_b$.

4) Thus,

$$p_{a_1}(x_b) \ll x_b, \quad p_{a_3-\delta}(x_b) \gg x_b.$$

But the internal continuous function $a \mapsto p_a(x_b)$ on the connected interval $[a_1, a_3 - \delta]$ must take all intermediate values. Therefore, there is an $a \in [a_1, a_3 - \delta]$ such that $p_a(x_b) = x_b$, as required. ■

Now by means of Theorem 4.1 we prove the existence of solution-ducks in the neighbourhood of a degenerate isolated minimum point. We assume that f is defined on the standard interval $[d_1, d_2]$ and has the following properties:

A. f is standard of class C^2 in $[d_1, d_2]$.

B. f has a unique extremum point on $[d_1, d_2]$, the minimum point x_0 , so that $f'(x) < 0$ for $x \in [d_1, x_0]$ and $f'(x) > 0$ for $x \in (x_0, d_2]$.

Theorem 4.2. *If f has the properties A and B, then there is an a such that the system (2.1) has a duck whose shadow passes along the whole part of the slow curve over the interval $[d_1, d_2]$.*

Proof. It is easy to extend a function of admissible form on some interval $[c_1, c_2] \supset [d_1, d_2]$ (Fig. 4.5). Now Theorem 4.2 follows from Theorem 4.1. ■

4.3. Duck-cycles of the van der Pol equation.

The van der Pol equation was the first example in which ducks were discovered and studied in detail. Here we describe properties of duck-cycles for the van der Pol equation, and also the tunnels, funnels, and showers surrounding them.

A. The existence of duck-cycles for the van der Pol equation follows from Theorem 4.1; we give here an alternative proof, based on the simple but important property of the entry-exit function defined in §3.3 (which we need later).

Lemma 4.1. *The entry-exit function Φ for the van der Pol equation has the following property: if $\Phi(p) = q$, then $f(p) < f(q)$ (that is, on the Liénard plane the entry point always lies higher than the exit point).*

Proof. We state our assertion in another way: it is known that

$$\int_q^p (x^2 - 1)(x + 1) dx = 0 \quad (\text{see } \S 3.3); \quad \text{we must prove that } \int_q^p (x^2 - 1) dx < 0;$$

here, $-1 \ll q \ll 1 \ll p$.

On the interval $[q, 1]$ we have $x+1 \leq 2$ and $x^2-1 \leq 0$, hence, $(x^2-1)(x+1) \geq 2(x^2-1)$. On the interval $[1, p]$ in turn we have $x+1 \geq 2$ and $x^2-1 \geq 0$, hence also $(x^2-1)(x+1) \geq 2(x^2-1)$. As a result we obtain

$$\int_q^p (x^2-1)(x+1) dx > 2 \int_q^p (x^2-1) dx,$$

as required. ■

Second proof of Theorem 4.1 for $f(x) = x^3/3 - x$. We start a trajectory from $(2-\delta, f(2-\delta))$, where $\delta > 0$ is standard and sufficiently small; we choose a value of $a \sim 1$ (see §2, Theorem 2.1) so that this trajectory is a duck leaving (to be definite, on the left) the slow curve in the halo of a standard point with the abscissa $q \ll 1$ (Fig. 4.6).

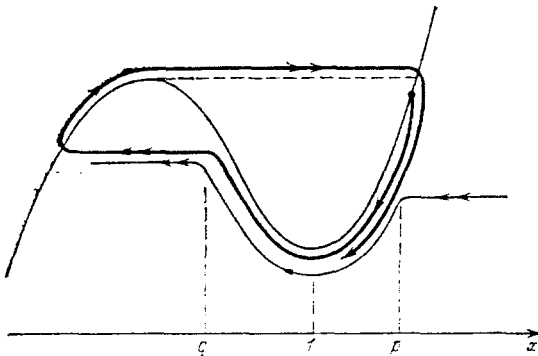


Fig. 4.6.

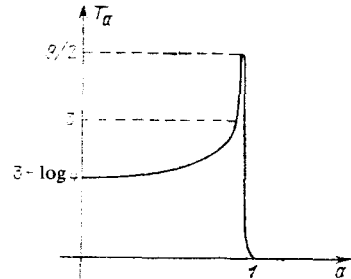


Fig. 4.7.

Let p be the inverse image of q under the map Φ . From the figure it is clear that after one turn this trajectory falls into a tunnel, one wall of which is the previous turn of the same trajectory, and the other wall is the trajectory that entered the halo of $(p, f(p))$ from the right along the horizontal, hence, left the slow curve in the halo of $(q, f(q))$. The same happens at each subsequent turn: our trajectory is a twisting spiral whose shadow is a “duck with a head”. The existence of a closed cycle with the same shadow now follows from the Poincaré-Bendixson theorem (see §1). ■

B. Period. It is of interest to find the period for all $a \leq 1$. In the formula

$$T_a = \int_{\Gamma} dt = \int_{\Gamma} \frac{1}{x} dx$$

the integral over the horizontal part of the cycle is infinitely small; therefore, if we wish to determine T_a to within equivalence it is sufficient to compute the integral only over the curvilinear part of the cycle. For $a \ll 1$ (a large cycle) we have $\dot{x} \sim \frac{a-x}{x^2-1}$, hence,

$$T_a \sim \int_{-2}^{-1} \frac{x^2-1}{a-x} dx + \int_{\frac{1}{2}}^1 \frac{x^2-1}{a-x} dx = 3 + (a^2-1) \log \frac{4-a^2}{1-a^2}.$$

The graph of T_a as a function of a is shown in Fig. 4.7; this function increases, and $a \nearrow 1$ implies that $T_a \nearrow 3$.

For a cycle-duck on the curvilinear parts we have $\dot{x} \sim -1/(x+1)$, hence, for the duck with a head

$$T_a \sim - \int_{x_b}^{-1} (x+1) dx - \int_2^{x_n} (x+1) dx = \frac{9}{2} + (x_b - x_n) \left(1 + \frac{x_b + x_n}{2} \right),$$

where x_b and x_n are the abscissas of the beak and the neck, respectively. It is easy to verify that $1 + (x_b + x_n)/2 > 0$, so that in this case $T_a \ll 9/2$.

Finally, for a duck without a head

$$T_a \sim - \int_{x_t}^{x_b} (x+1) dx = (x_t - x_b) \left(1 + \frac{x_t + x_b}{2} \right),$$

where x_t is the abscissa of the tail.

Thus, if the large cycle disappears for $a \sim 1$, the period does not simply decrease from its maximum value $T_a = 3$ to an infinitely small quantity, but at first (because of cycle-ducks) increases up to the value $9/2$ (for the largest duck without a head), and only then decreases to zero (on the graph of T_a there arises an infinitely thin needle of height $3/2$, see Fig. 4.7).

C. *Asymptotic stability of cycle-ducks.* To establish the fact of asymptotic stability of ducks we use the formula

$$E = \int_{\Gamma} \operatorname{div} \mathcal{Y} dt,$$

where \mathcal{Y} is the vector field corresponding to the system in question and E is the characteristic index of the cycle. Computing $\operatorname{div} \mathcal{Y} = -\varepsilon^{-1}(x^2 - 1)$ we obtain

$$-\varepsilon E = \int_{\Gamma} (x^2 - 1) dt = \int_{\Gamma} \frac{x^2 - 1}{\dot{y}} dy = \int_{\Gamma} \frac{x^2 - 1}{a - x} dy \sim - \int_{\Gamma} (x + 1) dy = S,$$

where S is the area bounded by the cycle Γ . The last equality is obtained by means of Green's formula.

The penultimate equality needs justification. The fact of the matter is that the relation $(x^2 - 1)/(a - x) \sim -(x + 1)$ holds not for all x , but for $x \neq 1$. Using one of the versions of Robinson's lemma, we conclude that this equivalence holds also on the set $|x - 1| > \delta$, where δ is some infinitely small number. Over the remaining set $|x - 1| \leq \delta$ we can estimate the integral $\int (x^2 - 1) dt$: the integrand is bounded, and the time in which the given part of the cycle is traversed (more precisely, the upper and the lower parts) is infinitely small; hence, so is the integral itself.

Thus, $E \ll 0$, which proves the asymptotic stability of the cycle.

Corollary 4.1. *For each value of $a \sim 1$ there is no more than one cycle-duck.*

For, the simultaneous existence of two cycles (one of which is necessarily situated inside the other) contradicts the asymptotic stability of both, since between them there are no positions of equilibrium. (The unique position of equilibrium $(a, f(a))$ is always situated inside any cycle-duck.)

Remark. By the footnote to (3.24), the assertion on the asymptotic stability of a duck without a head is equivalent to Lemma 4.1.

D. Evolution of the cycle. As a increases, the cycle for the van der Pol equation undergoes the following evolution. At first the lower segment of the large cycle starts to climb: there arises a duck with a head; as a increases further, the head shrinks and finally disappears completely: there arises the largest duck without a head. The upper segment of this duck starts to descend, that is, as a increases the body of the duck without a head shrinks; finally, there appears an infinitely small cycle.

We can prove an even stronger statement.

Theorem 4.3. *Suppose that the parameter \bar{a} corresponds to the largest duck without a head. Then:*

1) *a duck with a head whose neck has the abscissa c exists for*

$$a = \bar{a} - \exp \left\{ \varepsilon^{-1} \left(\frac{c^4}{4} + \frac{c^3}{3} - \frac{c^2}{2} - c + \frac{11}{12} \right) \right\};$$

more precisely: a necessary and sufficient condition for the existence of a duck with a head whose neck has the abscissa c for a given a is that $a < \bar{a}$ and

$$^{\circ}[\varepsilon \log (\bar{a} - a)] = \frac{c^4}{4} + \frac{c^3}{3} - \frac{c^2}{2} - c + \frac{11}{12};$$

2) *a duck without a head whose beak has the abscissa c exists for*

$$a = \bar{a} + \exp \left\{ \varepsilon^{-1} \left(\frac{c^4}{4} + \frac{c^3}{3} - \frac{c^2}{2} - c + \frac{11}{12} \right) \right\};$$

more precisely: a necessary and sufficient condition for the existence of a duck without a head whose beak has the abscissa c for a given a is that $a > \bar{a}$ and

$$^{\circ}[\varepsilon \log (a - \bar{a})] = \frac{c^4}{4} + \frac{c^3}{3} - \frac{c^2}{2} - c + \frac{11}{12}.$$

We do not give a detailed proof of Theorem 4.3, since it is in fact a combination of the proof of Theorem 3.3 (applied to the largest duck without a head) and the second proof of Theorem 4.1, and also some quite simple computations. We only mention that for the van der Pol equation the function $H(x)$ defined by (3.24) is $H(x) = x^4/4 + x^3/3 - x^2/2 - x + C$; the constant $C = 11/12$ is obtained from the condition $H(1) = 0$.

It is relevant to add that after the cycle-duck decreases to infinitely small dimensions in the infinitely small neighbourhood of $(1, -2/3)$ there occurs a classical Hopf bifurcation: the infinitely small cycle takes a form close to an ellipse that diminishes homothetically, and for $a = 1$ it becomes a stable singular point (see [63]).

E. *Tunnels, funnels, showers.* Using Lemma 4.1 as a principal tool it is easy to obtain tunnels, funnels, and showers corresponding to ducks with and ducks without a head for the van der Pol equation: they are shown in Figs. 4.8 and 4.9. The points P and Q (also P' and Q') are corresponding entry and exit points. The complete justification of the figures can be found in the second part of [19].

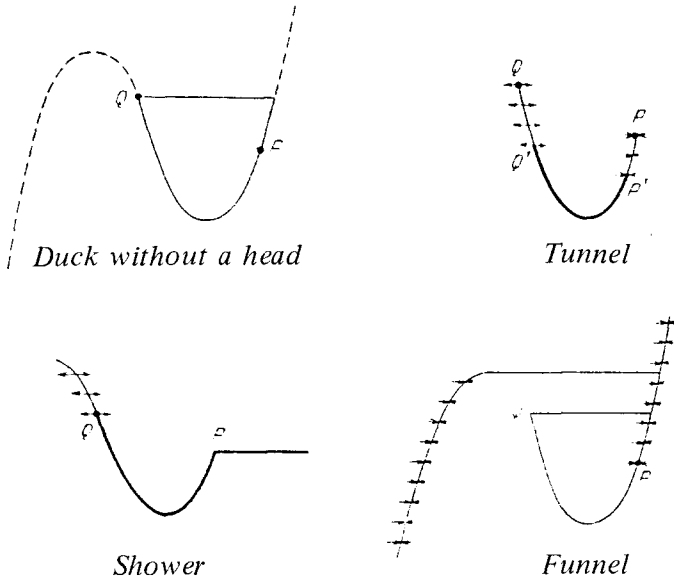


Fig. 4.8.

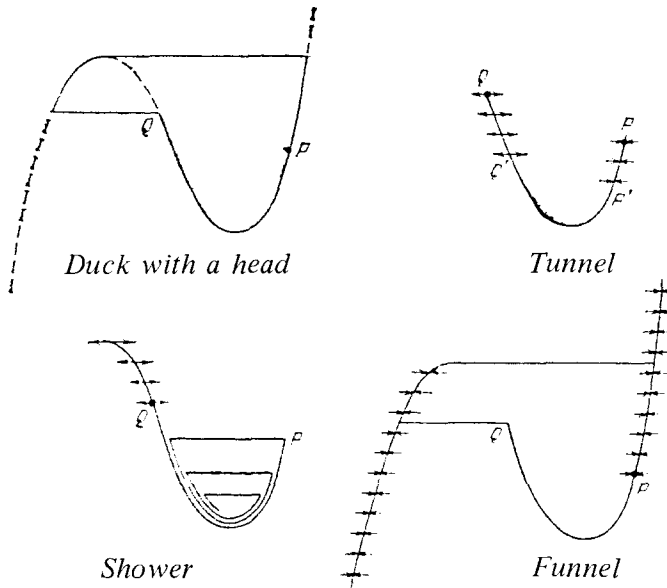


Fig. 4.9.

4.4. M. Diener's equation.

Lest the reader should think that all the properties of cycle-ducks for the van der Pol equation in §4.3 are determined only by the outward form of the graph of $f(x) = x^3/3 - x$, we give here one more example, studied in detail in the thesis of M. Diener [34]:

$$f(x) = x^5 - x^3 + x^2.$$

The outward form of the graph of $f(x)$ is the same as that for the van der Pol equation (Fig. 4.10), which implies that there is a large cycle for $-1 \ll a \ll 0$, and also the form of cycle-ducks is the same as for the van der Pol equation (that is, ducks with and without a head for $a \sim 0$).

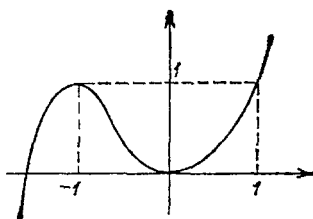


Fig. 4.10.

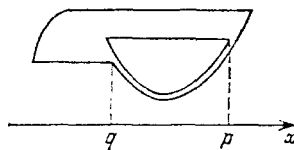


Fig. 4.11.

The qualitative behaviour of the solutions is essentially influenced by one circumstance that is insignificant at first sight: at the critical point $x = 0$ the third derivative of $f(x)$ is not positive, as for the van der Pol equation, but negative. Hence, firstly, by Theorem 3.1, the range of the parameter corresponding to ducks is situated not to the left of zero, but to the right:

$$a = \frac{3}{8} \varepsilon + \eta \varepsilon, \quad \eta \sim 0.$$

This means that for the values of $a > 0$ preceding ducks a stable large cycle coexists with an (also stable) position of equilibrium inside it. This is possible only when inside the stable cycle there is another unstable cycle, encircling the position of equilibrium. The rise of the unstable cycle is quite understandable: the appearance of ducks is preceded by the Hopf bifurcation for $a = 0$. However, according to §4.1, for $a = 0$ the singular point $(0, 0)$ is repelling; under the Hopf bifurcation only an asymptotically unstable cycle can arise from such a point (see [63]).

For a more detailed study of the situation we need the following lemma.

Lemma 4.2. *The entry-exit function Φ for M. Diener's equation is defined on $[0, 1]$, and its range is $[-1, 0]$ (that is, $\Phi(0) = 0$, $\Phi(1) = -1$). For all points p and q , $-1 \ll q \ll 0 \ll p \ll 1$, from $\Phi(p) = q$ it follows that $f(p) > f(q)$. (Thus, in contrast to the van der Pol equation, the entry point is higher than the exit point.)*

The elementary but cumbersome proof of this lemma is left to the reader.

Corollary 4.2. *Each duck without a head (except the largest one) is asymptotically unstable. Each duck with a head is asymptotically stable.*

Proof. We have to find the sign of the integral

$$\int_{\Gamma} \operatorname{div} y \, dt = -\varepsilon^{-1} \int_{\Gamma} f'(x) \, dt.$$

We estimate in this expression the coefficient of ε^{-1} , that is, $-\int_{\Gamma} f'(x) dt$.

For this purpose we note that the contribution to this integral of the horizontal parts of the cycle is infinitely small, that of the increasing parts of $y = f(x)$ is negative, and that of the decreasing parts is positive. Also, if $q = \Phi(p)$, where Φ is the entry-exit function, then on the segment of the cycle $\Gamma_{[q,p]}$ that projects to $[q, p]$ the integral $-\int_{\Gamma_{[q,p]}} f'(x) dt \sim \int_q^p f'(x) h(x) dx = 0$.

As the entry point it is natural to choose the tail of the duck. Then for ducks without a head, according to the lemma, the beak lies higher than the exit point, hence, the integral over the curvilinear part of the cycle is positive (and appreciable). For ducks with a head, according to the same lemma, the neck lies lower than the exit point, hence, the integral over the part from the tail to the neck is negative (and appreciable); the integral over the curvilinear part of the head is also negative. This proves the corollary. ■

As regards the largest duck without a head, there are many such ducks (but they are all infinitely close to one another, hence they have a common shadow). Among them there is one whose characteristic index is exactly zero. This duck is a *structurally unstable cycle*.

Now we have all the necessary facts to consider the evolution of ducks for a given equation. When a goes through the value $a_0 = 0$, the equation preserves its asymptotically stable large cycle; here the position of equilibrium situated inside the cycle becomes attracting, and from it there arises an infinitely small asymptotically unstable cycle encircling it (Hopf bifurcation). Later, a enters the domain of ducks: the large cycle becomes an asymptotically stable duck with a head; the small cycle becomes an asymptotically unstable duck without a head (which lies inside the duck with a head). It is important to remark that the abscissa of the neck of the duck with a head is none other than the image under the entry-exit map of the abscissa of the tail of the duck without a head (Fig. 4.11).

This can be explained by the fact that the duck with a head enters the halo of the duck without a head in the neighbourhood of the tail. Ducks continue to evolve: the duck without a head grows, and the head of the duck with a head shrinks. In the end both these two cycles (which have different stability) merge into one, lose their structural stability, and disappear.

Cycles disappear, but not ducks! Ducks cannot disappear suddenly, and we had the longest duck just now. What happens later? The behaviour of solutions of the equation is shown in Fig. 4.12.

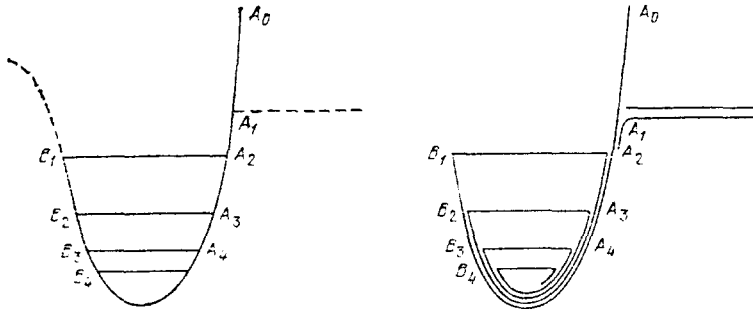


Fig. 4.12.

Suppose that the longest duck corresponding to the parameter value a reaches the point B at an appreciable distance from the maximum point⁽¹⁾ $(-1, 1)$. We consider a winding spiral $A_0A_1B_1A_2B_2A_3 \dots$; its shadow is represented in Fig. 4.12 on the left. Here A_{n+1} ($n = 1, 2, \dots$) is defined as the point on the graph of $y = f(x)$ that has the same ordinate as B_n (and a positive abscissa), and B_n ($n = 1, 2, \dots$) is the exit point corresponding to the entry point A_n . This curve is the neck of a funnel that draws in almost all the trajectories of our equation: after entering the halo of this curve they all have the same shadow. The only exception is an infinitely thin bundle of trajectories, approaching the spiral from the right along the ray CA_1 . Passing along the segments of the spiral $A_1B_1A_2B_2$, this bundle forms a shower, whose jets fill up the strip between B_1A_2 and B_2A_3 , then the tunnel A_3B_3 , again a shower, between B_2A_3 and B_3A_4 , and so on.

As a increases further, B_1 (the left upper corner of the spiral) descends lower and lower along the repelling branch of the slow curve; the curtain closes, and the ducks disappear.

§5. Brief guide to the literature

5.1. At the present time any complete list of publications on non-standard analysis and its applications seems impossible. However, we try to give, as far as possible, an exhaustive list of papers devoted (completely or partially) to applications of non-standard analysis to differential equations (see [1], [3], [9], [14]–[57], [64], [72], [75]–[77], [80]). The rest of the bibliography is divided into the following categories: firstly, instructive literature on non-standard analysis in Russian (see [2], [65]–[71]);

⁽¹⁾If B_1 is infinitely close to the maximum point, then the spiral considered below has infinitely many turns in the neighbourhood not only of the stationary point, but also of the (former, but disappearing) cycle $A_1B_1A_1$.

secondly, papers that develop a general theory of non-standard analysis and are closely connected with our topic (see [4]–[13]); finally, we have included a small list of “standard” papers on differential equations with small parameters, indicating results in those domains where one can expect further penetration of non-standard analysis ([58], [63], [73], [74]; see also the references in these papers).

The reader who wishes to get to the heart of the matter as quickly as possible is recommended to follow the sequence [65], [66], [2], [3].

5.2. The discovery and study of ducks is to a great extent the result of a collective effort. This is why it is not always easy to determine the author of a particular result. Among the pioneers of the application of non-standard methods to differential equations we name in the first place Eric Benoit, Jean-Louis Callot, Francine Diener, Marc Diener, Albert Troesch, and Emile Urlacher. The first papers in this direction are due to Troesch and Urlacher ([46], [47]). The first reference to ducks is contained, apparently, in [24]. The proof of the existence of ducks given in §2 is borrowed from M. Diener’s thesis [34]; however, some intermediate propositions of this proof appeared even earlier (for example, in [19]), and the phase plane is used in many papers. The existence of asymptotic expansions of ducks and the corresponding values of the parameter a in powers of ϵ is proved by F. Diener [29]. The second proof of the existence of these expansions (§3.5) is due to M.A. Shubin. Explicit recursion formulae for the coefficients of this expansion (§3.1) were obtained by A.K. Zvonkin and published here for the first time; also due to him are examples of “live ducks”. The exponential asymptotic of ducks (§3.3) was obtained by Benoit; he introduced the entry-exit function, as well as tunnels, funnels, and showers (see [17]–[19]). The “life of ducks is short” theorem is due to Callot; the principal method of obtaining these results, the so-called “exponential microscope”, was proposed by F. Diener. Cycle-ducks for the van der Pol equation were studied in detail in [19], from which we have borrowed all the results of §4.3 (except the second proof of the existence of cycles, which is due to Zvonkin). The example in §4.4 was studied in [34]. The proofs of Theorems 4.1 and 4.2 about the existence of duck-cycles and ducks in the case of any isolated (including degenerate) extremum point are due to Shubin and are published here for the first time (but the reference to the existence of ducks in the case of a degenerate extremum point occurs in [34]).

The proof in §4 of Theorem 4.1 about the existence of duck-cycles is based on the use of Poincaré’s successor function. Note that for the van der Pol equation this idea is used in [19] for the same purpose; however, we think that the proof of the existence of duck-cycles in [19] is incomplete, since, firstly, it does not even mention that the successor function may not be defined everywhere, and, secondly, it assumes implicitly

continuity of the successor function at the singular point, which does not happen for all values of a (this continuity depends on the character of the singular point).

5.3. We make some brief remarks about those papers that have not been mentioned in this survey.

In [20] Benoit and Lobry study the first example of a duck for a three-dimensional system.

In [22] and [23], Callot considers the following problem. The equation $\ddot{x} - \epsilon \dot{x} + ax = 0$ for integral a has as a particular solution a polynomial (namely, an Hermite polynomial); all the other solutions of this equation grow exponentially. How can such different asymptotics for the solutions coexist? By means of a *macroscope*, that is, compression of the scale by an infinitely large factor, the author reduces the question to the study of one specific class of ducks. In the paper by Troesch [45] the macroscope is used to prove the boundedness of solutions of the van der Pol equation.

The Liénard plane has an advantage over the phase plane in that on the Liénard plane the cycles can be “seen entirely”, whereas on the phase plane part of a cycle goes into the invisible (to us) domain of infinitely large values of the variables. In [25], [28], [29] F. Diener constructs the so-called “plane of observability”, which has the same property as the Liénard plane, but is suitable for a wider class of equations.

In [34] M. Diener studies ducks for the general class of systems

$$\begin{cases} \dot{x} = \epsilon^{-1} f(x, y, a), \\ \dot{y} = g(x, y, a). \end{cases}$$

In particular, he treats ducks in the neighbourhood of points of self-intersection (Morse points) of the slow curve $f(x, y, a) = 0$; an example of this situation is considered in [33].

In a paper by Harthong [35] a mathematical theory of moiré strips is constructed. The thesis [36] is devoted to applications of non-standard analysis to partial differential equations. One example of such an application is given in [3]. In [37] wave fronts and wave packets for the Schrödinger equation are studied.

In [39], [40], [42] boundary-value problems for ordinary differential equations with a small parameter are studied.

In his thesis [44] Troesch studies the qualitative behaviour of a four-dimensional system of differential equations corresponding to two connected van der Pol oscillators. Interesting multidimensional problems with a small parameter are also discussed in papers by Takens [60]–[62] and Chentsova [59]. A number of important problems in the theory of relaxation oscillations is studied in the book by Mishchenko and Rozov [58]. (In particular, one can find there a complete asymptotic expansion of the period of the large cycle in powers of the small parameter ϵ .)

Some interesting specific examples of equations or classes of equations are considered in [25]–[28], [30], [38], [50].

To conclude this short survey we note that many of the papers on the topics in question are published only in the form of preprints (mainly, Publications IRMA), and not by any means all of them were available to the authors of the present article.

After this article was sent to press, there appeared the paper by Eckhaus [78] and the collection of papers [79]. In [78], in particular, there is a standard account of some results concerning ducks.

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Remark added in proof

Recently S.N. Samborskii obtained for a multidimensional rapid-slow field with one rapid variable and for a curve consisting of two pieces of solutions of a degenerate system on the slow manifold, the first of which moves over the stable part of the slow manifold, and the second over its unstable part, necessary and sufficient conditions for obtaining an infinitely small perturbation of the field, so that this field has a duck whose shadow coincides with the given curve.

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