

# ON NON-COMMUTATIVE COMBINATORIAL INVERSE SYSTEM

J.-C. AVAL, N. BERGERON, AND H. LI

ABSTRACT. We introduce the notion of combinatorial inverse systems in non-commutative variables. We give two important examples in this setting and present some conjectures and results.

## 1. INTRODUCTION

Inverse systems [13] and Gröbner basis [15] are very useful tools to study finitely generated commutative algebras. In practice we are given a presentation of an algebra with generators and relations. That is the algebra is the quotient of a free commutative algebra (a polynomial ring in finitely many variables) by the ideal of relations. Inverse systems and Gröbner basis allow one to give explicit linear basis and to extract all the properties of the quotient.

We are interested in the case where the ideals of relations are obtained from a combinatorial construct of a family of algebras. More precisely, a *combinatorial inverse system* is a family of inverse systems obtained from a family of ideal  $\{I_n\}$  where  $I_n$  is generated by a combinatorial Hopf algebra [1] restricted to  $n$  variables. There are several important examples of combinatorial inverse systems (for some examples see [2, 4, 11, 12]). Coinvariants of the symmetric groups invariants are particularly well studied [10, 17] and more recently the spaces of diagonal harmonics [12] are still intensively studied.

For finitely generated non-commutative algebras, much less is known. In contrast with the commutative case, we are not guaranteed that the ideal of relations will be finitely generated. This may cause many problems. In particular, the problem of finding a Gröbner basis [15] is not decidable. A Gröbner basis in non-commutative variable is in general infinite and the Buchberger algorithm may not stop. In the non-commutative setting, even the inverse systems associated to the symmetric groups invariants are not well understood.

In this paper we introduce in Section 2 the basic notion of non-commutative inverse systems. We only consider homogeneous ideals since combinatorial inverse systems are generated by homogenous elements. In Section 3 we define more precisely the notion of combinatorial inverse systems and give some examples in commutative variables. We then return our attention to non-commutative combinatorial inverse systems. Section 4 is dedicated to the non-commutative combinatorial inverse system for the symmetric polynomials in  $n$  non-commutative variables (symmetric groups invariants). It is not known if this system is finite for all  $n$ . We have

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computed this system for  $n = 1, 2, 3$  and 4 and conjecture it is finite for all  $n$ . This conjecture would guaranty that the non-commutative Buchberger algorithm stops and that the ideals generated by non-commutative symmetric polynomials always have decidable finite Gröbner bases. We give some evidence of this and some weaker conjectures. In Section 5 we present the non-commutative combinatorial inverse system of Quasi-symmetric functions (Temperley-Lieb algebras invariants). In recent work, we have shown that this inverse system is finite and gives us here further evidence for our conjectures in Section 4.

## 2. ELEMENTS OF NON-COMMUTATIVE INVERSE SYSTEMS

Let  $R$  be the polynomials in non-commutating variables (or called an alphabet)  $\{x_1, x_2, \dots, x_n\}$ . That is

$$R = \mathbb{C}\langle x_1, x_2, \dots, x_n \rangle.$$

For  $a \in \{x_1, x_2, \dots, x_n\}$ , we define on  $R$  the operator  $d_a$  by

$$d_a \cdot w = \begin{cases} u & \text{if } w = au, \\ 0 & \text{otherwise,} \end{cases}$$

where  $w$  and  $u$  are monomials (or called words and  $a$  is refer to as a letter). We think of this operator as a derivative but it does not satisfies Leibniz's Rule. Let  $u = u_1 u_2 \cdots u_k$  be a word of degree  $k$  with  $u_j = x_{i_j}$  for some  $1 \leq i_j \leq n$ . We denote

$$\overleftarrow{u} = u_k u_{k-1} \cdots u_1$$

and

$$d_u \cdot w = (d_{u_1} \cdot (d_{u_2} \cdot (\cdots (d_{u_k} \cdot w) \cdots))).$$

A pairing  $\langle, \rangle$  on  $R$  is defined on the monomials (or words) by

$$\langle u, v \rangle = \delta_{u,v} = [d_{\overleftarrow{u}} \cdot v]_{c.t.},$$

where  $[f]_{c.t.}$  means the constant term of  $f$  and

$$\delta_{u,v} = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

For each  $f \in R$ , we define

$$f(d) = \sum_w c_w d_w,$$

when  $f = \sum_w c_w w$ , where  $c_w \in \mathbb{C}$ . Furthermore,

$$\overleftarrow{f}(d) = \sum_w c_w d_{\overleftarrow{w}} \text{ and } \langle f, P \rangle = [\overleftarrow{f}(d) \cdot P]_{c.t.}.$$

Each  $f \in R$  can be written as  $f = f_n + f_{n-1} + \cdots + f_0$ , where  $f_i$  is the component in  $f$  of degree  $i$  for all  $0 \leq i \leq n$ . We say that  $I \subseteq R$  is a *homogeneous ideal* if  $f = f_n + f_{n-1} + \cdots + f_0 \in I$  implies  $f_i \in I$  for all  $0 \leq i \leq n$ . In fact,  $I$  is a homogeneous ideal if and only if  $I$  is generated by homogeneous elements of  $R$ . For any homogeneous ideal  $I$  we define

$$I^\perp = \{P \in R : \langle f, P \rangle = 0, \forall f \in I\}.$$

If  $I$  is homogeneous clearly  $R = I \oplus I^\perp$ . In this case, we can compute  $I^\perp$  independently in each (finite dimensional) homogeneous component of  $R$ .

**Lemma 2.1.** *Let  $I$  be a homogeneous ideal. Then*

$$I^\perp = \{P \in R : \overleftarrow{f}(d) \cdot P = 0, \forall f \in I\}.$$

*This means that  $I$  is the solution to a system of (differential) equations.*

*Proof.* Let  $P \in R$  such that  $\overleftarrow{f}(d) \cdot P = 0$  for all  $f \in I$ . Then  $[\overleftarrow{f}(d) \cdot P]_{c.t.} = 0$ . So  $\langle f, P \rangle = 0$ , for all  $f \in I$ . Hence  $P \in I^\perp$ . Conversely, let  $P \in I^\perp$  and  $f \in I$ . Then  $fu \in I$  for any word  $u$ . So  $[d_{\overleftarrow{u}} \overleftarrow{f}(d) \cdot P]_{c.t.} = 0$ , for all  $u$ . This implies that  $\overleftarrow{f}(d) \cdot P = 0$ .  $\square$

**Lemma 2.2.** *Let  $I$  be a homogeneous ideal. Then  $I^\perp$  is closed under derivation, i.e.,  $d_u \cdot P \in I^\perp$  for all  $P \in I^\perp$  and  $u$ .*

*Proof.* Let  $P \in I^\perp$ . Then  $\langle f, P \rangle = 0$  for all  $f \in I$ . For any  $f \in I$ ,

$$\begin{aligned} \langle f, d_u \cdot P \rangle &= [\overleftarrow{f}(d) d_u \cdot P]_{c.t.} \\ &= \langle \overleftarrow{u} f, P \rangle \\ &= 0, \quad \text{since } \overleftarrow{u} f \in I. \end{aligned}$$

Hence  $d_u \cdot P \in I^\perp$ .  $\square$

Since  $R = I \oplus I^\perp$ ,  $R/I \cong I^\perp$ . Thanks to Lemma 2.1 and Lemma 2.2, the space  $I^\perp$  is the solution to a system of (differential) equations that is closed under differentiation. This is what we will refer to as the noncommutative inverse system of the homogeneous ideal  $I$ .

We are now interested in the relation between polynomials in commuting and non-commuting variables. Let  $u = u_1 u_2 \cdots u_k$  be a word of length  $k$  with  $u_j = x_{i_j}$  for some  $1 \leq i_j \leq n$ . Let  $\sigma \in S_n$  and  $\pi \in S_k$ . We define

$$\sigma \circ u = x_{\sigma(i_1)} x_{\sigma(i_2)} \cdots x_{\sigma(i_k)}$$

and

$$u \circ \pi = u_{\pi(1)} u_{\pi(2)} \cdots u_{\pi(k)}.$$

For any nonnegative integer vector  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , we define

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

and

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!.$$

Consider the maps

$$\begin{aligned} \chi : \mathbb{C}\langle x_1, x_2, \dots, x_n \rangle &\longrightarrow \mathbb{C}[x_1, x_2, \dots, x_n] \\ x_i &\longmapsto x_i \end{aligned}$$

and

$$\psi : \mathbb{C}[x_1, x_2, \dots, x_n] \longrightarrow \mathbb{C}\langle x_1, x_2, \dots, x_n \rangle$$

defined by

$$\psi(x^\alpha) = \sum_{\pi \in S_k} u \circ \pi,$$

where  $k = \alpha_1 + \alpha_2 + \cdots + \alpha_n$  and  $u$  is any word such that  $\chi(u) = x^\alpha$ . This is well defined since the letters of  $u$  in the definition of  $\psi$  are permuted in all possible ways, hence it does not depend on the choice of  $u$ . We know that  $\chi$  is an algebra homomorphism. On the other hand,  $\psi$  is a linear injection but it does not preserve products.

We define  $\partial_{x^\alpha} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$ , where  $\partial_i$  is the partial derivative operator over  $x_i$ , i.e., normally  $\frac{\partial}{\partial x_i}$ , and the product of these operators means the composition of them. For any  $Q \in \chi(R) = \mathbb{C}[x_1, x_2, \dots, x_n]$ , we define  $Q(\partial)$  by replacing  $x_i$  in  $Q$  with  $\partial_i$ . For  $P, Q \in \mathbb{C}[x_1, x_2, \dots, x_n]$ , the standard pairing is  $\langle Q, P \rangle = [Q(\partial) \cdot P]_{c.t.}$ . For an ideal  $J \subseteq \chi(R) = \mathbb{C}[x_1, x_2, \dots, x_n]$ , we define

$$\begin{aligned} J^\perp &= \{P \in \chi(R) : \langle Q, P \rangle = 0, \forall Q \in J\} \\ &= \{P \in \chi(R) : Q(\partial) \cdot P = 0, \forall Q \in J\}. \end{aligned}$$

This is the usual (commutative) inverse system for a (commutative) ideal.

**Lemma 2.3.** *Let  $P \in \chi(R)$  and  $f \in R$ . Then*

$$[\overleftarrow{f}(d) \cdot \psi(P)]_{c.t.} = [\chi(f)(\partial) \cdot P]_{c.t.}.$$

*Proof.* Let  $u$  and  $v$  be two words such that  $\chi(u) = x^\alpha$  and  $\chi(v) = x^\beta$ . First, we show that

$$[d_{\overleftarrow{v}} \cdot \psi(u)]_{c.t.} = \alpha! \delta_{\chi(u), \chi(v)} = [\partial_{\chi(v)} \cdot u]_{c.t.}.$$

$$\begin{aligned} [d_{\overleftarrow{v}} \cdot \psi(u)]_{c.t.} &= [d_{\overleftarrow{v}} \cdot \sum_{\pi \in S_k} u \circ \pi]_{c.t.} \\ &= \sum_{\pi \in S_k} [d_{\overleftarrow{v}} \cdot (u \circ \pi)]_{c.t.} \\ &= \sum_{\pi \in S_k} \delta_{v, u \circ \pi} \\ &= \#\{\pi \in S_k : v_i = u_{\pi(i)}\}. \end{aligned}$$

Clearly,  $\#\{\pi \in S_k : v_i = u_{\pi(i)}\}$  depends only on  $\chi(v)$  and  $\chi(u)$ . So

$$\begin{aligned} [d_{\overleftarrow{v}} \cdot \psi(u)]_{c.t.} &= \#\{\pi \in S_k : v = u \circ \pi\} \\ &= \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \alpha! & \text{otherwise} \end{cases} \\ &= \alpha! \delta_{\chi(u), \chi(v)}. \end{aligned}$$

On the other side,

$$\begin{aligned} [\partial_{\chi(v)} \cdot u]_{c.t.} &= [\partial_{\chi(v)} \cdot x^\alpha]_{c.t.} \\ &= \alpha! \delta_{\alpha, \beta}. \end{aligned}$$

Since  $\partial$ ,  $d$  and  $[\ ]_{c.t.}$  are linear maps,

$$[\overleftarrow{f}(d) \cdot \psi(P)]_{c.t.} = [\chi(f)(\partial) \cdot P]_{c.t.}.$$

□

**Corollary 2.4.** *Let  $P \in \chi(R)$  and  $I$  be any homogeneous ideal in  $R$ . Then*

$$P \in \chi(I)^\perp \Leftrightarrow \psi(P) \in I^\perp.$$

*Proof.*

$$\begin{aligned}
P \in \chi(I)^\perp &\Leftrightarrow \forall f \in I, \langle \chi(f), P \rangle = 0 \\
&\Leftrightarrow \forall f \in I, [\chi(f)(\partial) \cdot P]_{c.t.} = 0 && \text{by definition} \\
&\Leftrightarrow \forall f \in I, [\overleftarrow{f}(d) \cdot \psi(P)]_{c.t.} = 0 && \text{from Lemma 2.3} \\
&\Leftrightarrow \forall f \in I, \langle f, \psi(P) \rangle = 0 && \text{by definition} \\
&\Leftrightarrow \psi(P) \in I^\perp.
\end{aligned}$$

□

Corollary 2.4 gives us a linear inclusion  $\psi: \chi(I)^\perp \hookrightarrow I^\perp$  of inverse systems and a surjection of algebras  $\chi: R/I \rightarrow \chi(R)/\chi(I)$ .

### 3. COMBINATORIAL INVERSE SYSTEMS (COMMUTATIVE)

A combinatorial Hopf algebra as defined in [1] is a pair  $(\mathcal{H}, \zeta)$  where  $\mathcal{H}$  is a graded connected Hopf algebra and  $\zeta: \mathcal{H} \rightarrow \mathbb{C}$  is an algebra morphism. The map  $\zeta$  serve as a measure for some desired combinatorial invariants and will not be used here. For many examples,  $\mathcal{H}$  is described with a homogeneous basis  $\{b_\lambda\}$  such that all algebraic structure constants are non-negative integers. In the commutative case,  $\mathcal{H}$  is realized as a subalgebra of  $\mathbb{C}[[X]]$ , the homogeneous series in countably many variables  $X$ . Given this, we can restrict  $\mathcal{H}$  to finitely many variables  $X_n$  using an evaluation map  $\mathcal{H} \hookrightarrow \mathbb{C}[[X]] \rightarrow \mathbb{C}[X_n]$  where  $x = 0$  for all  $x \in X - X_n$ . If  $X_1 \subset X_2 \subset \dots$  and  $\lim_{n \rightarrow \infty} X_n = X$ , we obtain a family of ideals  $I_n = \langle b_\lambda(X_n) \rangle \subseteq \mathbb{C}[X_n]$  where  $b_\lambda(X_n)$  denote the image of a basis element of  $\mathcal{H}$  under the map  $\mathcal{H} \rightarrow \mathbb{C}[X_n]$  described above. We say that a family  $\{I_n^\perp\}_{n \geq 0}$  obtained in this way is a *combinatorial inverse system*.

To motivate our definition we present three key examples along with their main features.

**Example 3.1.** *Symmetric functions:* The Hopf algebra of symmetric functions [14] is  $Sym = \mathbb{C}[p_1, p_2, \dots]$  where the comultiplication is given by  $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$ . The degree of  $p_k$  is set to be  $k$ . There is an embedding  $Sym \hookrightarrow \mathbb{C}[[X]]$  with  $X = x_1, x_2, \dots$ , given by

$$p_k = \sum_{i \geq 1} x_i^k.$$

Taking  $X_n = x_1, \dots, x_n$  then  $Sym$  is a combinatorial Hopf algebra satisfying the criteria above. We can thus construct its combinatorial inverse system. That is the inverse systems  $H_n = \{I_n^\perp\}_{n \geq 1}$  corresponding to the ideals  $I_n = \langle p_k(X_n) : k \geq 1 \rangle \subseteq \mathbb{C}[X_n]$ . These spaces are central in mathematics and appear in a larger class of spaces in invariant and coinvariant theory [10, 17]. Since  $H_n = \{P \in \mathbb{C}[X_n] : \phi(\partial)P = 0, \forall \phi \in I_n\}$  it consists of polynomials  $P$  that are solutions of the equation  $p_k(\partial)P = 0$ , that is harmonic polynomials. The spaces  $H_n$  are also known as the symmetric harmonics in  $n$  variables, they have been extensively studied in several context and satisfy very fundamentals properties. Among them, for all  $n \geq 1$ ,

- (1) Let  $\Delta_n = \prod_{1 \leq i < j \leq n} (x_i - x_j)$  denote the VanDerMonde determinant. Then

$$H_n = \{P(\partial)\Delta_n : P \in \mathbb{C}[X_n]\};$$

- (2) The dimension of  $H_n$  is  $n!$ ;
- (3) The space  $H_n$  is the left regular representation of the symmetric group  $S_n$ ;

(4) As an  $S_n$ -module,

$$H_n = \bigoplus_{k=0}^{\frac{n(n-1)}{2}} H_n^{(k)}$$

is graded. The  $q$ -Frobenius characteristic is

$$\mathcal{F}_q(H_n) = \sum_{k=0}^{\frac{n(n-1)}{2}} q^k \text{char}(H_n^{(k)}) = H_n(X; q)$$

where  $\text{char}(H_n^{(k)})$  is the symmetric function associated with the representation  $H_n^{(k)}$  and  $H_n(X; q)$  is the Hall-Littlewood symmetric function [14].

**Example 3.2.** *Diagonal invariants and Diagonal harmonics:* The Hopf algebra of MacMahon symmetric functions (also known as diagonal invariants) is  $DSym = \mathbb{C}[p_{a,b} : (a,b) \in \mathbb{Z} \times \mathbb{Z}]$  where  $p_{0,0} = 1$  and the comultiplication is given by  $\Delta(p_{a,b}) = p_{a,b} \otimes 1 + 1 \otimes p_{a,b}$ . This space is bigraded and the degree of  $p_{a,b}$  is set to be  $(a,b)$ . There is an embedding  $Sym \hookrightarrow \mathbb{C}[X; Y]$  with  $X; Y = x_1, x_2, \dots, y_1, y_2, \dots$ , given by

$$p_{a,b} = \sum_{i \geq 1} x_i^a y_i^b.$$

Taking  $X_n; Y_n = x_1, \dots, x_n, y_1, \dots, y_n$  then  $DSym$  is a combinatorial Hopf algebra satisfying the criteria above. We can thus construct its combinatorial inverse system. That is the inverse system  $DH_n = \{I_n^\perp\}_{n \geq 1}$  corresponding to the ideals  $I_n = \langle p_{a,b}(X_n; Y_n) : (a,b) \in \mathbb{Z} \times \mathbb{Z} \rangle \subseteq \mathbb{C}[X_n; Y_n]$ . These spaces have been extensively studied in recent years. Here is a list of some of the results for these spaces [6, 12]. For all  $n \geq 1$ ,

(1) Let  $E_k = \sum_{i=1}^n y_i^k \partial_{x_i}$ . Then

$$DH_n = \{P(\partial, E)\Delta_n : P \in \mathbb{C}[X_n; Y_n]\},$$

where  $P(\partial, E)$  denote the operator we get by setting the variables  $x_k = \partial_{x_k}$  and  $y_k = E_k$  for  $1 \leq k \leq n$ .

(2) The dimension of  $DH_n$  is  $(n-1)^{n+1}$ ;

(3) The space  $DH_n$  is the so-called parking functions representation of the symmetric group  $S_n$ ;

(4) As an  $S_n$ -module,

$$DH_n = \bigoplus_{a,b} DH_n^{(a,b)}$$

is bigraded. The  $q, t$ -Frobenius characteristic is

$$\mathcal{F}_{q,t}(DH_n) = \sum_{a,b} q^a t^b \text{char}(DH_n^{(a,b)}) = \langle h_1^n, \nabla e_n \rangle$$

where  $\langle h_1^n, \nabla e_n \rangle$  is described in [6, 12] and is related to Macdonald symmetric functions  $H_\lambda(X; q, t)$  of [14].

There are still many open problems regarding  $DH_n$ . In particular it is not known how to construct an explicit linear basis.

**Example 3.3.** *Quasi-symmetric functions:* The Hopf algebra of Quasi-symmetric functions  $QSym$  plays a central role in the theory of combinatorial Hopf algebra in [1]. It is natural to study it in the context of combinatorial inverse system. Less is known about this system but some remarkable results have been obtained.

A composition  $\alpha \models n$  is a sequence of non-zero positive integer  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$  where  $k \geq 0$  and  $n = \alpha_1 + \alpha_2 + \dots + \alpha_k$ . For  $n = 0$  there is a unique composition  $\alpha = ()$  the empty composition. The Hopf algebra of quasi-symmetric functions  $QSym$  is the linear span of the functions  $\{M_\alpha : \alpha \models n \geq 0\}$ . The multiplication is given by quasishuffle

$$M_\alpha M_\beta = \sum_{\gamma \in \alpha \sqcup \beta} M_\gamma,$$

where for  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$  and  $\beta = (\beta_1, \beta_2, \dots, \beta_\ell)$  we define  $\alpha \sqcup \beta$  recursively as follow. If  $k = 0$  or  $\ell = 0$  then  $\alpha \sqcup \beta = \alpha \cdot \beta$  where  $\cdot$  denote the concatenation of lists. If not,

$$\begin{aligned} \alpha \sqcup \beta &= \alpha_1 \cdot ((\alpha_2, \dots, \alpha_k) \sqcup \beta) + \beta_1 \cdot (\alpha \sqcup (\beta_2, \dots, \beta_\ell)) \\ &\quad + (\alpha_1 + \beta_1) \cdot ((\alpha_2, \dots, \alpha_k) \sqcup (\beta_2, \dots, \beta_\ell)). \end{aligned}$$

The notation  $\gamma \in \alpha \sqcup \beta$  indicate that  $\gamma$  is in the support of  $\alpha \sqcup \beta$ . The comultiplication is defined by

$$\Delta(M_\alpha) = \sum_{\beta \cdot \gamma = \alpha} M_\beta \otimes M_\gamma.$$

The unit is  $M_{()}$  and the counit is  $\epsilon: QSym \rightarrow \mathbb{C}$  where  $\epsilon(f)$  is the coefficient of  $M_{()}$  in  $f$ . This space is graded by the size of  $\alpha$ . That is the degree of  $M_\alpha$  is  $n$  when  $\alpha \models n$ . Again there is an embedding  $QSym \hookrightarrow \mathbb{C}\langle X \rangle$  with  $X = x_1, x_2, \dots$  given by

$$M_\alpha = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k}.$$

Taking  $X_n = x_1, \dots, x_n$  then  $QSym$  is a combinatorial Hopf algebra satisfying the criteria above and we can construct its combinatorial inverse system. The inverse system corresponding to the ideals  $I_n = \langle M_\alpha(X_n) : \alpha \models n \geq 1 \rangle \subseteq \mathbb{C}\langle X_n \rangle$  is denoted  $SH_n = \{I_n^\perp\}_{n \geq 1}$ . These spaces have not been studied that much. Here is a list of some of the partial results and open problems for these spaces [2, 4]. For all  $n \geq 1$ ,

- (1) We conjecture that

$$SH_n = \{P(\partial)Q : P \in \mathbb{C}\langle X_n \rangle, Q \in \text{Soc}(SH_n)\},$$

for some nice set  $\text{Soc}(SH_n)$  of cardinality  $C_{n-1}$ , the  $(n-1)$ -th Catalan number;

- (2) The dimension of  $SH_n$  is  $C_n$  the  $n$ -th Catalan number;  
(3) The space  $SH_n$  seems to be related to the Temperley-Lieb algebra  $TL_n$ , yet it is not clear if this algebra acts or not on  $SH_n$ .

These three examples have very rich combinatorial results. More (commutative) combinatorial inverse systems are interesting to study and can be found in the literature [3, 17]. It is also interesting to compute the full resolution of these quotients (as in [9]) but we do not consider this here.

#### 4. SYMMETRIC FUNCTIONS IN NON-COMMUTING VARIABLES

To define non-commutative combinatorial inverse system, we start with a non-commutative combinatorial Hopf algebra  $\mathcal{H}$ . Let  $\{b_\lambda\}$  be a homogeneous basis for  $\mathcal{H}$  and assume there is a realization of  $\mathcal{H}$  as a subalgebra of  $\mathbb{C}\langle\langle X \rangle\rangle$  the homogeneous series in countably many non-commuting variables  $X$ . Given this, we can restrict  $\mathcal{H}$  to finitely many variables  $X_n$  using an evaluation map  $\mathcal{H} \hookrightarrow \mathbb{C}\langle\langle X \rangle\rangle \rightarrow \mathbb{C}\langle X_n \rangle$

where  $x = 0$  for all  $x \in X - X_n$ . Here  $\mathbb{C}\langle X_n \rangle$  is the free associative algebra finitely generated by  $X_n$ . If  $X_1 \subset X_2 \subset \dots$  and  $\lim_{n \rightarrow \infty} X_n = X$ , we obtain a family of ideals  $I_n = \langle b_\lambda(X_n) \rangle \subseteq \mathbb{C}\langle X_n \rangle$ . The family  $\{I_n^\perp\}_{n \geq 0}$  obtained in this way is a *non-commutative combinatorial inverse system*.

We trust that our motivating examples in the commutative case are convincing enough to the reader and that combinatorial inverse system are interesting objects to consider. We now want to look at non-commutative analogues of Example 3.1 and Example 3.3. We start with the combinatorial Hopf algebra  $NC\text{Sym}$  of symmetric functions in non-commuting variables [7, 16, 18]. It is simpler to first describe the embedding  $NC\text{Sym} \rightarrow \mathbb{C}\langle\langle X \rangle\rangle$  for  $X = x_1, x_2, \dots$ .

A set partition  $\Phi$  of  $k$  is a collection of nonempty subsets  $\Phi_1, \Phi_2, \dots, \Phi_n \subseteq [k] = \{1, 2, \dots, k\}$  such that  $\Phi_i \cap \Phi_j = \{\}$  for  $i \neq j$  and  $\Phi_1 \cup \Phi_2 \cup \dots \cup \Phi_n = [k]$ . We indicate that  $\Phi$  is a set partition of  $k$  by  $\Phi \vdash [k]$ . The subsets  $\Phi_i$  are called the parts of the set partition. The number of nonempty parts is referred to as the length and is denoted by  $\ell(\Phi)$ . A monomial with variables in  $X$  is a word  $w = w_1 w_2 \dots w_k$  with  $w_i \in X$ . The word  $w$  can be viewed as a function  $w : [k] \rightarrow X$ . Let  $\nabla(w) = \{w^{-1}(x) : x \in X\} \setminus \{\emptyset\}$ . Clearly  $\nabla(w)$  is a set partition of  $[k]$ . For  $\Phi \vdash [k]$  we define

$$M_\Phi(X) = \sum_{\nabla(w)=\Phi} w,$$

where the sum is over all  $w$  whose corresponding set partition is  $\Phi$ . For empty set partition, we define by convention  $M_\{\} = 1$ . For example, when  $k = 4$  and  $\Phi = 13.2.4 := \{\{1, 3\}, \{2\}, \{4\}\}$ ,

$$M_{13.2.4} = x_1 x_2 x_1 x_3 + x_1 x_3 x_1 x_2 + x_2 x_1 x_2 x_3 + x_2 x_3 x_2 x_1 + x_3 x_1 x_3 x_2 + x_3 x_2 x_3 x_1 + \dots$$

The vector space  $NC\text{Sym}$  has a linear basis  $\{M_\Phi(X)\}_{k \geq 0, \Phi \vdash [k]}$ . The multiplication is given by the multiplication in  $\mathbb{C}\langle\langle X \rangle\rangle$ . The comultiplication is given by

$$\Delta(M_\Phi) = \sum_{\Psi \cup \Xi = \Phi} M_\Psi \otimes M_\Xi.$$

Let  $X_n = x_1, \dots, x_n$  and let

$$I_n = \langle M_\Phi(X_n) : \Phi \vdash [k], k > 0 \rangle.$$

We are interested to compute  $\text{Har}_n = I_n^\perp$ . The space  $NC\text{Sym}_n$  linearly spanned by  $\{M_\Phi(X_n) : \Phi \vdash [k], k \geq 0\}$  is known to correspond to the invariants of the symmetric group  $S_n$  in non-commutative variable  $X_n$  [7, 16, 18]. From Lemma 2.1, in non-commuting variables  $\varphi \in \text{Har}_n \Leftrightarrow \{\overleftarrow{M_\Phi}(d) d_u \cdot \varphi = 0\}_{\Phi \neq \{\}}$  is a non-commutative system of (differential) equations. For this reason we call  $\text{Har}_n$  the *non-commuting Harmonic system*.

**Corollary 4.1.** *For a word  $w \in \mathbb{C}\langle X_n \rangle$ , define*

$$\Delta_w \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} \sum_{\pi \in S_k} (-1)^{\ell(\sigma)} \sigma \circ w \circ \pi. \quad (4.1)$$

*Then  $\Delta_w \in \text{Har}_n$  if and only if  $\Delta_w = 0$  or  $\chi(w) = x^\rho$ , where  $\rho = (n-1, n-2, \dots, 1, 0)$ .*



*Proof.* It is easy to see that  $\chi(I_n) = \langle p_k(X_n) \rangle$  the ideal considered in Example 3.1. Hence  $\chi(I_n)^\perp = H_n$ . Let

$$A_\alpha \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sigma \circ x^\alpha.$$

Then  $\Delta_w = \psi(A_\alpha)$  when  $\chi(w) = x^\alpha$ . Using Corollary 2.4,

$$\Delta_w \in \text{Har}_n \Leftrightarrow A_\alpha \in H_n.$$

Since  $H_n$  is isomorphic to the left regular representation of  $S_n$  (See Example 3.1), there is only one occurrence of the sign representation obtained by taking  $A_\rho$ . Hence  $A_\alpha \in H_n \Leftrightarrow \alpha = \rho$  or  $A_\alpha = 0$ .  $\square$

Thus  $d_u \Delta_w \in \text{Har}_n$  for all  $u$  and  $\chi(w) = x^\rho$ . This gives us a copy of the left regular representation in  $\text{Har}_n$ . For  $n = 1$  and  $2$ , this is all of  $\text{Har}_n$ . For  $n = 3$ , we see more.

We have computed  $\text{Har}_n$  for  $n = 1, 2, 3, 4$  and the following surprising fact arises  $\dim \text{Har}_n = 1, 2, 9, 946, \dots$ . The Hilbert series of  $\text{Har}_n$  is

$$\text{Hilb}_{\text{Har}_n}(t) = \sum_{d=0}^{\infty} \dim(\text{Har}_n^{(d)}) t^d,$$

where  $\text{Har}_n^{(d)}$  is the homogeneous component of degree  $d$  in  $\text{Har}_n$ . We list the following data obtained by computer

- $n = 0$ ,  $\text{Hilb}_{\text{Har}_0}(t) = 1$ ,
- $n = 1$ ,  $\text{Hilb}_{\text{Har}_1}(t) = 1$ ,
- $n = 2$ ,  $\text{Hilb}_{\text{Har}_2}(t) = 1 + t$ ,
- $n = 3$ ,  $\text{Hilb}_{\text{Har}_3}(t) = 1 + 2t + 3t^2 + 3t^3$ ,
- $n = 4$ ,  $\text{Hilb}_{\text{Har}_4}(t) = 1 + 3t + 8t^2 + 20t^3 + 47t^4 + 102t^5 + 197t^6 + 308t^7 + 248t^8 + 12t^9$ ,
- $n = 5$ ,  $\text{Hilb}_{\text{Har}_5}(t) = 1 + 4t + 15t^2 + 55t^3 + 199t^4 + 712t^5 + 2520t^6 + \dots$

**Conjecture 4.2.** *The non-commuting Harmonic system  $\text{Har}_n$  is finite dimensional for any  $n$ .*

This conjecture would imply that  $I_n$  has a decidable Gröbner basis for all  $n$ . This is a striking fact on its own.

In the commutative examples it turned out that the multiplicity of the alternating representation has very combinatorial behavior. It is only natural to do the same for  $\text{Har}_n$ . We now study the inverse systems  $\text{Har}_n$  intersecting alternating functions.

A set composition  $A$  of a set  $[n]$  is a list  $A = (A_1, A_2, \dots, A_k)$  such that  $A_i \neq \emptyset$  and  $\{A_1, A_2, \dots, A_k\} \vdash [n]$ . We denote this by  $A \models [n]$ . A function  $f$  in  $\mathbb{C}\langle X_n \rangle$  is alternating if  $\sigma \circ f = (-1)^{\ell(\sigma)} f$ , for all  $\sigma \in S_n$ . Let  $\text{Alt}_n$  be the set of all alternating functions in  $n$  non-commuting variables. We will construct a basis of  $\text{Alt}_n$  which is given by  $\{\mathcal{A}_\Phi\}$  indexed by set compositions  $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_k)$  with  $\min \Phi_i < \min \Phi_{i+1}$  and  $n - 1 \leq k \leq n$ . These set compositions may be identified with set partitions of size  $n - 1 \leq k \leq n$  where the parts are given in the prescribed order.

Conjecture 4.2 would imply that

$$\text{deg}(\text{Har}_n^{(d)} \cap \text{Alt}_n^{(d)}) = 0$$

when  $d$  is big enough, where  $Alt_n^{(d)}$  and  $Har_n^{(d)}$  denote the respective homogeneous component of degree  $d$  in in each space.

A *generalized set composition*  $A$  of  $[d]$  is a list of subsets  $(A_1, A_2, \dots, A_k)$ , where  $A_i \subseteq [d] = \{1, 2, \dots, d\}$  such that  $A_i \cap A_j = \emptyset$  for  $i \neq j$  and  $A_1 \cup A_2 \cup \dots \cup A_k = [d]$  (some parts may be empty). There is a one-to-one correspondence between words and generalized set compositions  $w \leftrightarrow A$ . Since we use generalized set compositions of  $d$  to decide the positions of variables in the corresponding words of degree  $d$ , we assume  $k = n$  the number of variables. For example, let  $n = 3$  and  $w = x_1 x_3 x_1$ . Then  $d = 3$ , the degree of  $w$ , and  $A = (\{1, 3\}, \emptyset, \{2\})$ . For convenience, we write  $A = 13.\emptyset.2$ .

Given a generalized set composition  $A = (A_1, A_2, \dots, A_n) \models [d]$  we define an alternating function

$$\mathcal{A}_A = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sigma \circ A,$$

where in the sum  $A$  is the corresponding word. For example, let  $d = 3$ ,  $n = 3$  and  $A = 13.2.\emptyset$ . Then

$$\mathcal{A}_A = x_1 x_2 x_1 - x_2 x_1 x_2 - x_3 x_2 x_3 - x_1 x_3 x_1 + x_2 x_3 x_2 + x_3 x_1 x_3.$$

If the number of empty sets in  $A$  is greater than one then  $\mathcal{A}_A = 0$ . Two properties hold:

- (1)  $\mathcal{A}_A = (-1)^{\ell(\pi)} \mathcal{A}_{\pi \circ A}$  for all  $\pi \in S_n$ , where  $\pi \circ A = (A_{\pi(1)}, A_{\pi(2)}, \dots, A_{\pi(n)})$ ;
- (2)  $\mathcal{A}_A = 0$  if  $\#\{A_i : A_i = \emptyset\} \geq 2$ ;

From (1) and (2) above, it is clear that any  $\mathcal{A}_A = \pm \mathcal{A}_\Phi$  for  $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_n)$  a generalized set composition where at most  $\Phi_n = \emptyset$  and the non-empty parts are ordered by  $\min \Phi_i < \min \Phi_{i+1}$ . For  $\Phi$  as described, each  $\mathcal{A}_\Phi$  has distinct support, hence  $\{\mathcal{A}_\Phi : \Phi \models [d] \text{ generalized, ordered, } \Phi_{n-1} \neq \emptyset\}$  is a basis of  $Alt_n^{(d)}$ .

For any generalized set composition  $A = (A_1, A_2, \dots, A_n)$  given a coloring function  $\epsilon : [n] \rightarrow \{\pm 1\}$  we get a two-colored generalized set composition

$$A^\epsilon = (A_1^{\epsilon_1}, A_2^{\epsilon_2}, \dots, A_n^{\epsilon_n}),$$

where  $\epsilon_i = \epsilon(i)$ . Let  $T = \epsilon^{-1}(1)$ . Define

$$\mathcal{A}_{A^\epsilon} = \sum_{\sigma \in S_T} (-1)^{\ell(\sigma)} \sigma \circ A^\epsilon,$$

where in the sum  $A^\epsilon$  is the word corresponding to  $A$  and  $\sigma$  only permutes indices given by  $T$  fixing the other parts. For convenience, we write  $2.\overline{13}.4.\overline{\emptyset}$  instead of  $2^1.13^{-1}.4^1.\emptyset^{-1}$ . Then

$$\mathcal{A}_{2.\overline{13}.4.\overline{\emptyset}} = x_2 x_1 x_2 x_3 - x_2 x_3 x_2 x_1.$$

**Proposition 4.3.** *Let  $\Psi \vdash [k]$  be a set partition,  $A \models [\ell]$  be a generalized set composition and  $\Phi \models [d]$  be an ordered generalized set composition with at most one  $\emptyset$ . Then  $\overleftarrow{M}_\Psi(d) d_{\overleftarrow{A}} \cdot \mathcal{A}_\Phi$  is 0 or  $\pm \mathcal{A}_{\Theta^\epsilon}$  for some two-colored generalized set composition  $\Theta^\epsilon$ .*

*Proof.* We prove it case by case. Suppose that

$$A_1 = A_1 \cap \Phi_{i_1}, \quad A_2 = A_2 \cap \Phi_{i_2}, \quad \dots, \quad A_n = A_n \cap \Phi_{i_n} \quad (4.2)$$

for some distinct  $i_1, i_2, \dots, i_n$ . Hence, there are some terms in  $\mathcal{A}_\Phi$  beginning with the word corresponding to  $A$ . Choose a fixed sequence  $i_1, i_2, \dots, i_n$  and reorder  $\Phi$  as  $\Phi^\tau = (\Phi_{i_1}, \Phi_{i_2}, \dots, \Phi_{i_n})$ , where  $\tau$  is the permutation corresponds to this

sequence. Then there is one word in the remaining terms in  $d_{\overleftarrow{A}} \cdot \mathcal{A}_{\Phi}$  corresponding to  $st(\Phi^\tau \setminus A)$ , where  $st(\Phi^\tau \setminus A)$  means deleting the numbers in  $A$  from  $\Phi^\tau$  then standardizing it (that is subtracting  $\ell$  from all the number in  $\Phi^\tau \setminus A$ ). Define a two-colored generalized set composition  $st(\Phi^\tau \setminus A)^\epsilon$  by  $\epsilon_i = -1$  if  $A_i \neq \emptyset$  and 1, otherwise. Then  $d_{\overleftarrow{A}} \cdot \mathcal{A}_{\Phi} = (-1)^{\ell(\tau)} \mathcal{A}_{st(\Phi^\tau \setminus A)^\epsilon}$ . If Equation (4.2) is not satisfied, then there is no term in  $\mathcal{A}_{\Phi}$  beginning with the word corresponding to  $A$ . Therefore,  $d_{\overleftarrow{A}} \cdot \mathcal{A}_{\Phi} = 0$ .

Hence, we may assume that Equation (4.2) is satisfied and let  $\Gamma = st(\Phi^\tau \setminus A)$ . Then  $\overleftarrow{M}_{\Psi}(d) d_{\overleftarrow{A}} \cdot \mathcal{A}_{\Phi} = (-1)^{\ell(\tau)} \overleftarrow{M}_{\Psi}(d) \cdot \mathcal{A}_{\Gamma^\epsilon}$ . The set partition  $\Psi = \{\Psi_1, \Psi_2, \dots, \Psi_r\}$  for some  $r \leq n$ . Suppose

$$\Psi_1 = \Psi_1 \cap \Gamma_{i_1}, \quad \Psi_2 = \Psi_2 \cap \Gamma_{i_2}, \quad \dots, \quad \Psi_r = \Psi_r \cap \Gamma_{i_r} \quad (4.3)$$

for some distinct  $i_1, i_2, \dots, i_r$ . Since  $\Psi$  is a set, we can reorder the  $\Psi_i$  in any order needed. For any monomial in  $M_{\Psi}$  there is a unique monomial in  $\mathcal{A}_{\Gamma^\epsilon}$  with the appropriate sign which gives  $\overleftarrow{M}_{\Psi}(d) \cdot \mathcal{A}_{\Gamma^\epsilon} = \mathcal{A}_{st(\Gamma \setminus \Psi)^\epsilon}$ . If Equation (4.3) is not satisfied, then there is no term in  $\mathcal{A}_{st(\Phi^\tau \setminus A)^\epsilon}$  beginning with the word  $\sigma \circ \Psi$  for all  $\sigma \in S_n/S_L$ . Hence  $\overleftarrow{M}_{\Psi}(d) \cdot \mathcal{A}_{st(\Phi^\tau \setminus A)^\epsilon} = 0$ .

In all cases, if any of Equation (4.2) or Equation (4.3) is not satisfied, then  $\overleftarrow{M}_{\Psi}(d) d_{\overleftarrow{A}} \cdot \mathcal{A}_{\Phi} = 0$ , Otherwise

$$\overleftarrow{M}_{\Psi}(d) d_{\overleftarrow{A}} \cdot \mathcal{A}_{\Phi} = (-1)^{\ell(\tau)} \mathcal{A}_{st(st(\Phi^\tau \setminus A) \setminus \Psi)^\epsilon}$$

We let  $\Theta = st(st(\Phi^\tau \setminus A) \setminus \Psi)$  and the proof is completed.  $\square$

We remark that in the proof of Proposition 4.3 we could reorder  $A$  so that  $\tau$  is the identity. That is we can assume that  $A$  is an ordered generalized composition. If we want to compute  $Har_n^{(d)} \cap Alt_n^{(d)}$ , we need to solve the equations

$$\overleftarrow{M}_{\Psi}(d) d_{\overleftarrow{A}} \cdot P = 0, \forall \Psi, A$$

and  $P \in Alt_n^{(d)}$ . It is easy to see that for fix  $A$  and  $\Psi$ , the possible  $\Theta^\epsilon$  are all linearly independent. This gives us a system of linear equations,

$$[\overleftarrow{M}_{\Psi}(d) d_{\overleftarrow{A}} \cdot P]_{\mathcal{A}_{\Theta^\epsilon}} = 0 \quad (4.4)$$

where  $[f]_{\mathcal{A}_{\Theta^\epsilon}}$  denote the coefficient of  $\mathcal{A}_{\Theta^\epsilon}$  in  $f$  and

$$P = \sum_{\Phi = [d] \text{ ordered}} c_{\Phi} \mathcal{A}_{\Phi}.$$

The System (4.4) is explicit and may be more easy to handle than the full space  $Har_n$ . As a weaker conjecture we propose

**Conjecture 4.4.** *The solution for the linear equations in the System (4.4) is all of  $Alt_n^{(d)}$  for large  $d$ .*

## 5. QUASI-SYMMETRIC FUNCTIONS IN NON-COMMUTATIVE VARIABLES

A second example of non-commutative combinatorial inverse system is given by quasi-symmetric functions in non-commutative variables [8]. We have recently showed [5] that the associated inverse systems are finite. The proof does not give the quotient, which is still an open problem. However, we present the result here as another evidence that combinatorial inverse systems are special and also to partially support the conjectures in Section 4.

The combinatorial Hopf algebra  $NCQSym$  of quasi-symmetric functions in non-commutative variables is the vector space freely generated by the set  $\{\mathcal{M}_A : A \models [d], d \geq 0\}$ . Again we let  $()$  be the unique set composition of  $[0]$ . The multiplication is given by quasi-shuffle of set composition. That is, for  $A = (A_1, A_2, \dots, A_k) \models [d]$  and  $B = (B_1, B_2, \dots, B_\ell) \models [q]$  we define  $A \overline{\sqcup} B^{\uparrow d}$  recursively. The notation  $B^{\uparrow d}$  means that we add  $d$  to each entry of  $B$ . If  $k = 0$  or  $\ell = 0$  then  $A \overline{\sqcup} B^{\uparrow d} = A \cdot B$  where  $\cdot$  denote the concatenation of lists. If not,

$$\begin{aligned} A \overline{\sqcup} B^{\uparrow d} &= A_1 \cdot ((A_2, \dots, A_k) \overline{\sqcup} B^{\uparrow d}) + B_1^{\uparrow d} \cdot (A \overline{\sqcup} (B_2^{\uparrow d}, \dots, B_k^{\uparrow d})) \\ &\quad + (A_1 \cup B_1^{\uparrow d}) \cdot ((A_2, \dots, A_k) \overline{\sqcup} (B_2^{\uparrow d}, \dots, B_k^{\uparrow d})). \end{aligned}$$

We then define

$$\mathcal{M}_A \mathcal{M}_B = \sum_{C \in A \overline{\sqcup} B^{\uparrow d}} \mathcal{M}_C$$

where  $C \in A \overline{\sqcup} B^{\uparrow d}$  indicate that  $C$  is in the support of  $A \overline{\sqcup} B^{\uparrow d}$ . The comultiplication is defined by

$$\Delta(\mathcal{M}_A) = \sum_{B \cdot C = A} \mathcal{M}_{st(B)} \otimes \mathcal{M}_{st(C)}.$$

Here,  $st(-)$  is the standardization map defined as follow. For  $B \cdot C = A \models [d]$ , it means that  $B \models S$  and  $C \models T$  where  $S \cup T = [d]$  and  $S \cap T = \emptyset$ . For any  $S \subseteq [d]$ , there is a unique order preserving map  $\phi: \{1, 2, \dots, |S|\} \rightarrow S$ . For any  $B \models S$ , we let  $st(B) \models \{1, 2, \dots, |S|\}$  the unique set composition we obtain using  $\phi$ , in other words  $st(B) = \phi^* B$ . The space  $NCQSym$  is graded by  $\deg(\mathcal{M}_A) = d$  when  $A \models [d]$ .

As seen in [8], there is an embedding  $QSym \hookrightarrow \mathbb{C}\langle\langle X \rangle\rangle$  with  $X = x_1, x_2, \dots$  given by

$$\mathcal{M}_A = \sum_{\tilde{\nabla}(w)=A} w,$$

where  $\tilde{\nabla}(w) = (w^{-1}(x_i))_{i=1}^{\infty} \setminus \emptyset$ . In other words, the infinite sequence  $(w^{-1}(x_i))_{i=1}^{\infty}$  has only finitely many non-empty parts. After removing the empty parts, we obtain a set composition denoted  $\tilde{\nabla}(w)$ .

Taking  $X_n = x_1, \dots, x_n$  then  $NCQSym$  is a combinatorial Hopf algebra satisfying the criteria above and we can construct its combinatorial inverse system. The inverse system  $SHar_n = \{I_n^\perp\}_{n \geq 1}$  corresponding to the ideals  $I_n = \langle \mathcal{M}_A(X_n) : A \models [d] \geq 1 \rangle \subseteq \mathbb{C}\langle X_n \rangle$ . These spaces have not been studied that much. The only know theorem is the following

**Theorem 5.1.**  $\dim(SHar_n) < \infty$ .

This is shown in [5]. In parallel with the commutative case, we believe that the non-commutative combinatorial inverse system will prove themselves to be very rich objects to study. It would not be surprising to discover that they all have very special Gröbner basis.

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(Jean-Christophe Aval) LABRI, UNIVERSITÉ BORDEAUX 1, 351 COURS DE LA LIBÉRATION, 33405  
TALENCE CEDEX, FRANCE

*E-mail address:* [aval@labri.fr](mailto:aval@labri.fr)  
*URL:* <http://www.labri.fr/perso/aval>

(Nantel Bergeron) DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, TO-  
RONTO, ONTARIO M3J 1P3, CANADA

*E-mail address:* [bergeron@mathstat.yorku.ca](mailto:bergeron@mathstat.yorku.ca)  
*URL:* <http://www.math.yorku.ca/bergeron>

(Huilan Li) DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PA 19104,  
U.S.A

*E-mail address:* [huilan.li@gmail.com](mailto:huilan.li@gmail.com)  
*URL:* <http://www.math.drexel.edu/~huilan/>