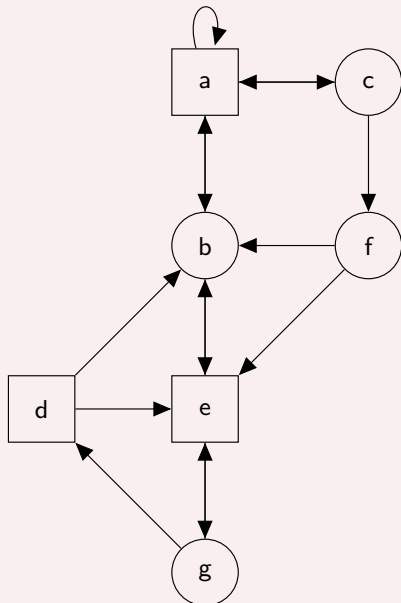


REACHABILITY AND BÜCHI GAMES

P_0 : CIRCLE, P_1 : SQUARE



Reachability game:

$F = \{a\}$: $W_0 = \{a, b, c, f\}$

$F = \{e\}$: $W_0 = V \setminus \{a\}$

Büchi games: same W_0 as above

11 / 107

REACHABILITY GAMES

ATTRACTORS

$$\begin{aligned} \text{Attr}_0^0(F) &= F \\ \text{Attr}_0^{n+1}(F) &= \text{Attr}_0^n(F) \cup \\ &\quad \{v \in V_0 \mid \exists w \in \text{Attr}_0^n(F), (v, w) \in E\} \\ &\quad \{v \in V_1 \mid \forall w \text{ t.q. } (v, w) \in E : w \in \text{Attr}_0^n(F)\} \\ \text{Attr}_0^0(F) &\subseteq \text{Attr}_0^1(F) \subseteq \dots \subseteq \text{Attr}_0^{|V|}(F) \end{aligned}$$

$\text{Attr}_0^i(F)$ is the set of vertices from which P_0 can reach F after at most i moves. $W_0 = \text{Attr}_0^{|V|}(F)$ is the **winning region** of P_0 (smallest fixpoint), and $V \setminus \text{Attr}_0^{|V|}(F)$ is the winning region of P_1 .

STRATEGIES

Reachability games have **positional** winning strategies (for both players), which can be computed in polynomial time.

Positional strategy (for P_0): mapping $\sigma : V_0 \rightarrow V$.

12 / 107

BÜCHI GAMES

STRATEGIES

Büchi games have **positional** winning strategies (for both players), which can be computed in polynomial time.

ALGORITHM

- $\text{Attr}_0^+(F)$: set of states from which P_0 can reach F in **at least** one move. We can compute $\text{Attr}_0^+(F)$, as well as a positional strategy, in polynomial time.
- $X^{(i)}$: set of states from which P_0 can go through F **at least** i times (without counting the starting state).

$$\begin{aligned} X^{(0)} &= V, & X^{(i+1)} &= \text{Attr}_0^+(X^{(i)} \cap F) \\ W_0 &= \bigcap_{i \geq 1} X^{(i)} \end{aligned}$$

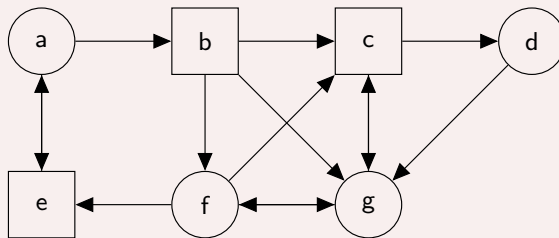
Notice that $X^{(i+1)} \subseteq X^{(i)}$. So there exists k with $W_0 = X^{(k)} = X^{(k+1)}$: W_0 is a largest fixpoint.

- Strategy for P_0 in the Büchi game: positional strategy used by P_0 from $W_0 = \text{Attr}_0^+(W_0 \cap F)$.

13 / 107

WEAK PARITY GAMES

NODE COLORING $\chi : V \rightarrow \{0, \dots, k\}$. WINNING CONDITION: MAXIMAL VISITED COLOR IS EVEN.



$$\chi(a) = \chi(e) = 0, \chi(b) = \chi(f) = 1, \chi(c) = \chi(g) = 2, \chi(d) = 3.$$

- $d \in W_1$ and $\text{Attr}_1(d) = \{d, c, b\} \subseteq W_1$
- $V' = V \setminus \{b, c, d\}$ is a trap for P_1 , defining a subarena for P_0 . So in V' , P_0 can avoid color 3, and $\text{Attr}_0(g) \upharpoonright_{V'} = \{g, f\} \subseteq W_0$
- $V'' = V' \setminus \{f, g\}$ is a trap for P_0 , so subarena for P_1 .
 $W_1 \upharpoonright_{V''} = \emptyset, W_0 \upharpoonright_{V''} = \{a, e\}$

Conclusion: $W_0 = \{a, e, f, g\}$, $W_1 = \{b, c, d\}$. Positional winning strategies for both players (attractor strategies).

14 / 107

WEAK PARITY GAMES

- The complement of an attractor C for P_i is a **trap** pour P_i : P_i has no outgoing transition from C , so P_{1-i} can force the play to stay in $V \setminus C$.
- A trap C for P_i is a **subarena** for P_{1-i} : every winning strategy for P_{1-i} in C is winning for P_{1-i} in V .

SOLUTION

$$V^{(k)} := \chi^{-1}(k).$$

- 1 Suppose that the maximal color k is odd (otherwise exchange P_0, P_1). Compute $\text{Attr}_1(V^{(k)})$ (a subset of W_1).
- 2 $V' := V \setminus \text{Attr}_1(V^{(k)})$ is subarena for P_0 and the maximal color in V' is at most $k - 1$. Compute inductively the winning region W_0 of P_0 in V' . Every vertex in W_0 is winning for P_0 in V (see above).
- 3 Every vertex in $V' \setminus W_0$ is winning for P_1 in V : P_1 applies his winning strategy from V' while staying in V' , and the attractor strategy as soon as the game enters $\text{Attr}_1(V^{(k)})$.
- 4 Strategies for both players are positional.

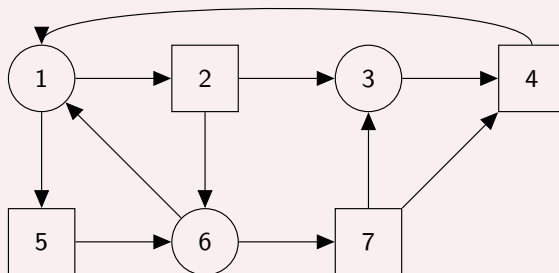
15 / 107

OBLIGATION GAMES

WINNING CONDITION

Win is given as a set $\mathcal{F} \subseteq \mathcal{P}(V)$ of sets of states. Player P_0 wins a play π if the set $\text{Occ}(\pi)$ of states visited in π belongs to \mathcal{F} .
Obligation games are thus Boolean combinations of reachability games.

EXAMPLE: CONJUNCTION OF REACHABILITY CONDITIONS



Play $\pi \in \text{Win}$ iff $\{2, 7\} \subseteq \text{Occ}(\pi)$. Player P_0 needs 1 bit of memory in state 1: he needs to visit alternatively states 2 and 5.

16 / 107

OBLIGATION GAMES (WEAK MULLER GAMES)

SOLUTION

Reduce the obligation game to a simpler (and larger) weak parity game.

- Recall: P_0 wins in a weak parity game if the maximal visited color is even.
- General idea of the **reduction**: start with an obligation game $G = (\mathcal{A}, \mathcal{F})$ and define a weak parity game $G' = (\mathcal{A}', \chi)$ such that:
 - 1 $\mathcal{A}' = (V'_0, V'_1, E')$, $V'_i = V_i \times M$, with M finite (memory).
 - 2 Transitions: $((v, m), (w, m')) \in E'$ if $(v, w) \in E$ and $m' = f(m, v, w)$ ($f =$ memory update function).
 - 3 With each play π in G starting in v , and each initial value $m_0 \in M$, we associate a unique play π' in G' from (v, m_0) . In addition, π and π' are won by the same player.
- Since weak parity games have positional strategies, such a reduction then provides finite memory strategies in the obligation game.

17 / 107

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REDUCTION

- $M = 2^V$, $E' = \{(v, R) \rightarrow (w, R \cup \{v\}) \mid (v, w) \in E\}$.
- Coloring: $\chi(v, R) = 2|R|$ if $R \in \mathcal{F}$; $\chi(v, R) = 2|R| - 1$ if $R \notin \mathcal{F}$.
- For every play $\pi' = (v_0, R_0), (v_1, R_1), \dots$ in G' we have $R_0 \subseteq R_1 \subseteq R_2 \dots$, so for some $i \geq 0$ we have $R_j = R_i$ for all $j \geq i$. For $R_0 = \emptyset$, the set R_i equals $\text{Occ}(\pi)$, where $\pi = v_0, v_1, \dots$

18 / 107

MULLER AND PARITY GAMES

Let $\text{Inf}(v_0v_1\cdots) = \{v \in V \mid \exists^\infty i : v = v_i\}$. For simplicity, we assume that there are no dead-ends in (V, E) .

PARITY GAMES

Node coloring $\chi : V \rightarrow \{1, \dots, k\}$.

$$\text{Win} = \{\pi \mid \max(\chi(\text{Inf}(\pi))) \text{ is even}\}.$$

MULLER GAMES

Let $\mathcal{F} \subseteq \mathcal{P}(V)$.

$$\text{Win} = \{\pi \mid \text{Inf}(\pi) \in \mathcal{F}\}$$

REM.

Every game $(\mathcal{A}, \text{Win})$ with $\text{Win} \subseteq V^\omega$ regular reduces to a Muller game. We will show how to reduce Muller games to parity games, then how to solve parity games.

19 / 107

A PREPARATORY EXAMPLE

(DZIEMBOWSKI, JURDZINSKI, WALUKIEWICZ)

- $V_0 = \{A, B, C, D\}$, $V_1 = \{1, 2, 3, 4\}$
- $E = V_0 \times V_1 \cup V_1 \times V_0$
- Winning condition for *player 1*:

$$|\text{Inf}(\pi) \cap V_0| = \max(\text{Inf}(\pi) \cap V_1)$$

How can P_1 win? solution: Last Appearance Record (LAR).

20 / 107

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How can P_1 win? solution: Last Appearance Record (LAR).

LAR

The order of visited V_0 states is recorded: the last visited state is moved to the front of LAR, and the position where it occurred previously (if any) is underlined – the hit position.

Example:

$$\epsilon \xrightarrow{A} A \xrightarrow{D} DA \xrightarrow{C} CDA \xrightarrow{C} \underline{C}DA \xrightarrow{D} D\underline{C}A \xrightarrow{D} D\underline{C}A \xrightarrow{C} CD\underline{A} \dots$$

If $\text{Inf}(\pi) \cap V_0 = \{C, D\}$ then infinitely often, the hit position in the LAR is 2 (we start to count from 1). P_1 's (winning) strategy is to play the hit position of the LAR.

21 / 107

FROM MULLER TO PARITY GAMES

REDUCTION

- Let (G, \mathcal{F}) be a Muller game with $G = (V_0, V_1, E)$ and $\mathcal{F} \subseteq 2^V$.
- LAR: pairs $\langle v_1 \dots v_r, h \rangle$, where $0 \leq h \leq r$ and v_1, \dots, v_r are distinct states from V .

Let Q be the set of LAR's. The initial LAR is $q_0 = \langle \epsilon, 0 \rangle$, and the transition function $\delta : Q \times V \rightarrow Q$ is given by:

$$\delta(\langle v_1 \dots v_r, h \rangle, v) = \begin{cases} \langle v v_1 \dots v_r, 0 \rangle & \text{if } v \neq v_p \text{ for all } p \\ \langle v v_1 \dots v_{p-1} v_{p+1} \dots v_r, p \rangle & \text{if } v = v_p \end{cases}$$

Hit set of $\langle v_1 \dots v_r, h \rangle$: $\{v_1, \dots, v_h\}$.

- For an infinite play π , let $\pi = \pi_0 \pi'$ be such that $\text{Inf}(\pi)$ is the set of states occurring in π' . If h is the maximal hit position repeated infinitely often, then $\text{Inf}(\pi)$ equals the hit set for the LAR's with hit position h in π' .

$$\chi(\langle v_1 \dots v_r, h \rangle) = \begin{cases} 2h & \text{if } \{v_1, \dots, v_h\} \in \mathcal{F} \\ 2h - 1 & \text{else} \end{cases}$$

$\text{Inf}(\pi) \in \mathcal{F}$ iff the maximal LAR-color repeated infinitely often is even. Thus, Muller games can be simulated by parity games (with $n! \cdot n$ additional memory).

22 / 107

SOLVING PARITY GAMES

INDUCTIVE SOLUTION

- Let $G = (V_0, V_1, E)$ and $\chi : V \rightarrow \{0, \dots, k\}$ be a parity game.
- Induction over $|V|$. Suppose w.l.o.g. that the maximal color k is even and let v_0 be some state with $\chi(v) = k$.
- Let $A_0 = \text{Attr}_0(v_0)$, so $V \setminus A_0$ is a subgame for P_1 . By induction, $V \setminus A_0$ is partitioned into W'_0 and W'_1 , and P_i has a positional winning strategy on W'_i .
- 2 cases (and exactly one applies):
 - 1 From v_0 , P_0 can get into $A_0 \cup W'_0$ in one step.
 - 2 From v_0 , P_1 can get into W'_1 in one step.
- Case 1: $W_0 = W'_0 \cup A_0$, $W_1 = W'_1$.
- Case 2: $v_0 \in \text{Attr}_1(W'_1)$. Let $A_1 = \text{Attr}_1(W'_1 \cup \{v_0\})$. $V \setminus A_1$ is a subgame for P_1 . Let W''_i be the winning region of P_i in this subgame (with positional strategies).
 $W_0 = W''_0$ and $W_1 = W''_1 \cup A_1$.

23 / 107

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COMPLEXITY

The above algorithm has exponential running time. Since strategies are positional, computing the winning regions is in $\text{NP} \cap \text{co-NP}$.

24 / 107