

ON THE SYMMETRY OF THE PARTITION FUNCTION OF SOME SQUARE ICE MODELS

JEAN-CHRISTOPHE AVAL

ABSTRACT. We consider the partition function $Z(N; x_1, \dots, x_N, y_1, \dots, y_N)$ of the square ice model with domain wall boundary. We give a simple proof of the symmetry of Z with respect to all its variables when the global parameter a of the model is set to the special value $a = \exp(i\pi/3)$. Our proof does not use any determinantal interpretation of Z and can be adapted to other situations (for examples to some symmetric ice models).

1. INTRODUCTION

An *alternating sign matrix* (ASM) is a square matrix with entries in $\{-1, 0, 1\}$ and such that in any row and column: the non-zero entries alternate in sign, and their sum is equal to 1. Their numbers appear in the so-called *Razumov-Stroganov conjecture* related to the $O(1)$ loop model (7; 8; 2). Their enumeration formula was conjectured by Mills, Robbins and Rumsey (6), and proved by Zeilberger (12), and almost simultaneously by Kuperberg (4). Kuperberg used a bijection between the ASM's and the states of a statistical square ice model, for which he studied and computed the partition function. He also used these tools in (5) to obtain many enumeration or equinumeration results for various classes of symmetries of ASM's, most of them having been conjectured by Robbins (10). The same method was recently used to obtain the enumeration of ASM's invariant (9) or quasi-invariant (1) under a quarter-turn rotation. A property useful in all these works, which was established by Stroganov (11), states that the partition function $Z(N; x_1, \dots, x_N, y_1, \dots, y_N)$ of the (unrestricted) square ice model with domain wall boundary is symmetric in all its variables when the global parameter a of the model is set to the special value $a = \exp(i\pi/3)$. We give here another proof of this result. This proof is somehow “elementary” since it does not use any determinantal interpretation. Moreover, it can be adapted to other cases, for example to some symmetric models, to models in which there can be lines that carry more than one spectral parameter, or in which some fixed oriented edges are specified.

This paper is organized as follows: in Section 2, we recall the definitions of the ice models and of their associated weights; in Section 3, we give and prove the main result, *i.e.* the symmetry of Z ; in the last section, we present how we can obtain in the same way symmetry properties for other ice models.

2. DEFINITIONS

2.1. Notations. We recall here the main definitions and refer to (5) for details and many examples.

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2.3. Yang-Baxter equation. To deal with partition functions of ice models, the crucial tool is Yang-Baxter equation, that we recall below.

Lemma 1. [Yang-Baxter equation] *If $xyz = \bar{a}$, then*

$$(2) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}$$

3. MAIN RESULT

The following result has been obtained by Stroganov (11).

Theorem 2. (Stroganov) *When $a = \omega_6 = \exp(i\pi/3)$, the partition function $Z(N; X_N, Y_N)$ is symmetric in all its variables $x_1, \dots, x_N, y_1, \dots, y_N$.*

To prove this result Stroganov (11) uses a determinantal interpretation of Z . We want here to give a proof that only uses Yang-Baxter equation to study the partition function.

The method used was introduced by Kuperberg (5): observe that Z is a Laurent polynomial, then give enough specialization of one of its variable to imply the desired property.

3.1. Laurent polynomial. Since the weight of any vertex is a Laurent polynomial in the variables x_i 's and y_i 's, the partition function Z is a Laurent polynomial in these variables. Moreover it is a centered Laurent polynomial, *i.e.* its lowest degree is the opposite of its highest degree (called the half-width of the polynomial). Since any row and column of an ASM has at least one non-zero entry, which corresponds to a constant $\sigma(a^2)$, we get the following property.

Lemma 3. *The partition function $Z(N; X_N, Y_N)$ is a Laurent polynomial in any of its variables of half-width $N - 1$ and of parity the parity of $N - 1$.*

3.2. Partial symmetry. The following lemma gives a (now classical) example of use of the Yang-baxter equation.

Lemma 4.

$$(3) \quad \begin{array}{c} y \rightarrow \text{---} | \text{---} | \text{---} | \leftarrow \\ x \rightarrow \text{---} | \text{---} | \text{---} | \leftarrow \end{array} = \begin{array}{c} x \rightarrow \text{---} | \text{---} | \text{---} | \leftarrow \\ y \rightarrow \text{---} | \text{---} | \text{---} | \leftarrow \end{array}$$

Proof. We multiply the left-hand side by $\sigma(a\bar{z})$, with $z = \bar{a}x\bar{y}$. We get

$$\sigma(a\bar{z}) \begin{array}{c} y \rightarrow \text{---} | \text{---} | \text{---} | \leftarrow \\ x \rightarrow \text{---} | \text{---} | \text{---} | \leftarrow \end{array} = \begin{array}{c} x \rightarrow \text{---} | \text{---} | \text{---} | \leftarrow \\ y \rightarrow \text{---} | \text{---} | \text{---} | \leftarrow \end{array}$$

□

As a consequence, we get the “partial symmetry” of Z , which is true whatever the value of the global parameter a .

Lemma 5. *The functions $Z(N; X_N, Y_N)$ is symmetric separately in the two sets of variables X_N and Y_N .*

3.3. Specialization.

Lemma 6. [specialization of Z ; Kuperberg] *If we denote*

$$\begin{aligned} A(y_1, X_N \setminus x_1, Y_N \setminus y_1) &= \prod_{2 \leq k \leq N} \sigma(ax_k \bar{y}_1) \prod_{1 \leq k \leq N} \sigma(a^2 y_1 \bar{y}_k), \\ \bar{A}(y_1, X_N \setminus x_1, Y_N \setminus y_1) &= \prod_{2 \leq k \leq N} \sigma(ay_1 \bar{x}_k) \prod_{1 \leq k \leq N} \sigma(a^2 y_k \bar{y}_1), \end{aligned}$$

then we have:

$$\begin{aligned} (4) \quad Z(N; \bar{\mathbf{a}}\mathbf{y}_1, X_N \setminus x_1, Y_N) &= \bar{A}(y_1, X_N \setminus x_1, Y_N \setminus y_1) Z(N-1; X_N \setminus x_1, Y \setminus y_1) \\ (5) \quad Z(N; \mathbf{a}\mathbf{y}_1, X_N \setminus x_1, Y_N) &= A(y_1, X_N \setminus x_1, Y_N \setminus y_1) Z(N-1; X_N \setminus x_1, Y_N \setminus y_1). \end{aligned}$$

Proof. We recall the method to prove equation (4). We observe that when $x_1 = \bar{a}y_1$, the parameter of the vertex at the crossing of the two lines of parameter x_1 and y_1 is \bar{a} . Thus the weight of this vertex is $\sigma(a\bar{a}) = \sigma(1) = 0$ unless the orientation of this vertex is the second one on Figure 1. But this orientation implies the orientation of all vertices in the row x_1 and in the column y_1 , as shown on Figure 3. The fixed part gives the partition function Z in size $N-1$, without parameters x_1 and y_1 , and the weights of the fixed part gives the factor $\bar{A}(\dots)$.

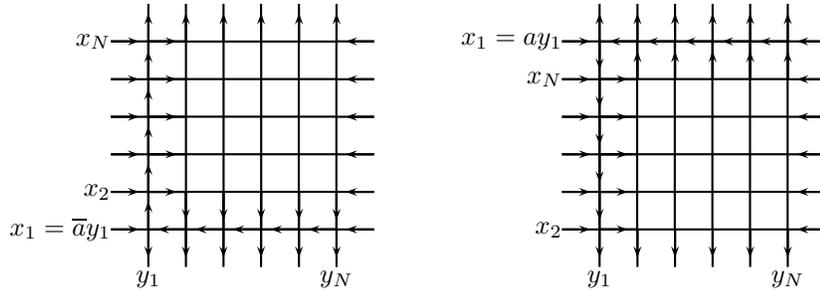


FIGURE 3. Fixed edges for (4) on the left and (5) on the right

The case of (5) is similar, after using Lemma 5 to put the line x_1 at the top of the grid.

□

3.4. Conclusion. We are now in a position to complete the proof of Theorem 2. From now on we set the global parameter to the special value $a = \exp(i\pi/3)$. This special value of the parameter implies the following equalities:

$$(6) \quad \sigma(a) = \sigma(a^2) \quad \sigma(a^2 x) = -\sigma(\bar{a}x) = \sigma(a\bar{x}).$$

The proof of Theorem 2 is done by induction on N . We easily check the property for $N = 1$. Now if $N \geq 2$, because of Lemma 5, we want to obtain the symmetry of Z with respect to x_2, y_2 .

We use Lemma 3 to reduce the proof of Theorem 2 to the proof of the symmetry of Z in at least N (independent) specializations of the variable x_1 . When $x_1 = ay_1$, Lemma 6 gives:

$$(7) \quad Z(N; \mathbf{a}y_1, X_N \setminus x_1, Y_N) = A(y_1, X_N \setminus x_1, Y_N \setminus y_1) Z(N-1; X_N \setminus x_1, Y_N \setminus y_1).$$

By recurrence, we have that $Z(N-1; X_N \setminus x_1, Y_N \setminus y_1)$ is symmetric in x_2, y_2 . The terms of $A(y_1, X_N \setminus x_1, Y_N \setminus y_1)$ involving the variables x_2, y_2 are:

$$\sigma(ax_2\bar{y}_1) \sigma(a^2y_1\bar{y}_2).$$

We use equation (6) to write:

$$\sigma(ax_2\bar{y}_1) \sigma(a^2y_1\bar{y}_2) = \sigma(ax_2\bar{y}_1) \sigma(ay_2\bar{y}_1)$$

which is clearly symmetric in the variables x_2, y_2 . We obtain in the same way the symmetry when $x_1 = \bar{a}y_1$; and by Lemma 5 we get this symmetry for the $2(N-1)$ special values $x_1 = a^{\pm 1}y_k$ for $k = 1, 3, \dots, N$, which is more than enough to conclude the proof of Theorem 2.

4. OTHER ICE MODELS

The method used to prove the global symmetry of Z may be adapted to other ice models. We illustrate this with half-turn symmetric ASM's (HTASM's). This example shows how our method can be used to prove global symmetries, or partial symmetries (for example when a line of the ice model carries two different spectral parameters -which breaks the homogeneity of the partition function-, or when the orientation of an edge is fixed).

4.1. HTASM's – notations and results. The ice models corresponding to HTASM's were introduced by Kuperberg (5) for the even size and by Razumov and Stroganov (9) for the odd size. We recall these models on Figure 4. The spectral parameters on this figure are slightly different from the original ones to better suit the proof. The dotted lines mean a change of parameter: on one side the parameter is x , whereas on the other side it is y .

We will prove the following theorem, for the special value $a = \exp(i\pi/3)$, and for the specialization $x = y$ in the even case (which corresponds to the original definition of $Z_{\text{HT}}(2N)$).

Theorem 7. *When the global parameter a is set to the special value $= \exp(i\pi/3)$, the function $Z_{\text{HT}}(2N; X_{N-1}, (x_N, x_N), Y_N)$ is symmetric with respect to all its $2N$ variables, and the function $Z_{\text{HT}}(2N+1; X_N, (x, y), Y_N)$ is symmetric with respect to its $2N$ variables $x_1, \dots, x_N, y_1, \dots, y_N$.*

The property on $Z_{\text{HT}}(2N+1)$ may be deduced from the main result of (9). The assertion about $Z_{\text{HT}}(2N)$ is new.

4.2. Proofs. Since the method is essentially the same as in the case of unrestricted ASM's, we shall give the main steps of the proof and only insist on the difference with the previous case.

4.2.1. Laurent polynomials. To deal with Laurent polynomials of given parity in the variable y , and thus divide by two the number of required specializations of this variable, we shall group together the states with a given orientation (indicated as subscripts in the following notations) at the edge where the parameters x and y meet.

So let us consider the partition functions $Z_{\text{HT}}^{\rightarrow}(2N; X_{N-1}, (x, y), Y_N)$ and $Z_{\text{HT}}^{\leftarrow}(2N; X_{N-1}, (x, y), Y_N)$, respectively parts with the parity of $N-1$ and of N of $Z_{\text{HT}}(2N; X_{N-1}, (x, y), Y_N)$ in y ; and $Z_{\text{HT}}^{\leftarrow}(2N+1; X_N, Y_N, (x, y))$ and $Z_{\text{HT}}^{\rightarrow}(2N+1; X_N, Y_N, (x, y))$, respectively parts with the parity of N and of $N-1$ of $Z_{\text{HT}}(2N+1; X_N, Y_N, (x, y))$ in y .

We deduce from this lemma the following property of pseudo-symmetry in (x, y) for the functions $Z_{\text{HT}}^{\curvearrowright}(2N; X_{N-1}, (x, y), Y_N)$ and $Z_{\text{HT}}^{\curvearrowleft}(2N; X_{N-1}, (x, y), Y_N)$.

Lemma 11. *For $\star = \curvearrowright, \curvearrowleft$ and $\square = \curvearrowright, \curvearrowleft$ respectively, we have*

$$(12) \quad Z_{\text{HT}}^{\star}(2N; X_{N-1}, (x, y), Y_N) = \frac{1}{\sigma(a^2 y \bar{x})} \sigma(a^2) Z_{\text{HT}}^{\star}(2N; X_{N-1}, (y, x), Y_N) \\ + \sigma(x \bar{y}) Z_{\text{HT}}^{\square}(2N; X_{N-1}, (y, x), Y_N).$$

4.2.3. *Specializations.* We now give specializations of the functions Z_{HT} in the variable x or y .

Lemma 12. [specialization of Z_{HT}] *If we denote*

$$A_H^1(x_1, X_N \setminus x_1, Y_N) = \prod_{1 \leq k \leq N} \sigma(a^2 x_1 \bar{x}_k) \prod_{1 \leq k \leq N} \sigma(a y_k \bar{x}_1) \\ \bar{A}_H^1(x_1, X_N \setminus x_1, Y_N) = \prod_{1 \leq k \leq N} \sigma(a^2 x_k \bar{x}_1) \prod_{1 \leq k \leq N} \sigma(a x_1 \bar{y}_k) \\ A_H^0(y_1, X_{N-1}, Y_N \setminus y_1) = \prod_{1 \leq k \leq N-1} \sigma(a x_k \bar{y}_1) \prod_{1 \leq k \leq N} \sigma(a^2 y_1 \bar{y}_k) \\ \bar{A}_H^0(y_1, X_{N-1}, Y_N \setminus y_1) = \prod_{1 \leq k \leq N-1} \sigma(a y_1 \bar{x}_k) \prod_{1 \leq k \leq N} \sigma(a^2 y_k \bar{y}_1),$$

then for $\star = \curvearrowright, \curvearrowleft$ and $\square = \curvearrowright, \curvearrowleft$ respectively, we have

(13)

$$Z_{\text{HT}}^{\star}(2N+1; X_N, Y_N, (x, \mathbf{ax}_1)) = A_H^1(x_1, X_N \setminus x_1, Y_N) Z_{\text{HT}}^{\square}(2N; X_N \setminus x_1, (x_1, x), Y_N)$$

(14)

$$Z_{\text{HT}}^{\square}(2N+1; X_N, Y_N, (x, \bar{\mathbf{ax}}_1)) = \bar{A}_H^1(x_1, X_N \setminus x_1, Y_N) Z_{\text{HT}}^{\star}(2N; X_N \setminus x_1, (x, x_1), Y_N)$$

(15)

$$Z_{\text{HT}}^{\star}(2N; X_{N-1}, (x, \mathbf{ay}_1), Y_N) = \sigma(a x \bar{y}_1) A_H^0(y_1, X_{N-1}, Y_N \setminus y_1) Z_{\text{HT}}^{\square}(2N-1; X_{N-1}, Y_N \setminus y_1, (x, y_1))$$

(16)

$$Z_{\text{HT}}^{\square}(2N; X_{N-1}, (\bar{\mathbf{ay}}_1, y), Y_N) = \sigma(a y_1 \bar{y}) \bar{A}_H^0(y_1, X_{N-1}, Y_N \setminus y_1) Z_{\text{HT}}^{\star}(2N-1; X_{N-1}, Y_N \setminus y_1, (y, y_1))$$

Proof. The method is almost the same as the one used to prove Lemma 6. An extended proof may be found in (1). \square

4.2.4. *Conclusion.* We are now in a position to conclude the proof of Theorem 7. From now on we set the global parameter a to the special value $a = \exp(i\pi/3)$.

Lemma 9 allows us to reduce the proof of the theorem to the proof of the following assertion: the functions $Z_{\text{HT}}^{\curvearrowright}(2N; X_{N-1}, (x, y), Y_N)$, $Z_{\text{HT}}^{\curvearrowleft}(2N; X_{N-1}, (x, y), Y_N)$, $Z_{\text{HT}}^{\curvearrowright}(2N+1; X_N, Y_N, (x, y))$ and $Z_{\text{HT}}^{\curvearrowleft}(2N+1; X_N, Y_N, (x, y))$ are symmetric with respect to x_2, y_2 .

The proof is done by induction on N . We easily check the property for small values of N . We suppose the property true in size $2N$, and we consider the odd case $2N+1$.

Because of Lemma 8, we have to check the property for enough specializations of the variable y . We use Lemma 12: equations (13) and (13) give us the specializations $y = a^{\pm 1} x_1$. As in the case of Z , the recurrence gives us the symmetry of the factor $Z_{\text{HT}}(2N)$, and the symmetry of the factor A is settled through (6) which implies:

$$\sigma(a x_2 \bar{y}_1) \sigma(a^2 y_1 \bar{y}_2) = \sigma(a x_2 \bar{y}_1) \sigma(a y_2 \bar{y}_1)$$

the right-hand being clearly symmetric in x_2, y_2 .

Using Lemma 9 we get $2(N - 1)$ specializations satisfying the symmetry, which is enough to imply the symmetry in full generality as soon as $N > 2$.

Now we use the property in size $2N + 1$ to prove it in size $2N + 2$. The method is the same, with the only difference that we have to use Lemma 10 because equation (16) gives a specialization for the variable x .

The proof of Theorem 7 is now complete.

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JEAN-CHRISTOPHE AVAL, LABRI, UNIVERSITÉ BORDEAUX 1, CNRS, 351 COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE