

TD2 “Logique et langages”

Exercise 1 Show that every regular word language can be expressed by an MSO formula of the form $\exists X.\varphi$, where φ is first-order – i.e., a formula with a single second order (existential) quantifier.

Hint: Start with an automaton \mathcal{A} accepting the language. A vector $(b_1, \dots, b_k) \in \{0, 1\}^k$ (corresponding to $\exists X_1 \cdots \exists X_k$) can be encoded by a 0-1 string of the form

$$11 b_1 0 b_2 0 \cdots b_k 0$$

In addition, you need to “speed-up” the formula for \mathcal{A} by the “factor” $2k + 2$.

Exercise 2 Give a bottom-up tree automaton for the set E of binary trees that encode Boolean expressions (with \vee, \wedge at inner nodes and $0, 1$ at leaves) that evaluate to 1.

Is it possible to have a deterministic top-down tree automaton for E ? Justify.

Exercise 3 Let L be a regular word language. Give a bottom-up tree automaton that checks that the left-to-right ordered sequence of leaf labels belongs to L .

Exercise 4 The logic *anti-chain MSO* over trees allows the second-order quantification only on sets S that are anti-chains, i.e., their elements are two by two incomparable w.r.t. descendant relation. Thus, $t \models \exists X.\varphi$ if for some anti-chain S of t we have $t \models \varphi(X \leftarrow S)$.

We consider here only binary trees where each node has 0 or 2 successors (called proper binary trees). For such trees, an inner node can be encoded by the leaf that is reached by taking the right successor, and then only left successors.

Show that MSO and anti-chain MSO are equivalent over proper binary trees.

Exercise 5 Chain MSO is defined as anti-chain MSO, replacing “anti-chains” by “chains”. Let $L = \{\underbrace{f(f(\dots f(a, a)\dots))}_n \mid n \text{ even}\}$. Give a chain MSO and an anti-chain MSO formula defining L . Show that L is not first-order definable.

Exercise 6 1. Show that the set of trees over $\Sigma = \{a, b\}$ that contain at least one a cannot be accepted by a deterministic top-down automaton.

2. Show that any tree language that is accepted by a deterministic top-down automaton is chain MSO definable.

Exercise 7 The following property can be expressed in $\text{FO}[S_0, S_1]$ over proper binary trees:

There exists a leaf at even depth.

To show this, define the zigzag of a node v as the maximal path from the set $v(01)^*(0 + \epsilon)$.

1. Show the following: there exists a leaf at even depth if and only if either (1) the zigzag of the root has even length or (2) there is a non-leaf node w such that the zigzag of $w0$ and the zigzag of $w1$ have lengths of different parity.

2. Write an $\text{FO}[S_0, S_1]$ formula $Z(x)$ stating that the zigzag of node x has even length.