

PhD subject: infinite graphs and mathematical programming

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Abstract

Infinite graphs arise from different contexts and stimulate various research branches, from classical extremal graph theory to the emerging field of graph limits and graphons [5]. The purpose of this proposal is to study how some of the main topics studied by the "Graph and Applications" team behave in the infinite case. The potential research scope is therefore wide and include some key problems from mathematical programming. In this scope, Lovász ϑ 's function [4] is expected to play a main role: it is a polynomial time (with given accuracy) computable function for every graph G , via a semi-definite program. This graph parameter is sandwiched between the clique number ω (the maximal size of sets of pairwise adjacent vertices) and the chromatic number χ (the minimum number of colours needed to colour the vertices, subject to adjacent vertices get distinct colours). Analogs of Lovász theta number for infinite graphs have been introduced in the case of infinite graphs associated to the Euclidean space \mathbb{R}^n (see [3, 1, 7, 8]). The relationship with the sphere packing problem and related problems in geometry has been addressed in the team "Théorie des Nombres" of IMB (see [1, 2, 9]).

Let us outline some relevant subjects that could be studied among many:

- *countable graphs and identifying codes:* identifying codes satisfy both domination constraints and separating constraints by definition. Our team studied extensively these graph structures during the ANR project IDEA devoted to this graph parameter. Though there is a rich literature about identifying codes, the study of identifying codes of infinite (countable) graphs is yet in its early stages. For instance, there are some preliminary work on the minimal density of identifying codes of some infinite grids, including the square grid, the hexagonal mesh and the king grid [10].
- *independent set of uncountable graphs:* the famous sphere packing problem (what is the maximum fraction of the Euclidean space \mathbb{R}^n that can be covered by unit balls with pairwise disjoint interiors?) has a natural reformulation in graph theoretic terms as follows: let G denote the graph whose vertices are the points of the Euclidean space and edges are pair of vertices at distance at most 2 one from the other. The independent sets of G are the sphere packings: so, finding a maximum-density sphere packing is the same as finding a maximum-density independent set in this infinite graph. Good approximations of the optimal sphere packing density can be obtained through a theta-like number (see [3]). The study of geometric packing problems through mathematical programming is at the time a very active area (see e.g. [1, 8] and the references therein).
- *chromatic number of infinite uncountable graphs:* determining the chromatic number of the unit distance graph G_d of \mathbb{R}^d that connects pairs of points at distance exactly equal to 1 is a widely open problem. For $d = 2$, the chain of inequality $4 \leq \chi(G_2) \leq 7$ (Nelson and Isbell) has not been improved since 1950. For the asymptotic behaviour with respect to d , deep results have been recently achieved by extending the Lovász theta function to the unit distance graph and considering the measurable chromatic number of the Euclidean space (in this setting, colour classes are required to be measurable sets) [7, 2, 9]. A possible direction for further improvements on this problem would be to define and compute some steps of a suitable hierarchy of semidefinite programs starting from the above mentioned generalization of Lovász theta number.
- *polytopes* The above mentioned packing and coloring problems in Euclidean space can be generalized to a non Euclidean metric in \mathbb{R}^n . In particular, the problems will have a more combinatorial flavor if the metric is defined by a convex polytope. It seems that, in this case, it is completely relevant to consider the restriction to suitable discrete subgraphs in order to obtain an insight on the initial problem. In this area, the open problems that could be studied during the thesis are numerous. To give an exemple, the optimal packings of regular pentagons in the plane are not known.

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