

# On sets represented by partitions

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**Abstract:** We prove a lemma that is useful to get upper bounds for the number of partitions without a given subsum. From this we can deduce an improved upper bound for the number of sets represented by the (unrestricted or into unequal parts) partitions of an integer  $n$ .

## 1 Introduction

Let  $n$  be an integer and let

$$n = n_1 + n_2 + \dots + n_j, \quad n_i \in \mathbb{N}^*, \quad 1 \leq n_1 \leq n_2 \leq \dots \leq n_j$$

be a partition  $\Pi$  of  $n$ . We shall say that this partition represents an integer  $a$  if there exist  $\epsilon_1, \epsilon_2, \dots, \epsilon_j \in \{0, 1\}$  such that  $a = \sum_{i=1}^j \epsilon_i n_i$ . Let  $\mathcal{E}(\Pi)$  denote the set of these integers; we shall call it the set represented by  $\Pi$ . One can easily see that  $\mathcal{E}(\Pi)$  is included in  $[0, n]$  and symmetric (if it contains  $a$ , it also contains  $n - a$ ). For fixed  $n$ , let us introduce  $p(n)$  the number of partitions of  $n$  and  $\hat{p}(n)$  the number of different sets amongst the sets  $\mathcal{E}(\Pi)$  (where  $\Pi$  runs over the  $p(n)$  partitions of  $n$ ).

Let  $k$  be a positive integer. We shall say that a partition is  $k$ -reduced if and only if each summand appears at most  $k$  times; for instance the 1-reduced partitions are the partitions into unequal parts. We shall use  $q(n, k)$  the number of  $k$ -reduced partitions of  $n$  and  $\hat{q}(n, k)$  the number of different  $\mathcal{E}(\Pi)$  where  $\Pi$  runs over the  $q(n, k)$   $k$ -reduced partitions of  $n$ . When  $k$  equals 1, we shall note:  $q(n) = q(n, 1)$  and  $\hat{q}(n) = \hat{q}(n, 1)$ .

Following an idea due to P. Erds, the sets represented by the partitions of an integer  $n$  were first studied at the end of the 80's. P. Erds, J.-L. Nicolas and A. Sárkzy (cf. [3]) obtained upper bounds for the number of partitions without a given subsum. P. Erds then proposed to study the asymptotic behaviour of  $\hat{p}(n)$  and  $\hat{q}(n)$ . In [1] and [5], M. Delglise, P. Erds, J.-L. Nicolas and A. Sárkzy proved the following estimates:

**Theorem 1:** For  $n$  large enough, one has

$$q(n)^{0.51} \leq \hat{q}(n) \leq q(n)^{0.96}$$

and

$$p(n)^{0.361} \leq \hat{p}(n) \leq p(n)^{0.773}.$$

We shall obtain the following improved upper bounds:

**Theorem 2:** For  $n$  large enough, one has

$$\hat{q}(n) \leq q(n)^{0.955} \quad \text{and} \quad \hat{p}(n) \leq p(n)^{0.768}.$$

To get these new exponents, we shall prove in part 2 a lemma improving a result due to J. Dixmier [2], whose application in part 3 gives the announced improvements.

## 2 The main lemma

Let  $a$  be an integer,  $a \leq n$ . We introduce  $\mathcal{R}(n, a)$ , the set of partitions of  $n$  that do not represent  $a$ , and  $R(n, a)$  shall denote its cardinality. In the case of partitions into unequal parts, we shall need the same notions, with the similar notations  $\mathcal{Q}(n, a)$  and  $Q(n, a)$ . We shall also define  $Q(n, a, 2)$  as the number of 2-reduced partitions  $\Pi$  of  $n$  such that  $a$  is not represented by  $\Pi$ .

**Lemma 1:** Let  $\epsilon > 0$ . Assume there exists  $\delta \in ]0, 1[$  such that, for any integer  $n$  and for any integer  $a$ , the following property holds

$$(1) \quad \epsilon\sqrt{n} - 1 \leq a \leq 2\epsilon\sqrt{n} \Rightarrow R(n, a) \leq p(n)^\delta.$$

Then, for  $n$  large enough, one has

$$\frac{j}{2}\epsilon\sqrt{n} \leq a \leq \frac{(j+1)}{2}\epsilon\sqrt{n} \Rightarrow R(n, a) \leq (2p(\epsilon\sqrt{n}))^{j-2}p(n)^\delta$$

- for  $j = 2, 3, \dots, 2\lfloor\sqrt{n}/2\rfloor$  if  $\epsilon < 1$ ,

- for  $j = 2, 3, \dots, \tau(n)$  with  $\tau(n) = o(\sqrt{n})$  for every  $\epsilon$ .

**Remark 1:** To obtain a similar conclusion, J. Dixmier [2] assumed that hypothesis (1) is true for  $\epsilon\sqrt{n} \leq a \leq 3\epsilon\sqrt{n}$ .

**Proof:** We shall prove Lemma 1 by induction on  $j$ . It is true for  $j = 2, 3$  by (1). Let us suppose that  $j \geq 4$  and that the result is true up to  $j - 1$ . Let  $a$  be such that  $\frac{j}{2}\epsilon\sqrt{n} \leq a \leq \frac{(j+1)}{2}\epsilon\sqrt{n}$ . Let  $\Pi \in \mathcal{R}(n, a)$  and  $b = \lfloor\epsilon\sqrt{n}\rfloor$ .

If  $b$  is not represented by  $\Pi$ , then  $\Pi$  belongs to a set  $\mathcal{E}$  such that  $|\mathcal{E}| \leq p(n)^\delta$ .

If  $b$  is represented by  $\Pi$ , then we can write  $\Pi = (\Pi', \Pi'')$ , where  $S(\Pi') = b$ ,  $S(\Pi'') = n - b$  and  $\Pi''$  does not represent  $a - b$ . We get

$$a - b \geq \frac{j}{2}\epsilon\sqrt{n} - \epsilon\sqrt{n} = \frac{j-2}{2}\epsilon\sqrt{n} \geq \epsilon\sqrt{n-b}$$

since  $j \geq 4$ , and

$$a - b \leq \frac{j+1}{2}\epsilon\sqrt{n} - \epsilon\sqrt{n} + 1 = \frac{j-1}{2}\epsilon\sqrt{n} + 1.$$

Moreover we have

$$\frac{j}{2}\epsilon\sqrt{n-b} \geq \frac{j}{2}\epsilon\sqrt{n}\left(1 - \frac{b}{n}\right) \geq \frac{j}{2}\epsilon\sqrt{n} - \epsilon\frac{j}{2}\frac{\epsilon\sqrt{n}}{\sqrt{n}}.$$

We still have to show (at least for  $n$  large enough)

$$\frac{j-1}{2}\epsilon\sqrt{n} + 1 \leq \frac{j}{2}\epsilon\sqrt{n} - \frac{j}{2}\epsilon^2.$$

- If  $\epsilon < 1$ , since  $j/2 \leq \sqrt{n}/2$ , we have to check the inequality

$$-1/2\epsilon\sqrt{n} + 1 \leq -\epsilon^2\sqrt{n}/2,$$

which is true when  $n$  is large enough.

- In the second case, we want to show

$$-1/2\epsilon\sqrt{n} + 1 \leq -1/2\epsilon^2\frac{\tau(n)}{\sqrt{n}}.$$

This is true when  $n$  is large enough by using the hypothesis on  $\tau(n)$ .

We finally get

$$\frac{j}{2}\epsilon\sqrt{n-b} \geq a - b.$$

We deduce from the induction hypothesis that  $\Pi''$  belongs to a set  $\mathcal{F}$  such that

$$|\mathcal{F}| \leq (2p(\epsilon\sqrt{n}))^{j-3}p(n)^\delta.$$

This implies that  $\Pi$  belongs to a set  $\mathcal{G}$  such that

$$|\mathcal{G}| \leq p((\epsilon\sqrt{n}))(2p(\epsilon\sqrt{n}))^{j-3}p(n)^\delta.$$

Hence we have

$$R(n, a) \leq p(n)^\delta + p((\epsilon\sqrt{n}))(2p(\epsilon\sqrt{n}))^{j-3}p(n)^\delta \leq (2p(\epsilon\sqrt{n}))^{j-2}p(n)^\delta$$

which completes the proof of the lemma.

**Remark 2:** It is easy to see that the result remains true when we replace all the  $R(n, a)$ 's by  $Q(n, a)$ 's or by  $Q(n, a, 2)$ 's, i.e. when we deal with partitions into unequal parts or with 2-reduced partitions (in the proof, if  $\Pi$  is into unequal parts, then  $\Pi'$  and  $\Pi''$  are also into unequal parts; the same phenomenon occurs when we are dealing with 2-reduced partitions).

### 3 Applications

This lemma is useful to get upper bounds for  $\hat{p}(n)$  and  $\hat{q}(n)$  improving those obtained in [1]. Lemma 1 allows us to prove the following lemma:

**Lemma 2:** *When  $n \rightarrow \infty$  we have:*

1. for  $1.07\sqrt{n} \leq a \leq n - 1.07\sqrt{n}$ ,

$$Q(n, a) \leq \exp((1 + o(1))1.732\sqrt{n}),$$

2. for  $0.81\sqrt{n} \leq a \leq n - 0.81\sqrt{n}$ ,

$$Q(n, a, 2) \leq \exp((1 + o(1))1.969\sqrt{n}).$$

To get Lemma 2 (the method is developed in [1]), we find upper bounds for  $Q(n, a)$  and  $Q(n, a, 2)$  when  $a$  ranges over the interval  $[\epsilon\sqrt{n}, 2\epsilon\sqrt{n}]$  and we choose the best  $\epsilon$ ; then we use Lemma 1 and the results in [3].

From Lemma 2, we get Theorem 2 as in [1]. For instance, when studying  $\hat{q}(n)$ , we distinguish two cases according to whether the partition represents all integers between  $1.07\sqrt{n}$  and  $n - 1.07\sqrt{n}$  or not. We get this way

$$\hat{q}(n) \leq n \exp((1 + o(1))1.732\sqrt{n}) + 2^{1.07\sqrt{n}} \leq q(n)^{0.955}$$

since  $q(n) = \exp((1 + o(1))\pi\sqrt{n/3})$  (cf. [4]).

The method is the same for  $\hat{p}(n)$ , since  $\hat{p}(n) = \hat{q}(n, 2)$  [1, Thorme 1].

**Remark 3:** The improvement on the exponents in the Theorem 2 is small ( $5.10^{-3}$ ). This comes from the fact that the functions (cf. [1]) we bound on an interval  $[x, 2x]$  (and not  $[x, 3x]$ , see Remark 1) have slow variations around their minimum value. Indeed, even replacing  $[x, 2x]$  by  $[x, (1+\eta)x]$  with  $\eta$  decreasing to 0 would only lead to another small improvement ( $4.10^{-3}$  less than our results). To make the exponents in the upper bounds really smaller, we need to find another method.

## 4 References

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