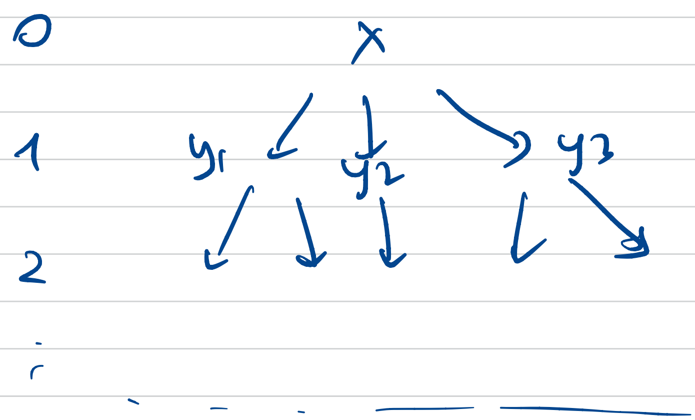
All-reach(x) : $\forall y . \left(x \xrightarrow{*} y \right)$

We want a $\underline{\exists SO}$ formula for All-reach(x)

$$\exists R_1 \ldots \exists R_k \; \underbrace{\varphi(R_1, \ldots, R_k)}_{\text{FO formula}}$$



$P(y_i, x)$

parent relation

$\exists P$ ("P is a parent relation" & x is the root & P has no loop)

binary

1 every vertex $\neq x$ has parent

2 if the parent is def., then it's unique

A relation satisfying (1) & (2) may have loops :

$$\begin{tikzcd} x_1 \arrow[rr, bend right, "P"'] & x_2 \arrow[l, "P"'] & x_3 \arrow[l, "P"'] \end{tikzcd}$$

3 x has no parent

$\underline{\varphi}$ P has no loop

→ use levels !

(4.1) $P(z, z')$: z' is parent of z

$\Rightarrow$ level(z') < level(z)

(4.2) level(x) = smallest level

$$\exists \underbrace{L, P}_{\text{binary}} \big(L \text{ is total order } \wedge$$

L ↓ total order, P ↓ parent

P is parent relation ∧

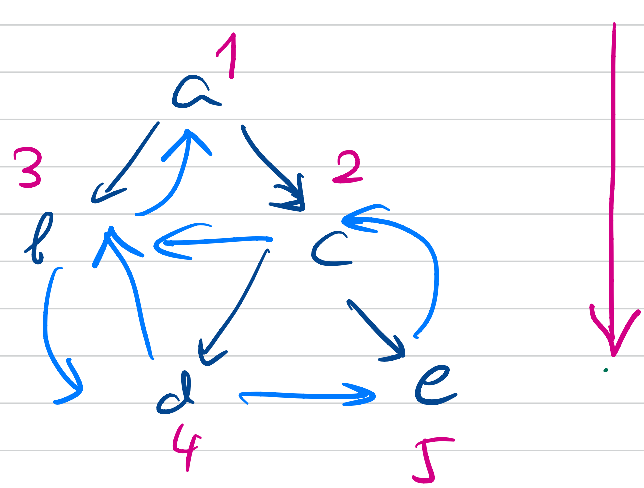
x has no parent ∧

the L-rank decreases along P-"edges")

.

$P := \{ (b, a), (c, a), (d, c), (e, c) \}$

$x = a$



φ_1 1) <u>L is total order</u> $\stackrel{?}{=}$

L is refl. & transitive & anti-symm. & total

$\forall x.\ L(x,x) \wedge \forall x,y,z.\ (L(x,y) \wedge$
$L(y,z) \Rightarrow L(x,z)) \wedge$
$\forall x,y\ (L(x,y) \wedge L(y,x) \to x = y) \wedge$
$\forall x,y\ (L(x,y) \vee L(y,x))$

φ_2 2) L-rank decreases along P - "edges"

$z \xrightarrow{p} z'$ $\Rightarrow$ z' is before z in L

$\forall z,z'.\ [\,P(z,z') \Rightarrow L(z',z)$
$\&\ z \neq z'\,]$

$\rightsquigarrow$ no loops in P

φ_3 3) x is root $\stackrel{?}{=}$ x is minimal wrt L.

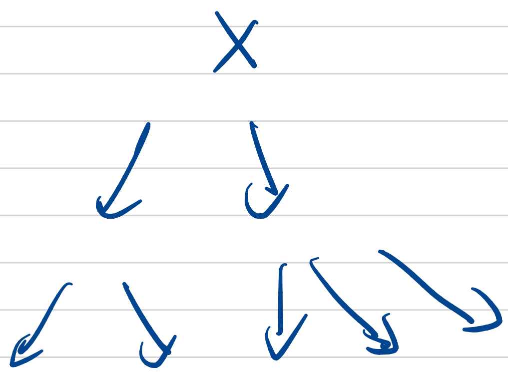
$\forall y.\ L(x,y)$

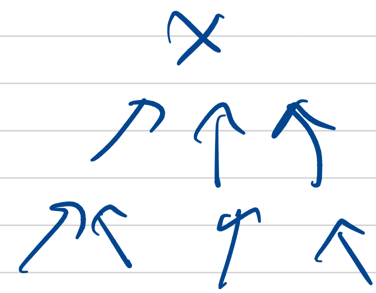
φ_4 4) $\forall z,z'.\ (P(z,z') \Rightarrow E(z',z))$

$\exists L.\ \exists P.\ (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)$

$\stackrel{?}{\equiv}$ All-reach(x)

$\exists SO$





All-co-reach(x)

Graph is strongly connected:

$\exists x.\ (\text{All-reach}(x) \wedge \text{All-co-reach}(x))$

$\exists L\ \exists P \ldots$

$\exists L'\ \exists P' \ldots$

$\exists x.\ \exists L\ \exists P\ \exists L'\ \exists P'\ (\ldots \wedge \ldots)$

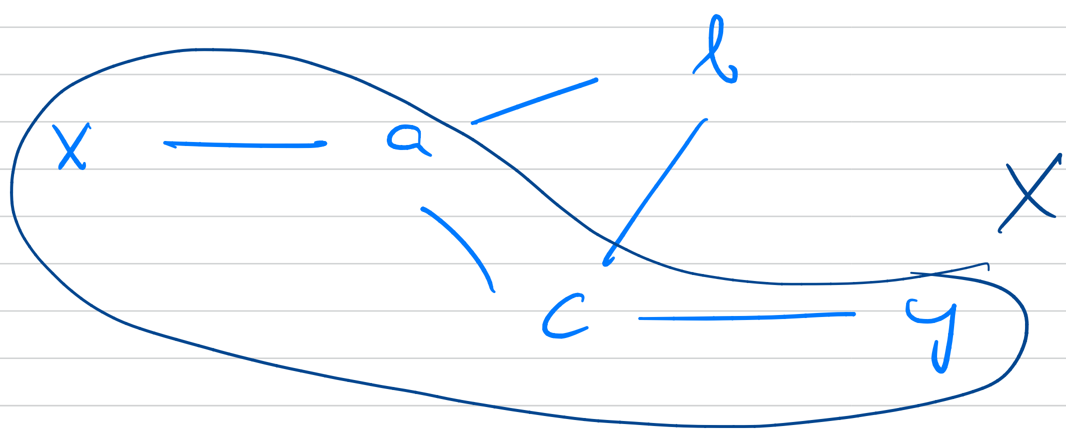
$\exists SO$ formula

G undirected, x, y vertices

Reach(x, y) $\stackrel{?}{=}$ there is a path between x & y

MSO formula: $\exists X. \varphi$

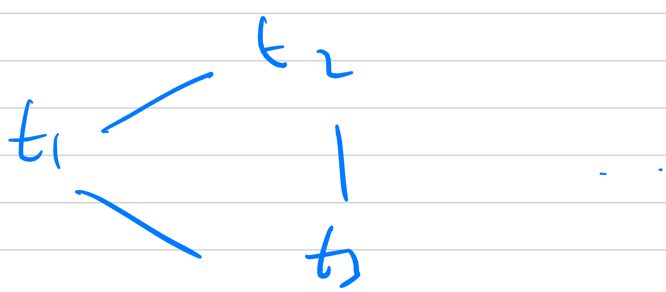
X = set of vertices on a simple path between x and y



$\exists X.$ (x has exactly 1 neighbor in X, y — — , and $\forall z \in X' \setminus \{x, y\}$ has exactly 2 neighbors in X and $x, y \in X'$)

If such X exists then there is a path between x and y.

$$\begin{array}{ccccccc} x & - & z_1 & - & z_2 & \cdots & - & y \\ \in X & & \in X & & \in X & & & \in X \end{array}$$

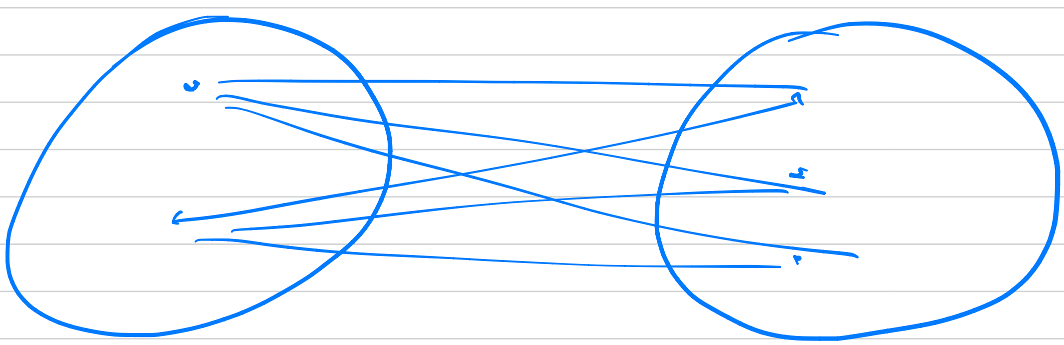


If there is a path between x, y then X with all properties exists. $\rightarrow$ Take a simple path between x and y.

$$\begin{tikzcd} x \arrow[r] & z_1 \arrow[r] & z_2 \arrow[r] \arrow[ll, bend right] & y \end{tikzcd}$$

$$X = \{x, z_1, z_2, y\}$$

$K_{m,n}$: full bipartite graph



$m = 2$ $\qquad$ $n = 3$ vertices

$K_{m,n}$ has HC $\iff$ $m = n \geq 2$

MSO over words $=$ REG

$L = \{a^n b^n : n \geq 0\} \notin REG$

word $w = a^m b^n \rightarrow$

construct a full bipartite

graph G_w s.t.

G_w has HC $\Leftrightarrow$

$w \in L \Leftrightarrow m = n$

$G_w = K_{m,n}$

$w =$ a a a a b b b

$K_{4,3}$

V : positions of the word

$E(x, y) = (a(x) \wedge b(y)) \vee (b(x) \wedge a(y))$

$$\exists X\ \forall x,y\ (x \in X \wedge y \notin X \rightarrow E(x,y))$$

talks about G_w

$$\exists X.\ \forall x,y\ (x \in X \wedge y \notin X \rightarrow ((a(x) \wedge b(y)) \vee (b(x) \wedge a(y))))$$

talks about w

If φ is a formula MSO expressing Hamiltonicity, then $\hat{\varphi} = \varphi \mid$ where $E(x,y)$ is replaced is an MSO formula that expresses $\{w : |w|_a = |w|_b\}$

EF-games: used for non-expressiveness results

EF-games for FO, MSO, . . .

FO(succ) FO(<)

over words

FO(E)

$(aa)^* \notin FO(<)$

EF-games: 2 players S, D.

They play over 2 words (or graphs . . .).

$w_1 = a\,a\,a\,a$ $\qquad\qquad$ $a\,a\,a = w_2$

In k rounds: S, D choose positions $i_1, \ldots, i_k$ in w_1, $j_1, \ldots, j_k$ in w_2.

D wins iff

$\forall m, n \in \{1, \ldots, k\}$

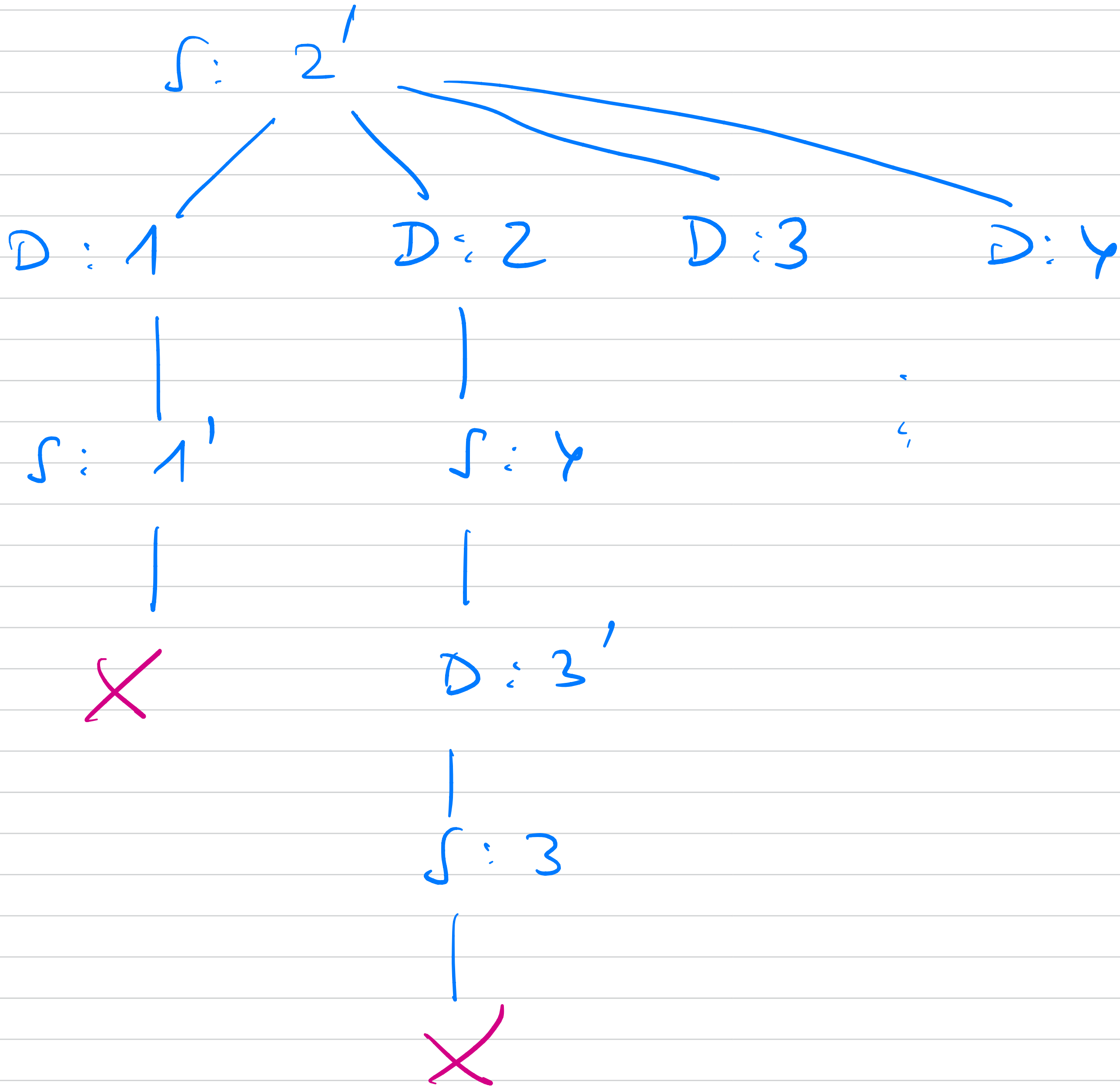
1) $i_m < i_n \iff j_m < j_n$

2) $i_m = i_n \iff j_m = j_n$

Game for FO(<)

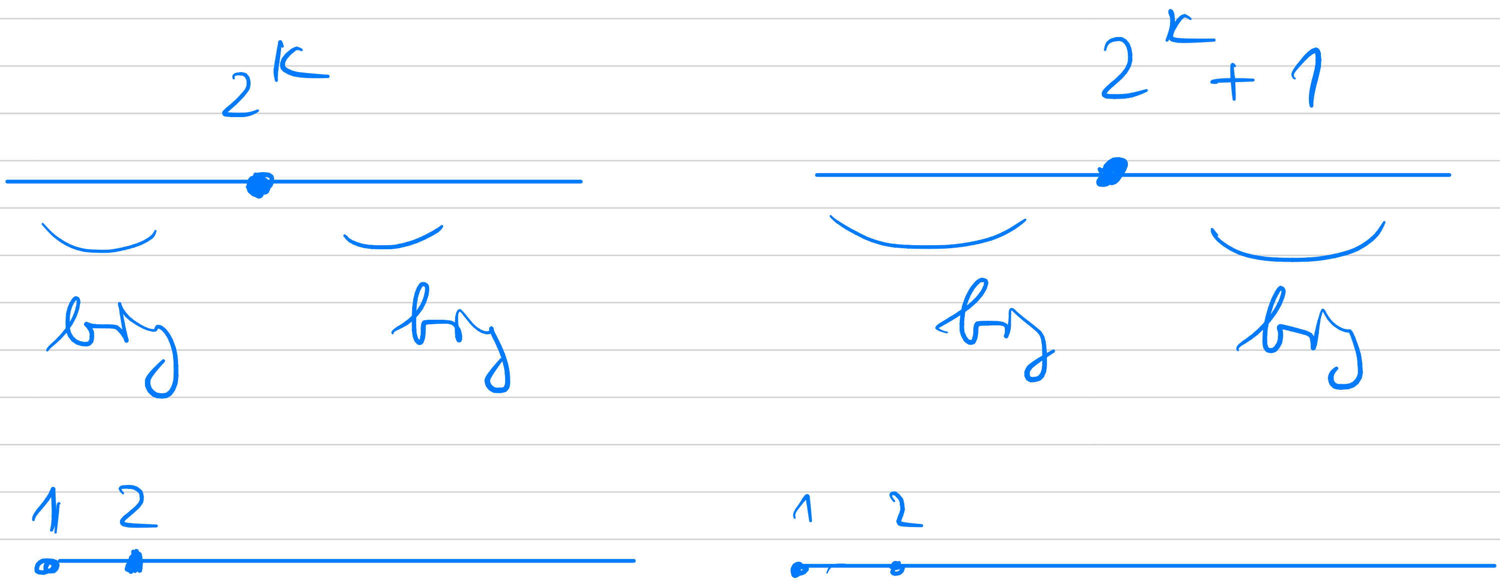
D1 S3 S2 S1 D2

1 2 3 4 $1'$ $2'$ $3'$



S wins in 3 rounds

$w_1 = a^{2^K}$ $w_2 = a^{2^K+1}$



$\forall K$: Dupl has a winning strategy on $\underbrace{w_1, w_2}_{\text{depend on } K}$

s.t. $w_1 \in L$ and $w_2 \notin L$

$\Rightarrow$ $L = (aa)^*$ is not expressible in FO