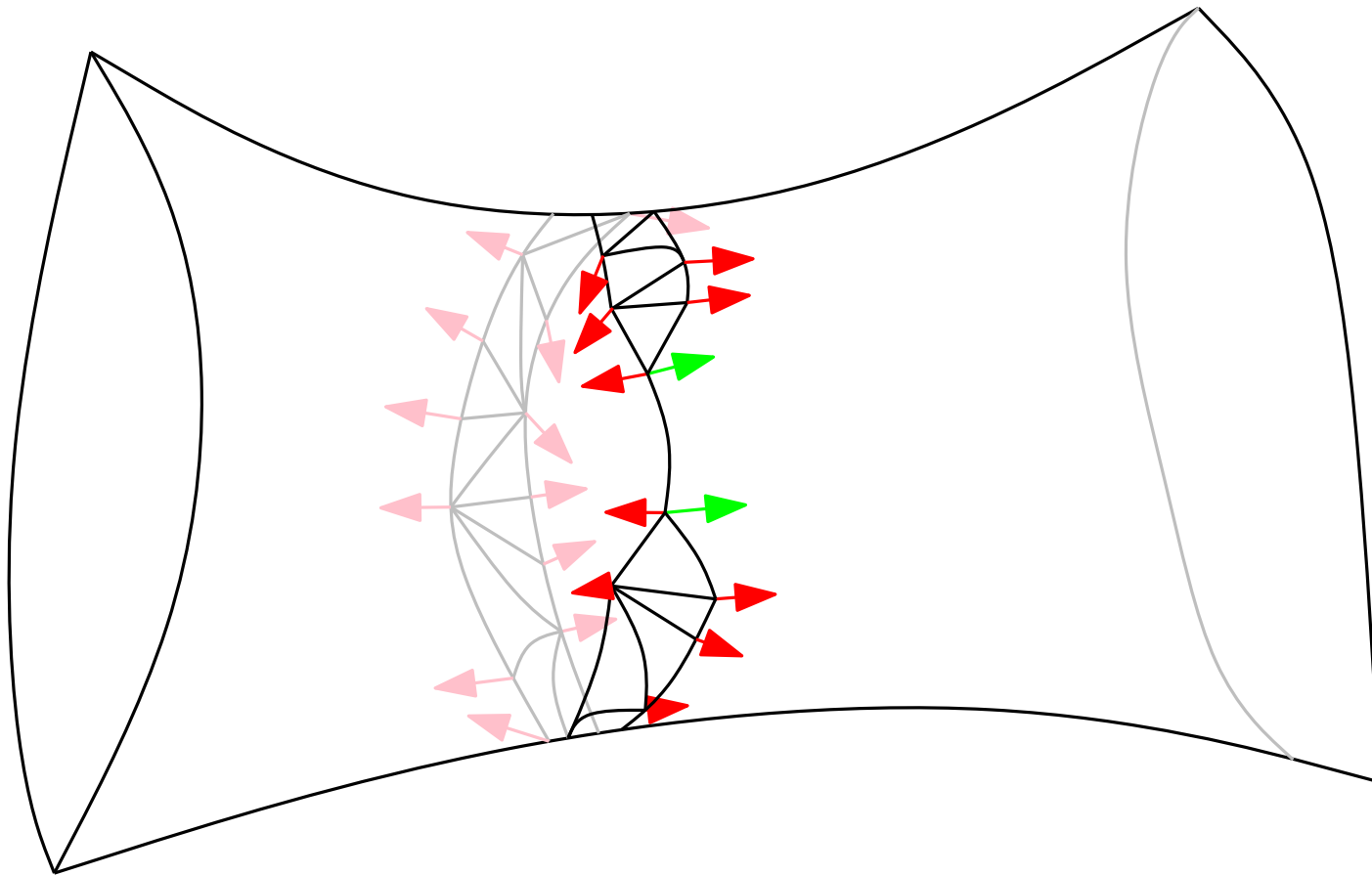


Orienting triangulations: on a conjecture of Barát and Thomassen

Kolja Knauer
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Boris Albar, Daniel Gonçalves
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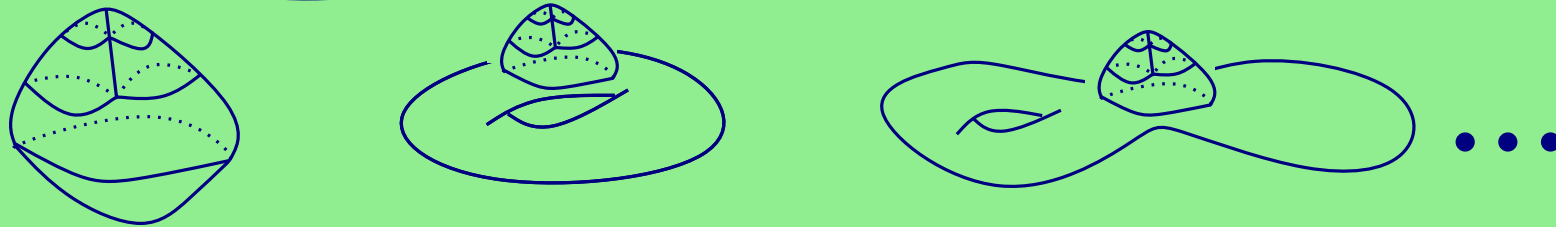
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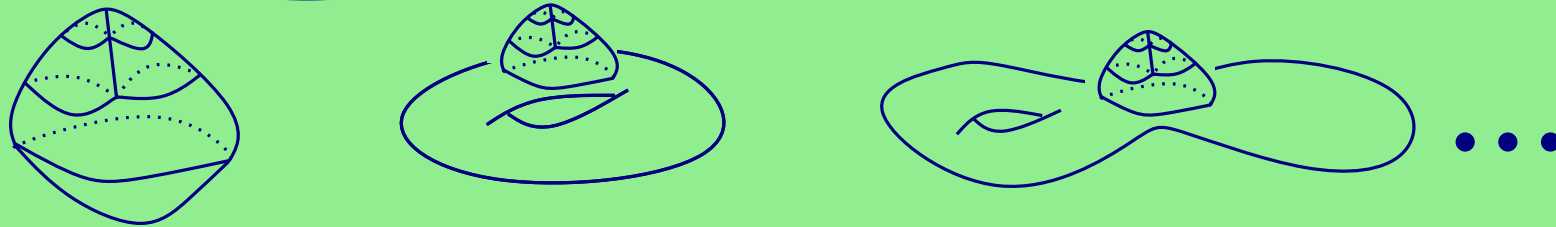
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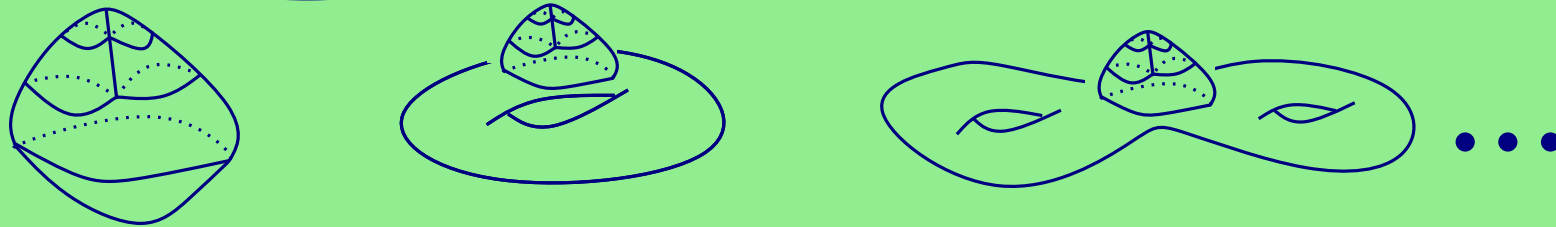
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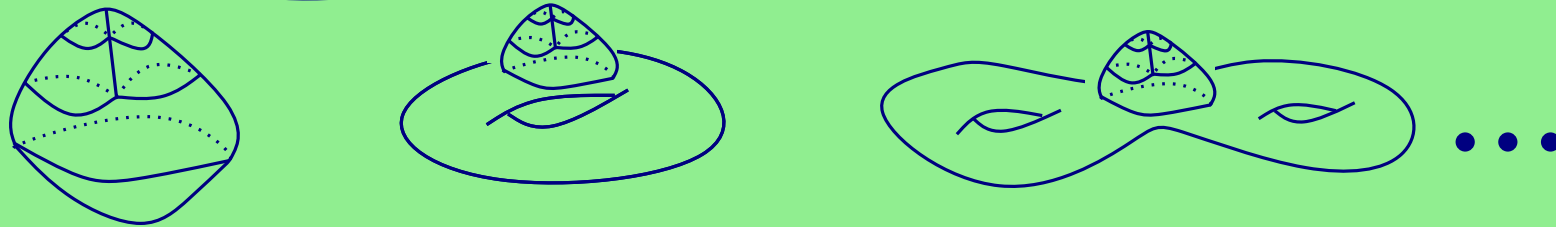
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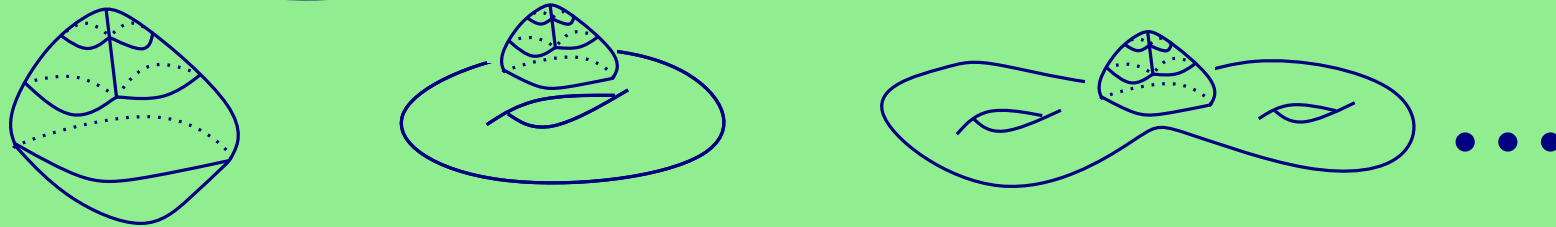
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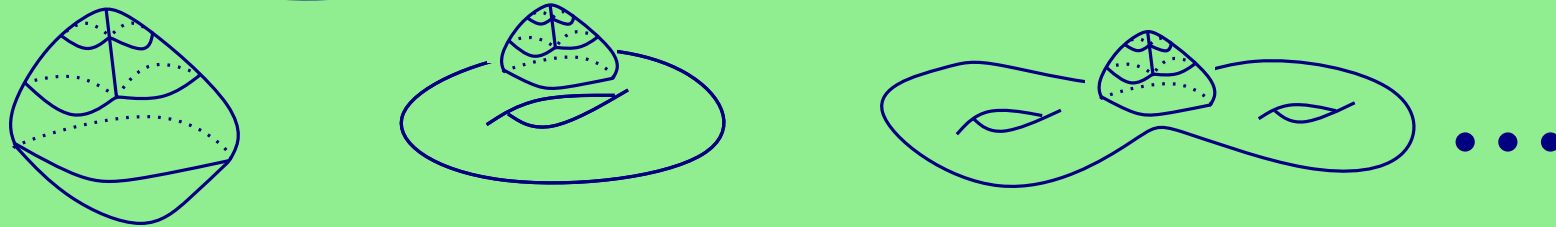
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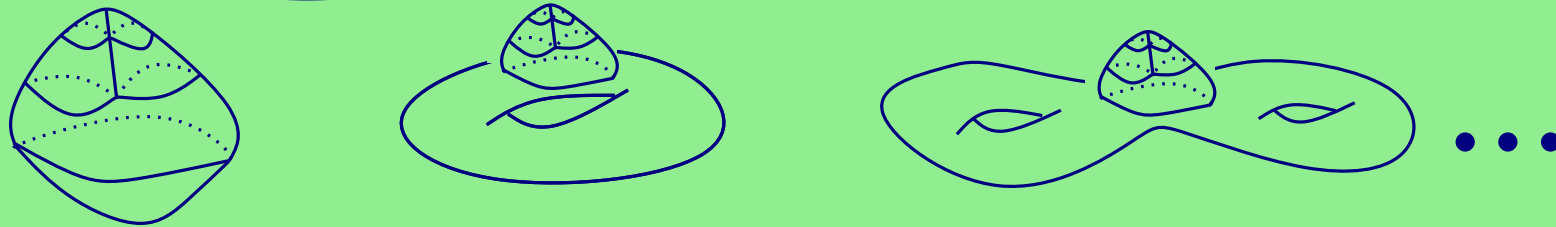
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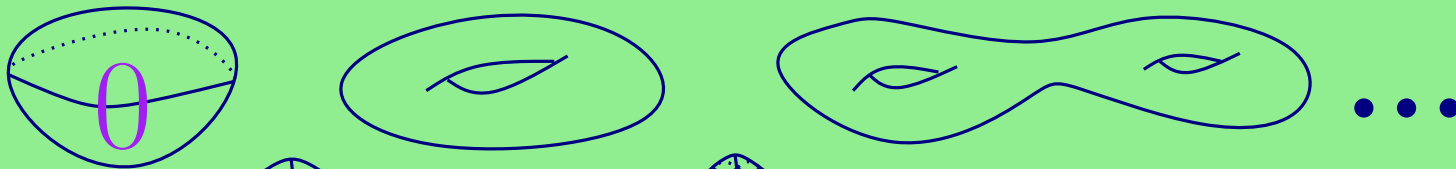
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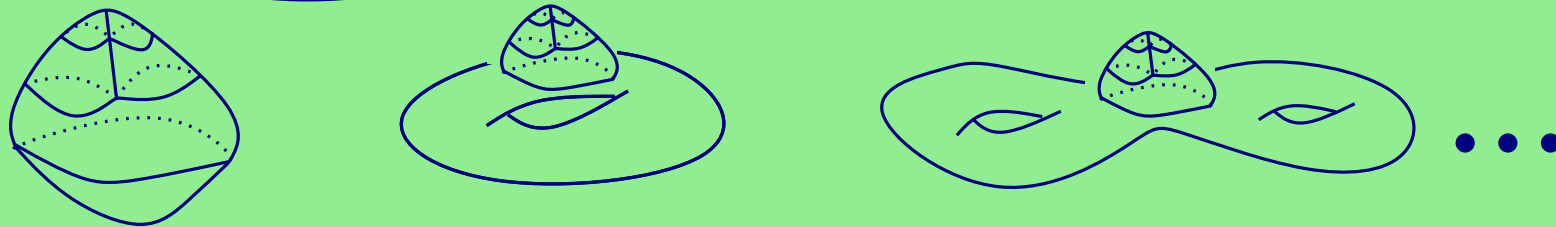
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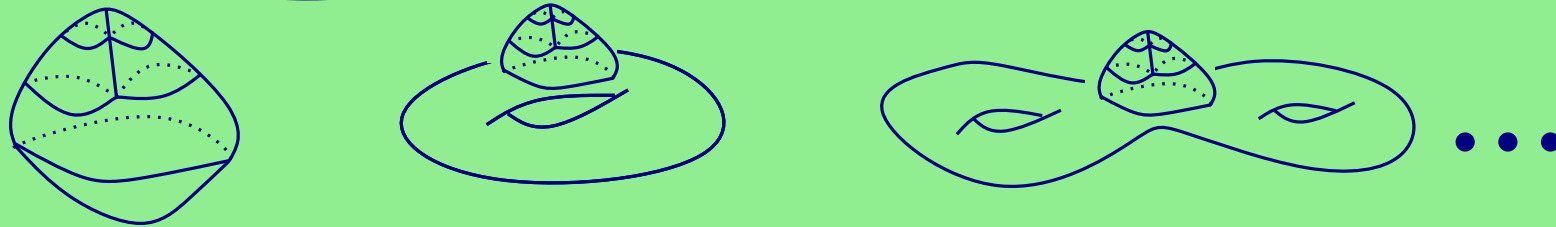
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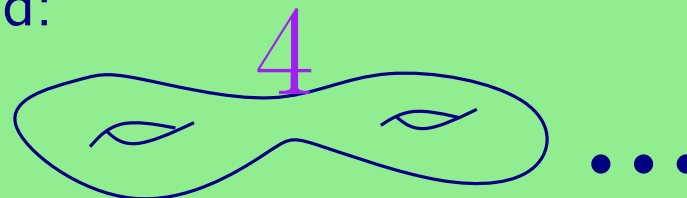
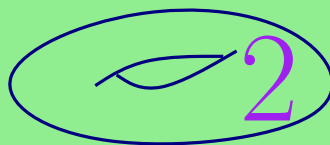
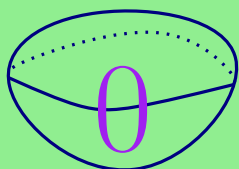
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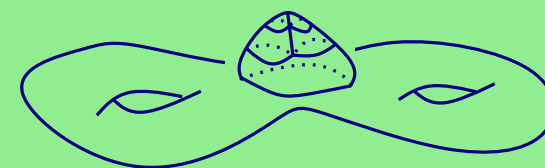
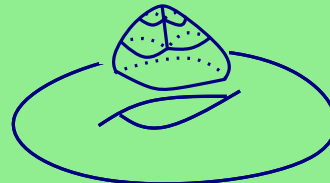
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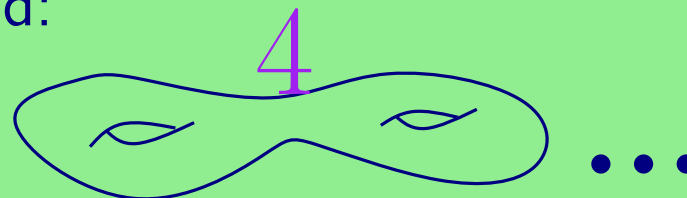
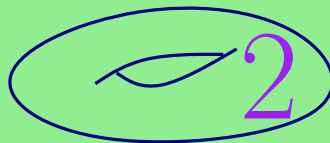
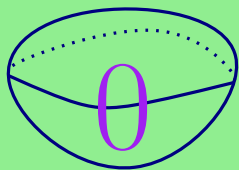
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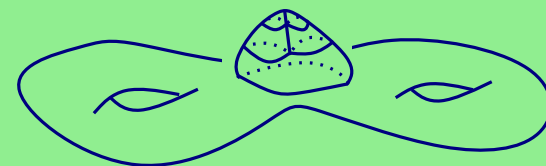
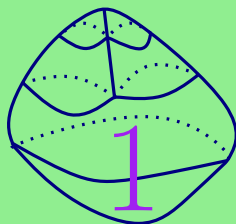
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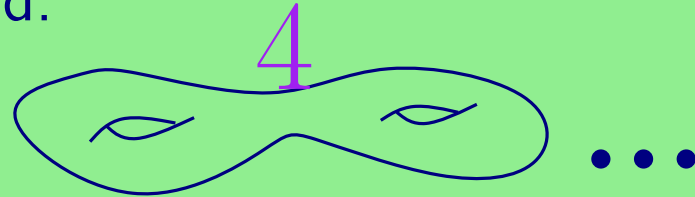
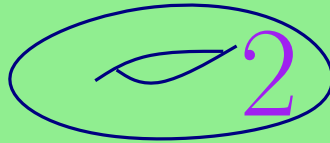
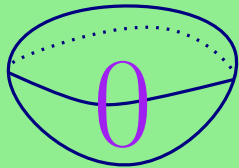
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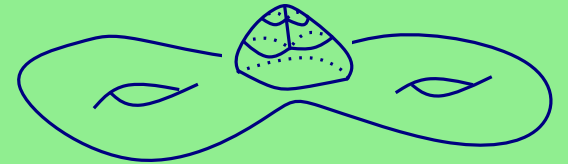
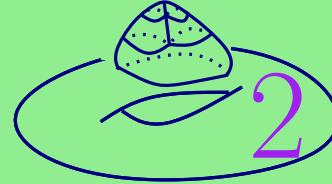
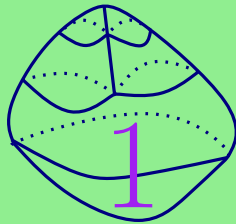
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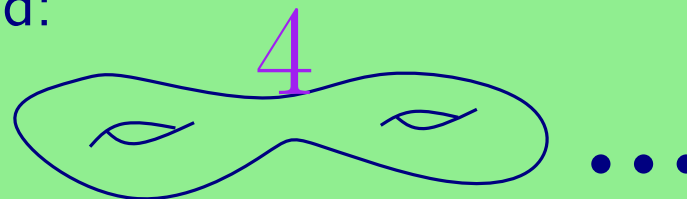
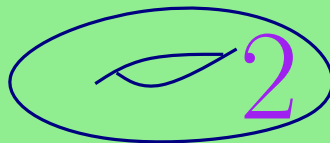
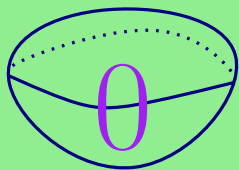
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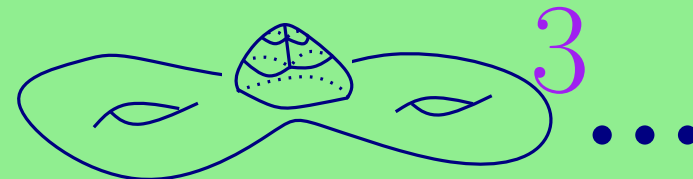
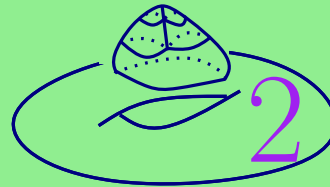
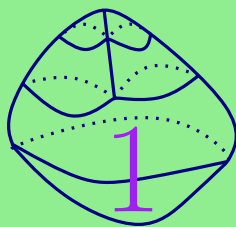
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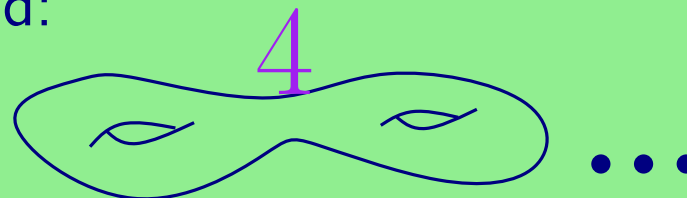
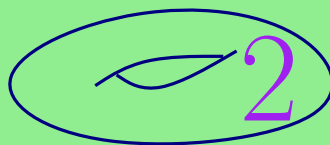
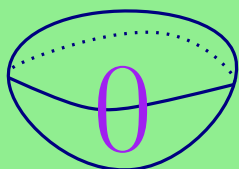
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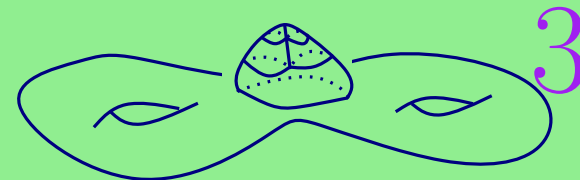
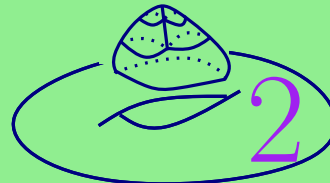
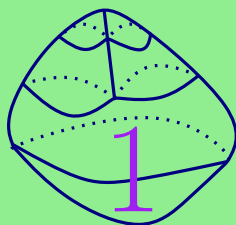
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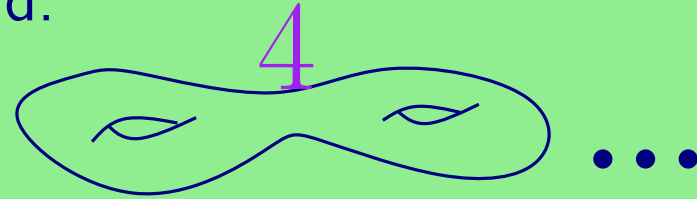
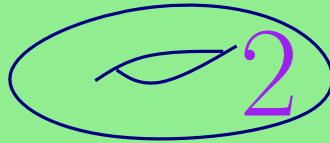
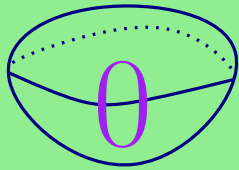
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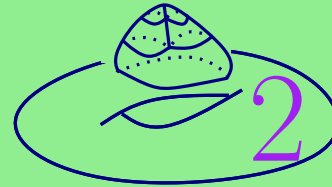
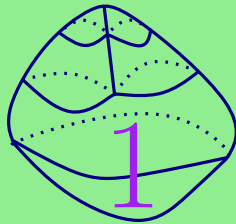
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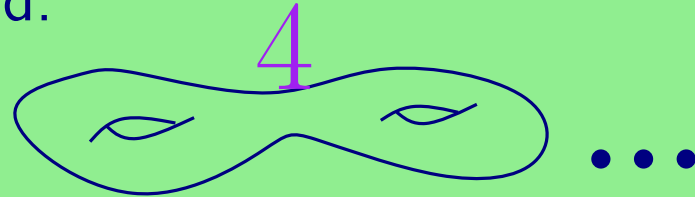
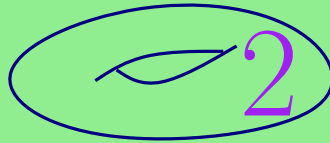
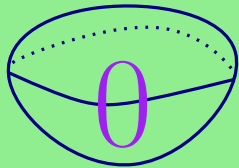
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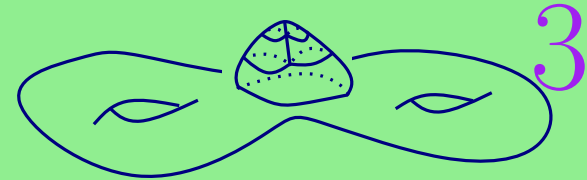
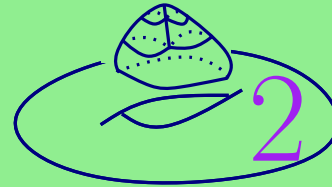
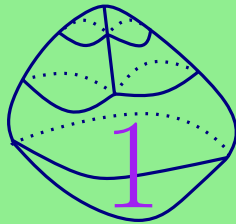
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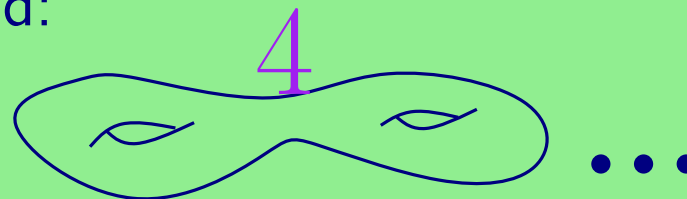
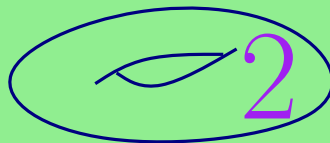
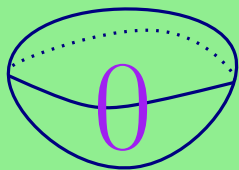
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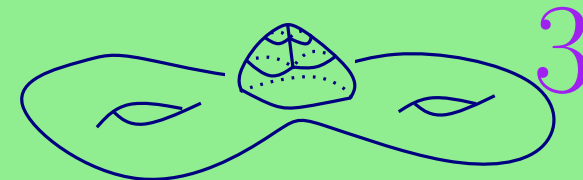
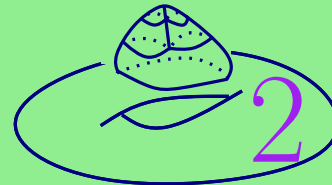
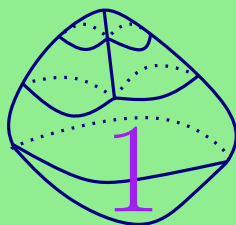
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Euler genus requirement necessary and Conjecture best-possible.

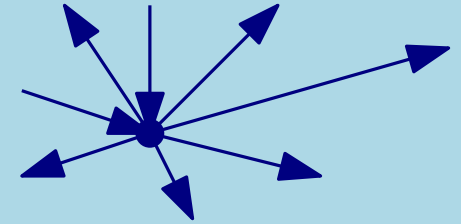
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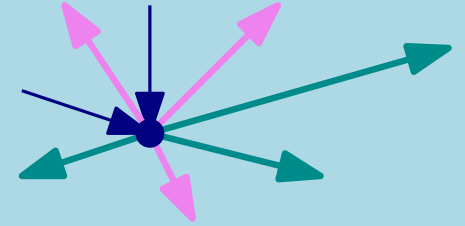
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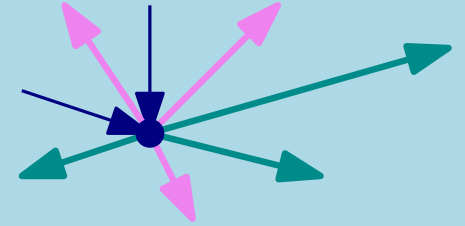
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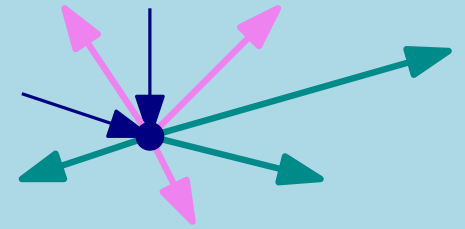
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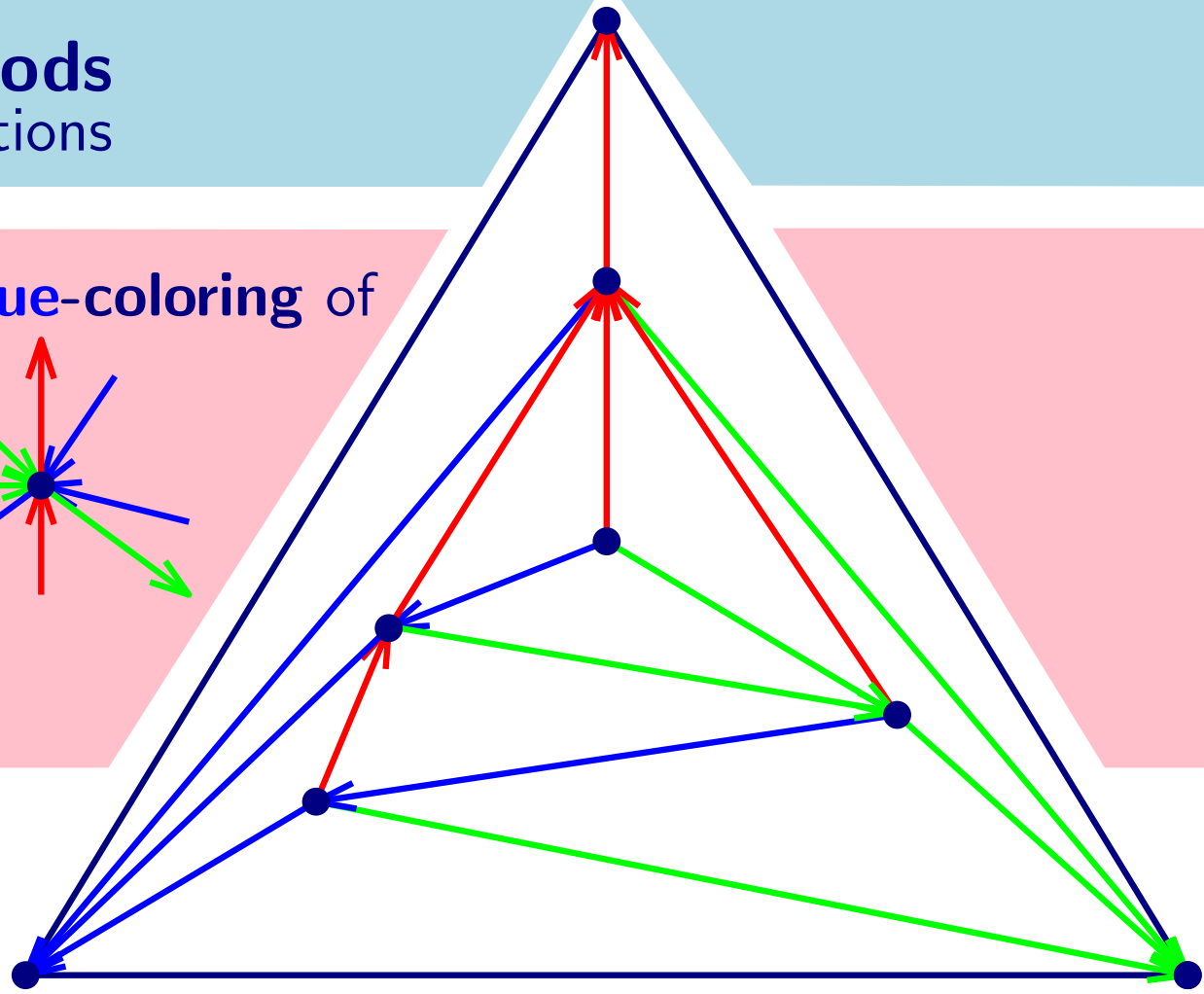
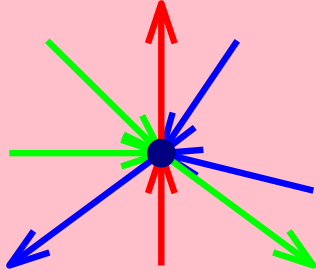
Thm[Barát, Thomassen '06]:

Every triangulation of a closed surface admits an orientation with all vertices of outdegree *divisible by three*. On torus and Klein-bottle outdegrees non-zero.

additional motivation: *generalizations of Schnyder Woods*

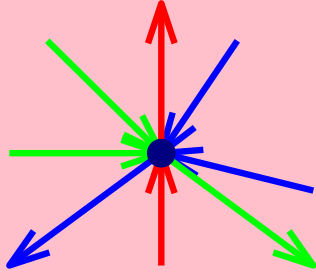
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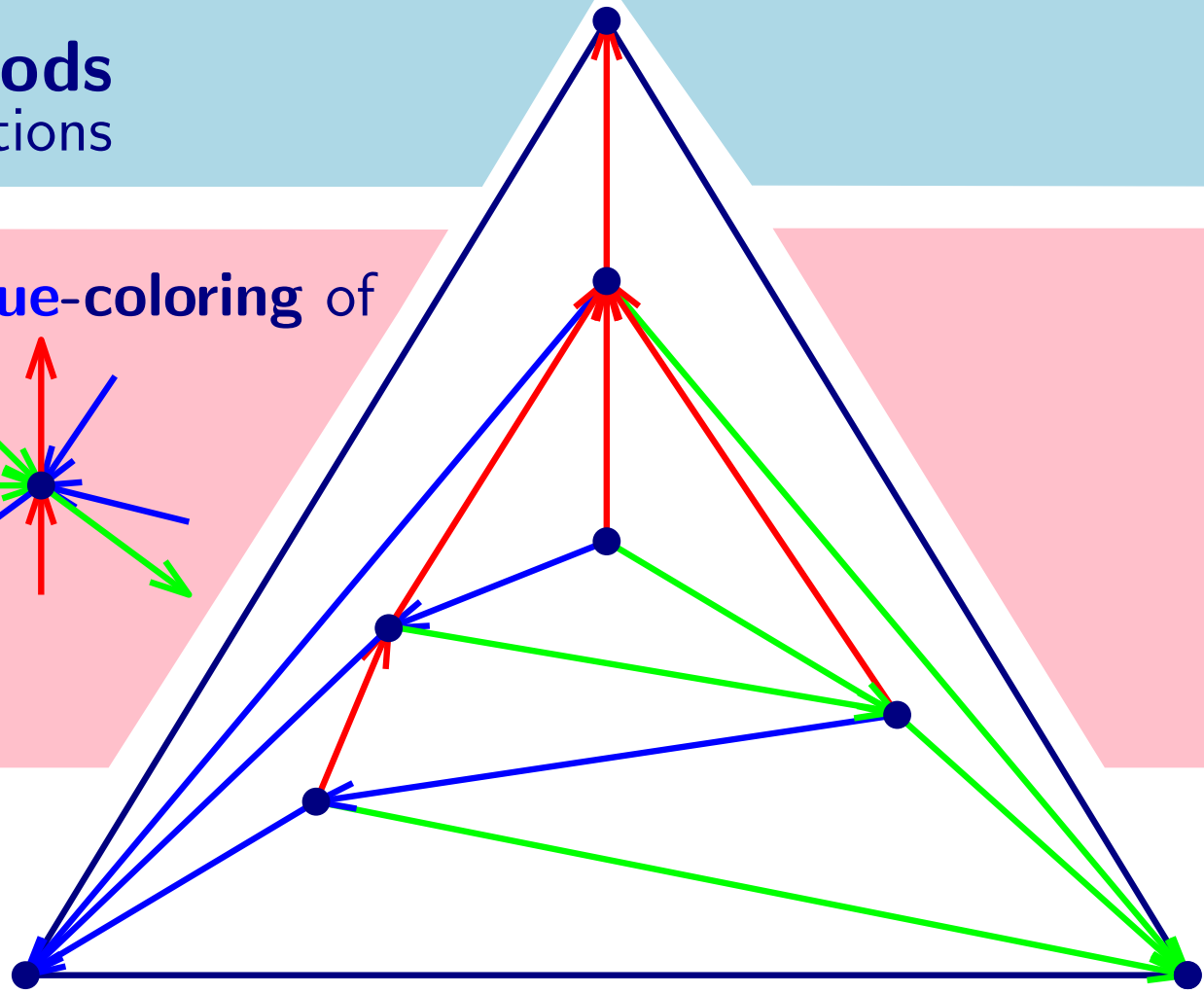


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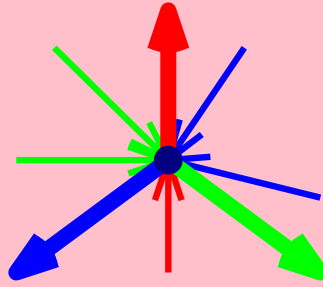


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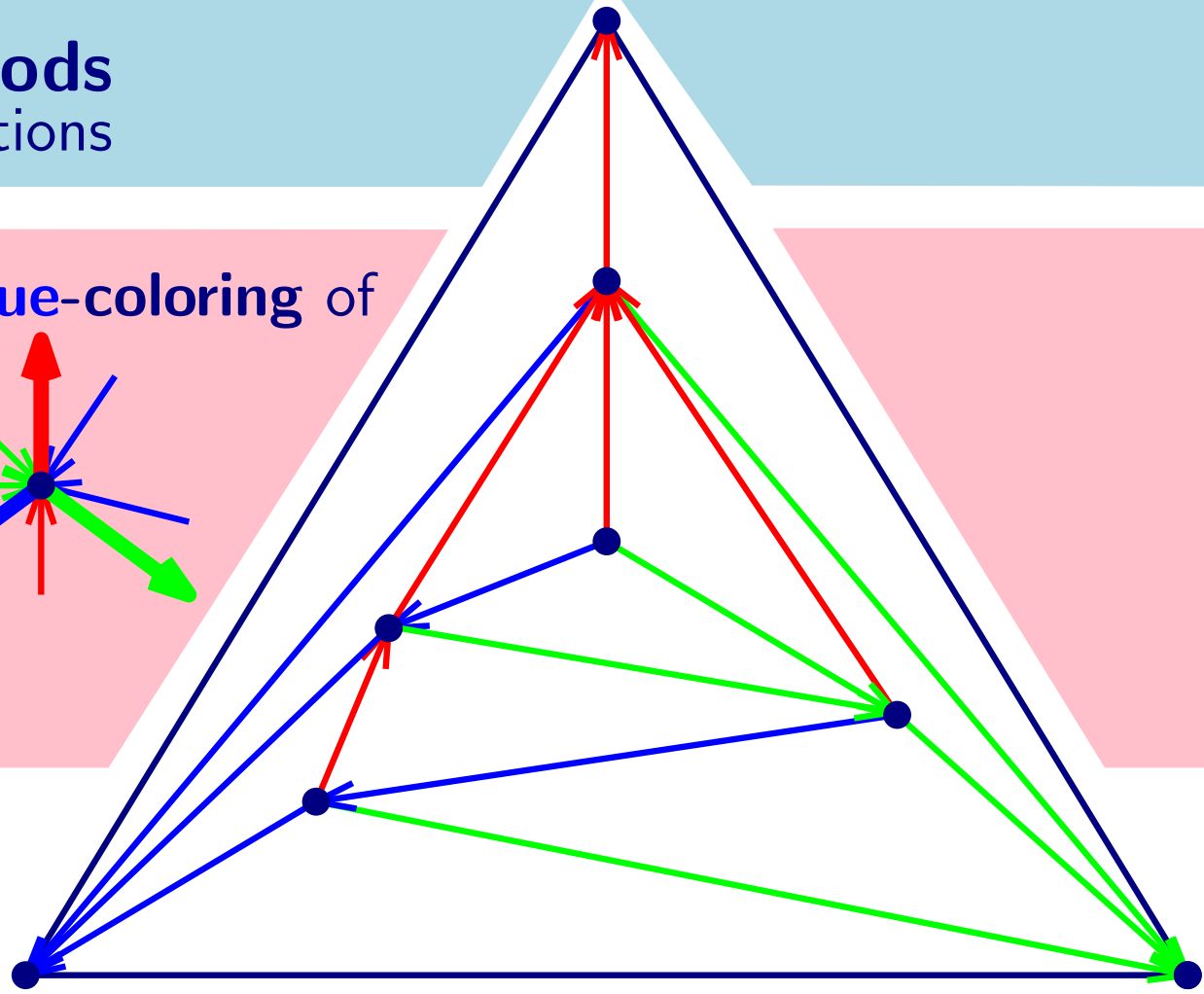


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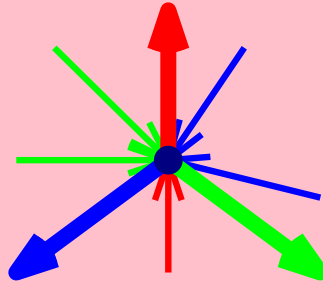


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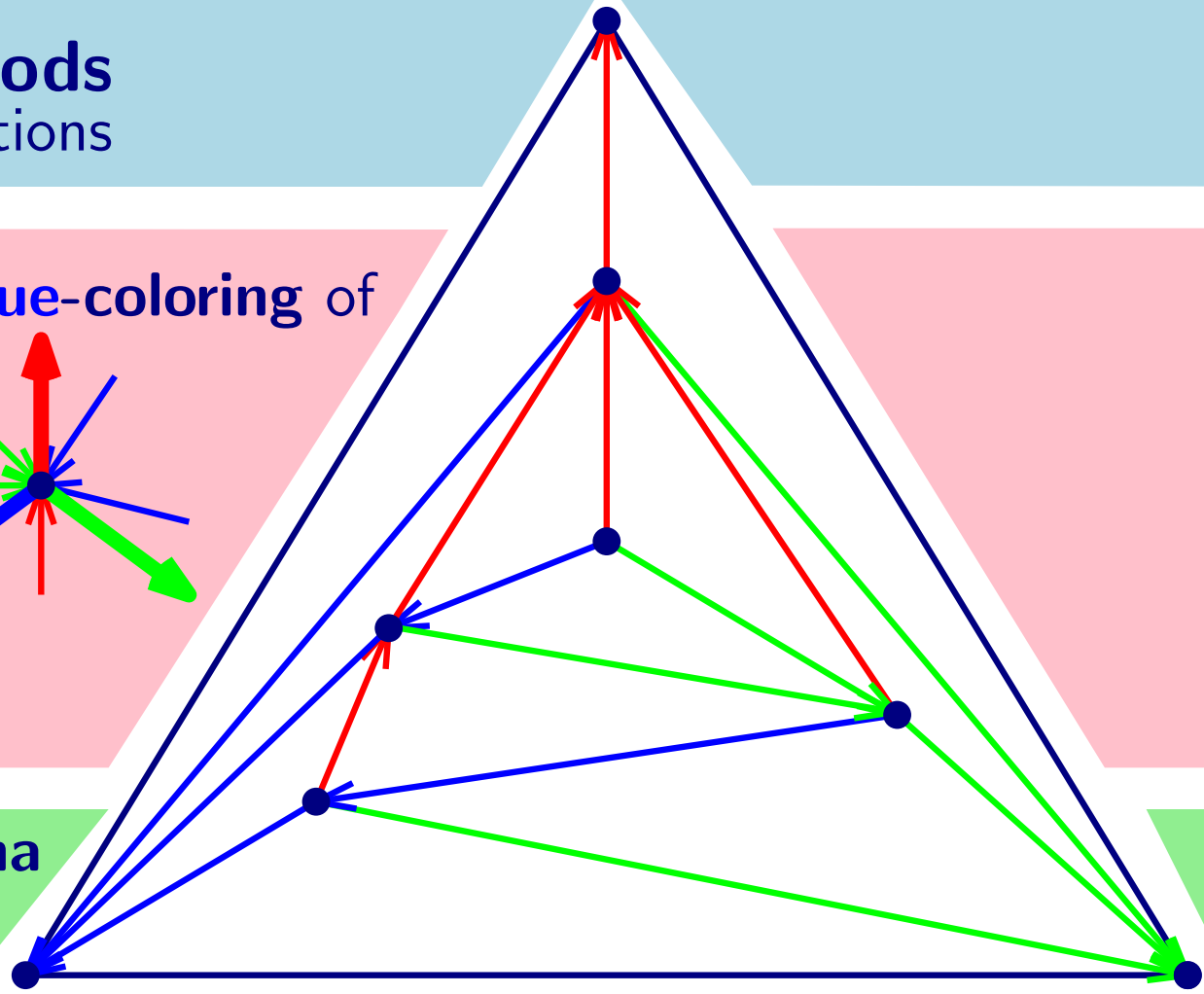
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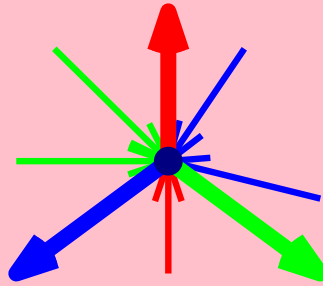
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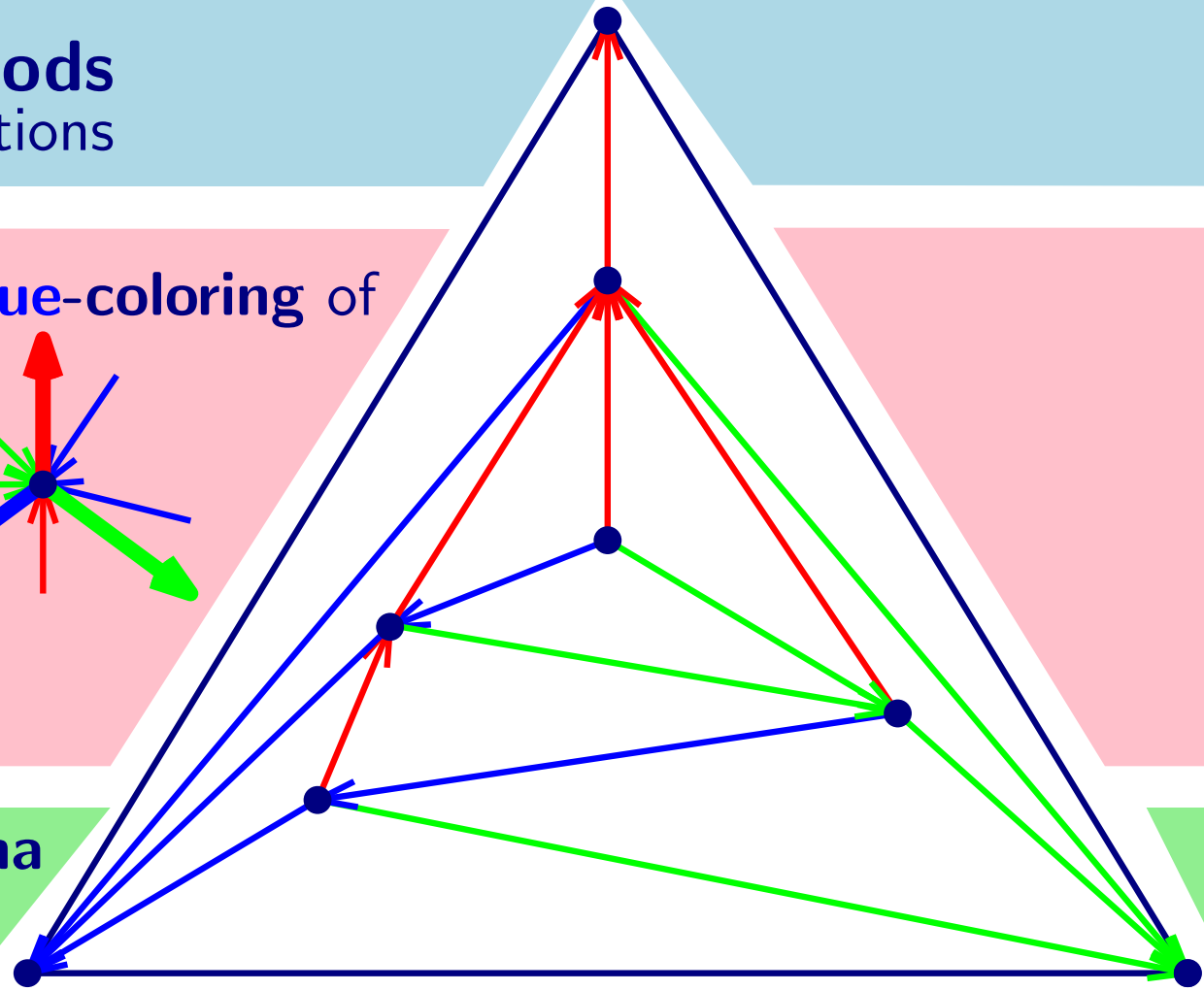
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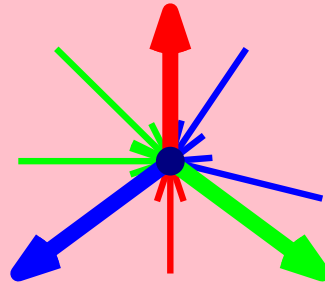
Applications in:

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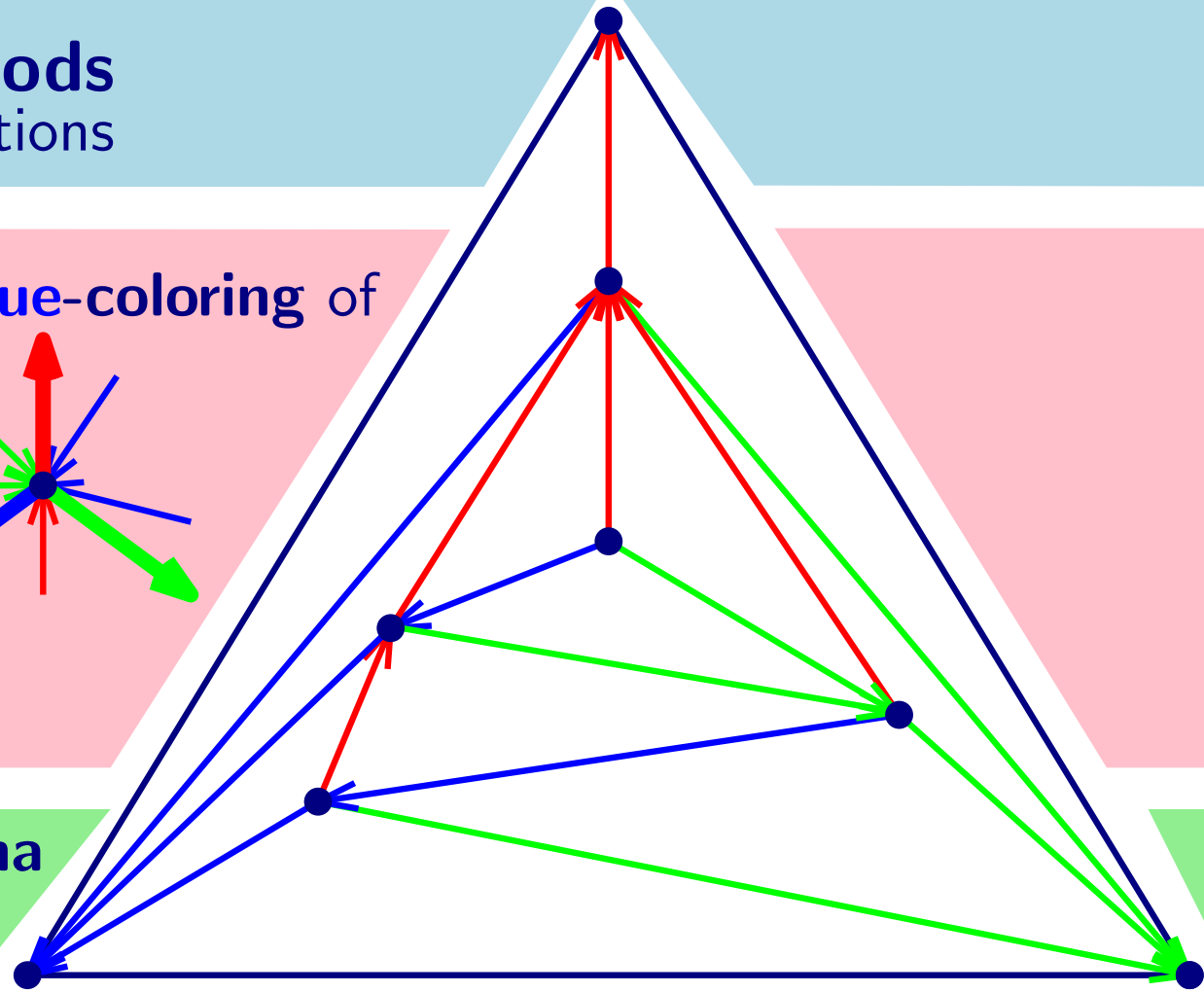
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How about generalizations to higher genus?



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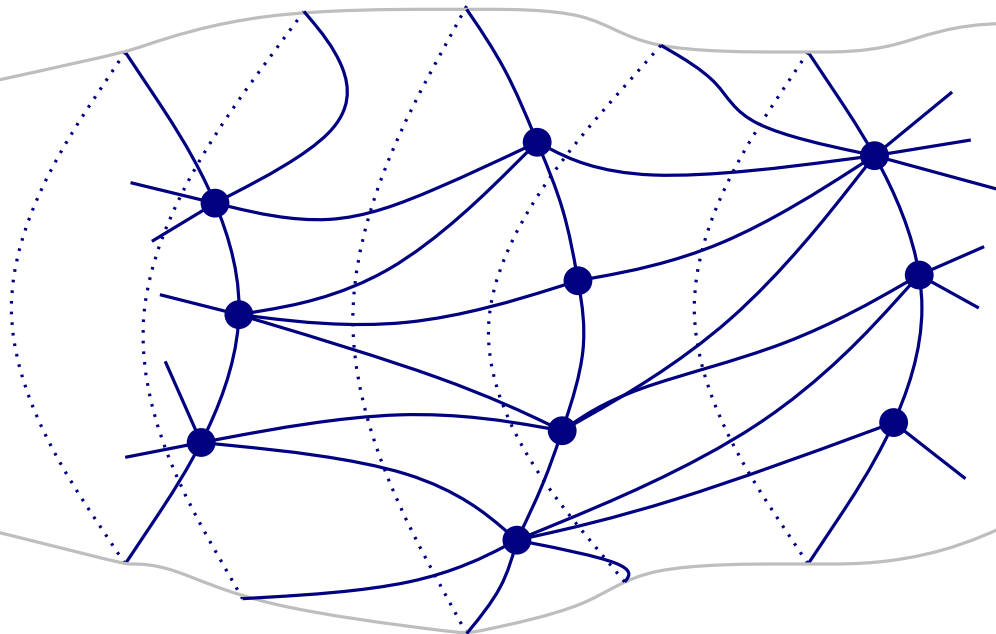
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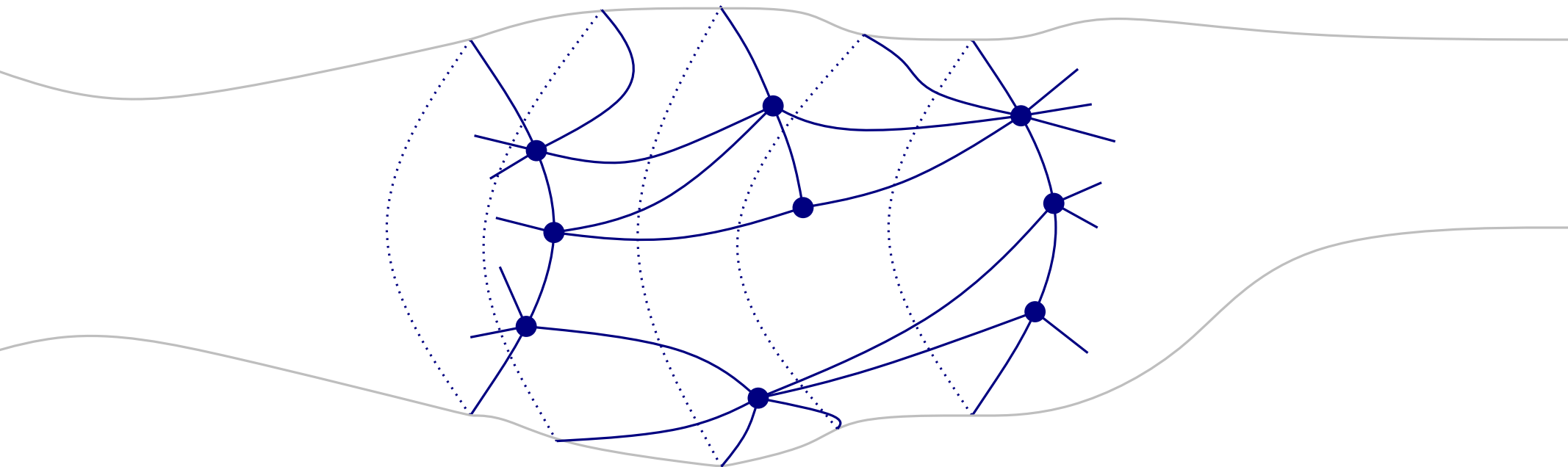
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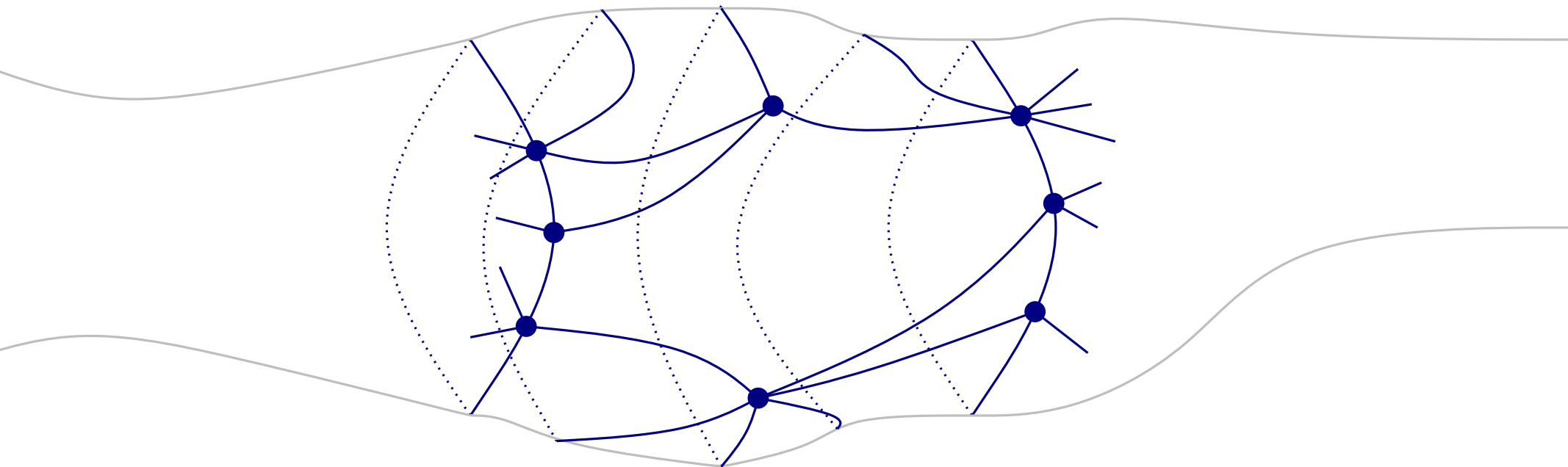
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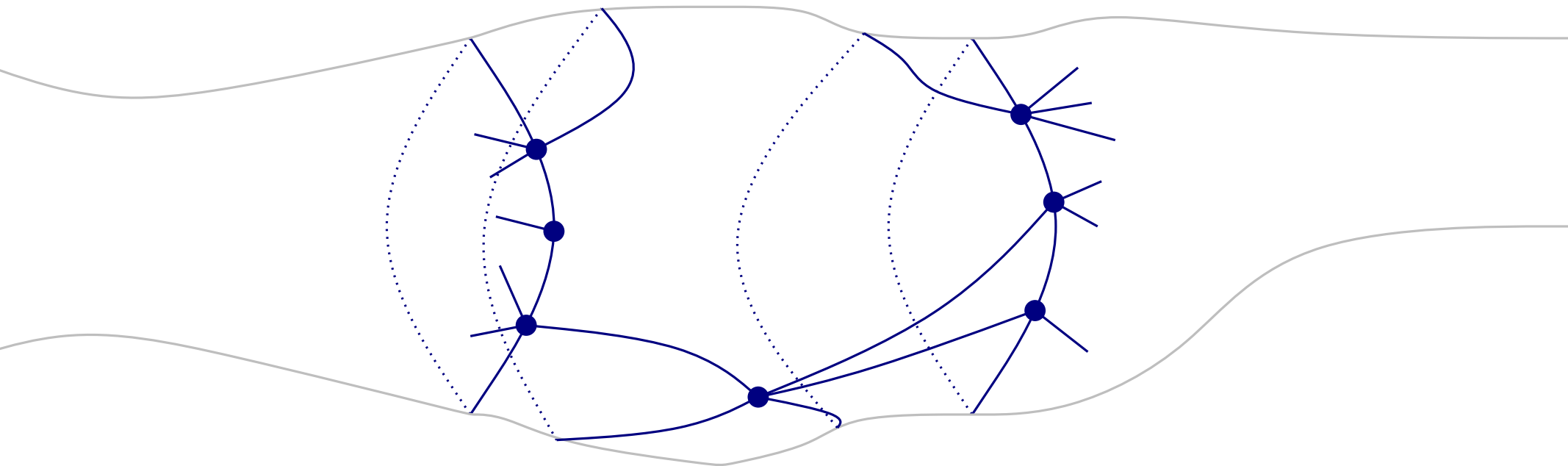
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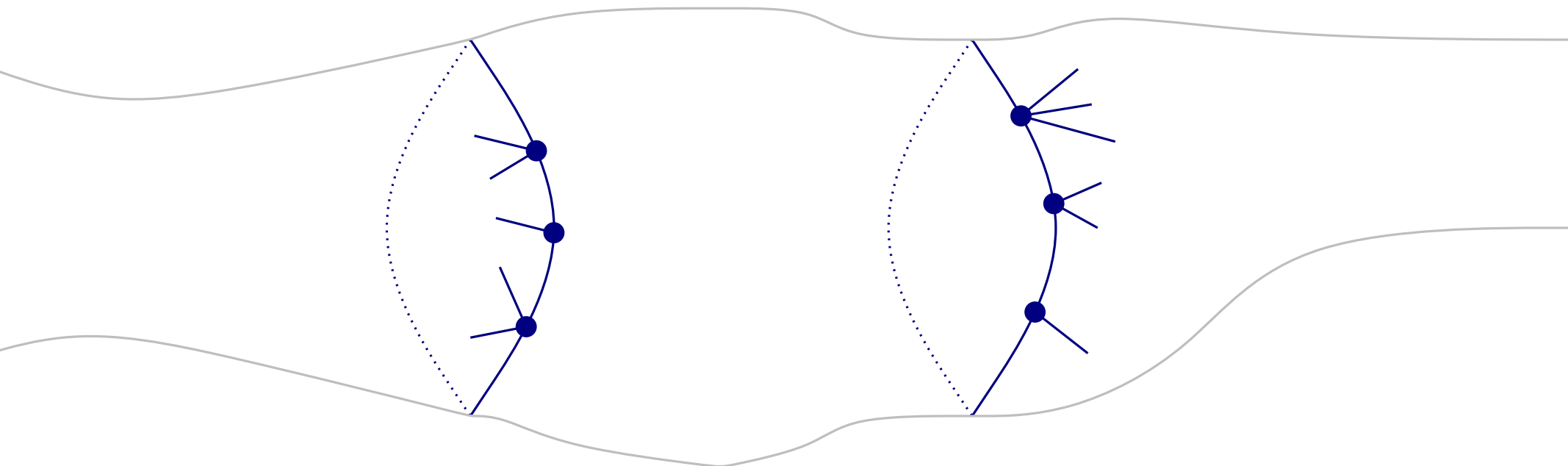
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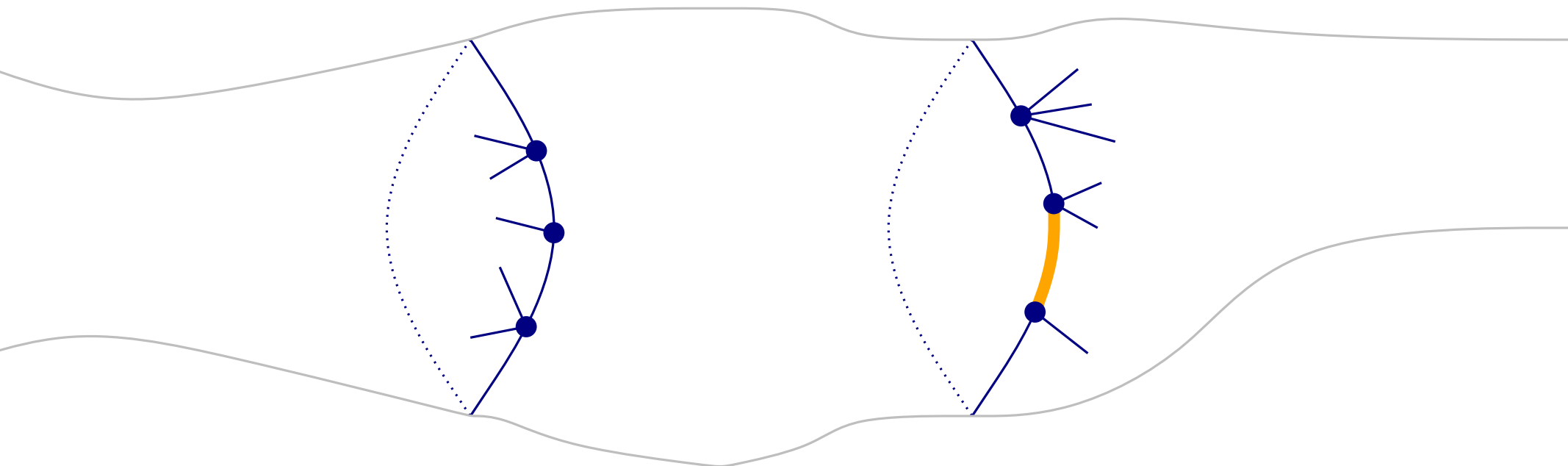
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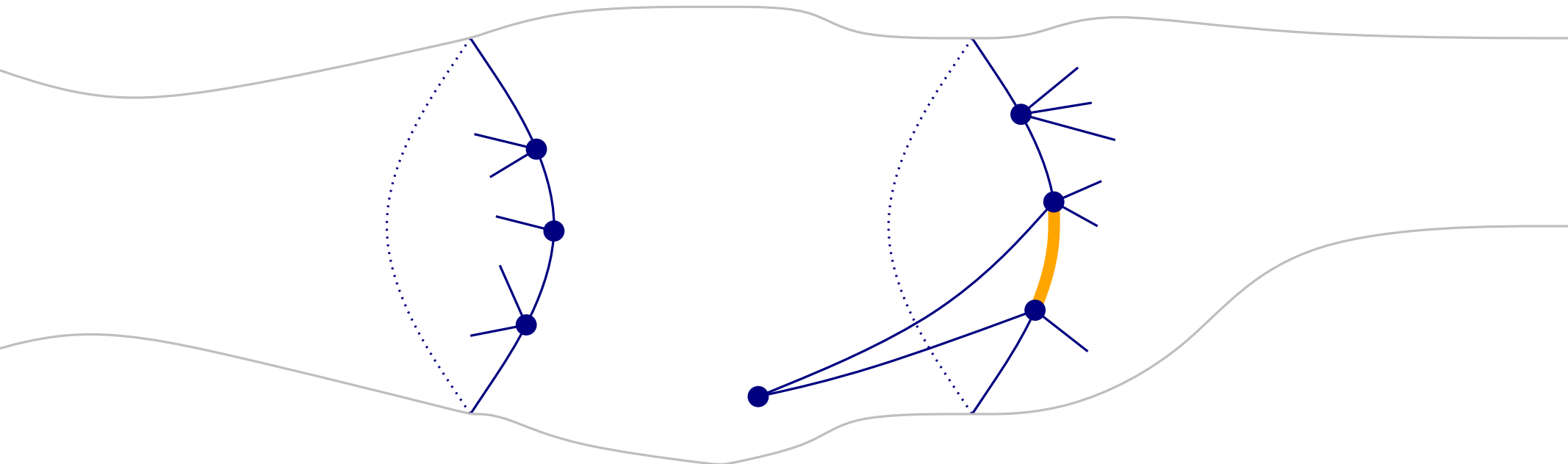
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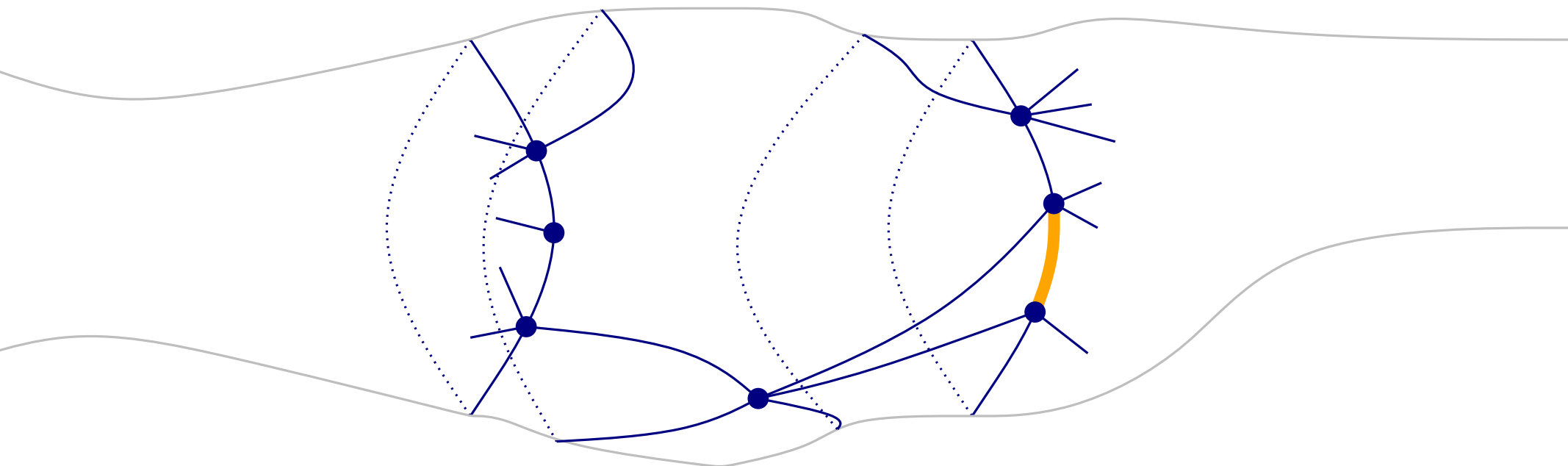
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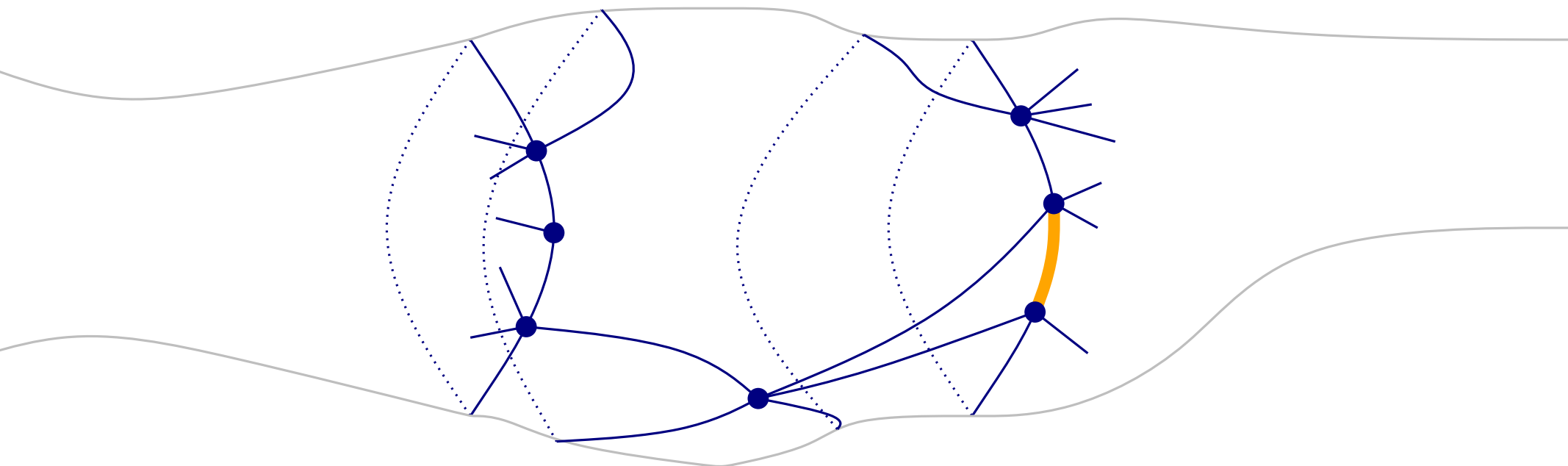
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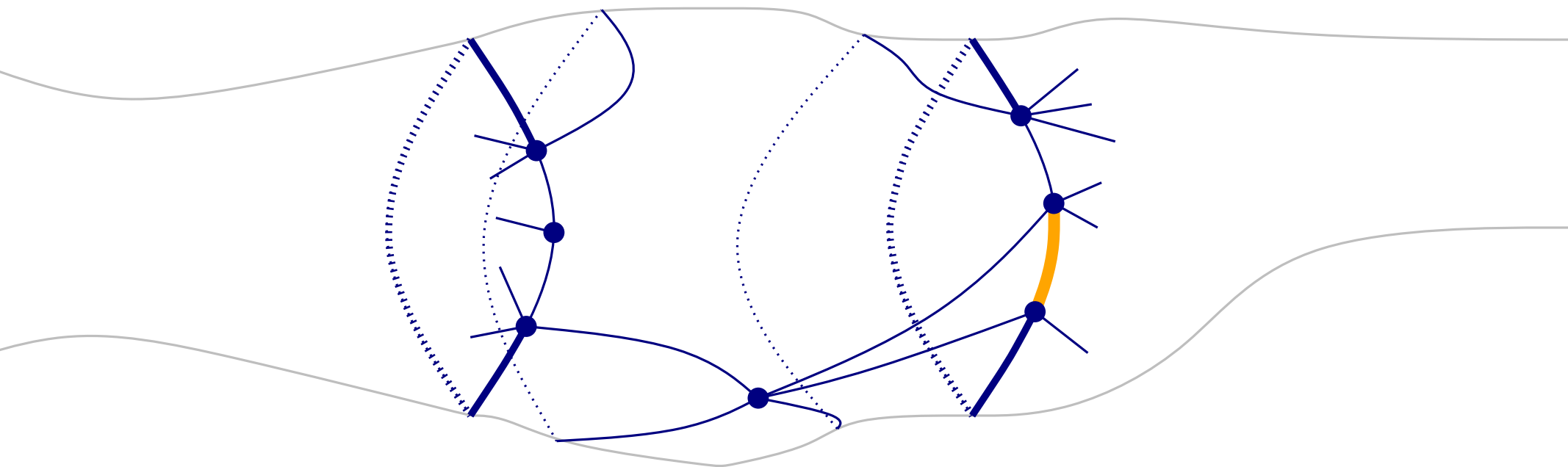
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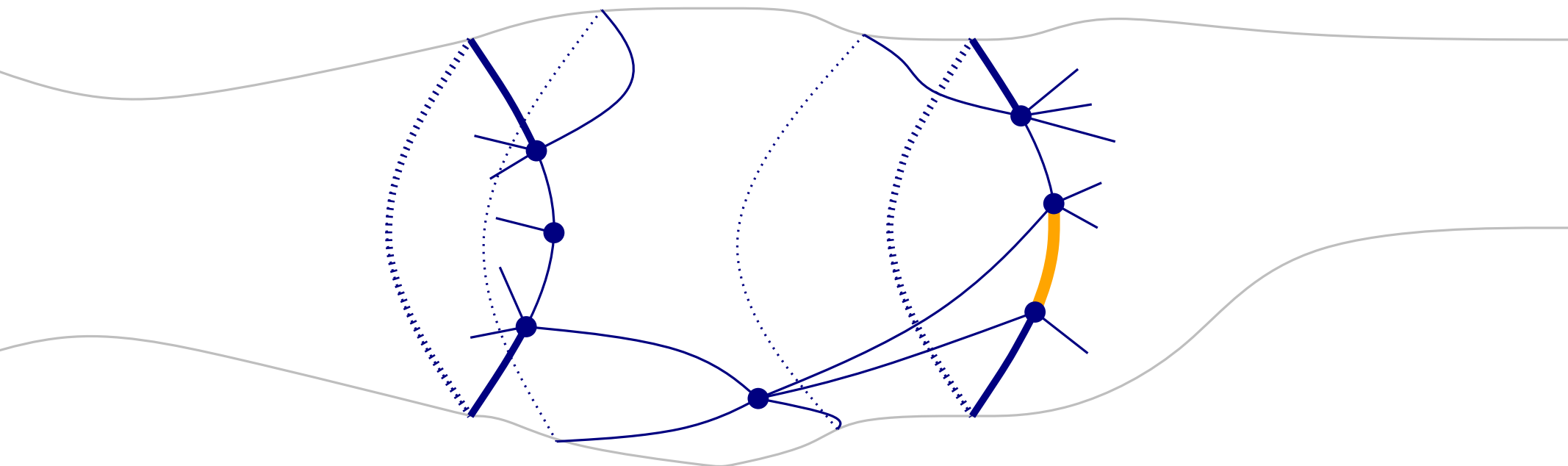
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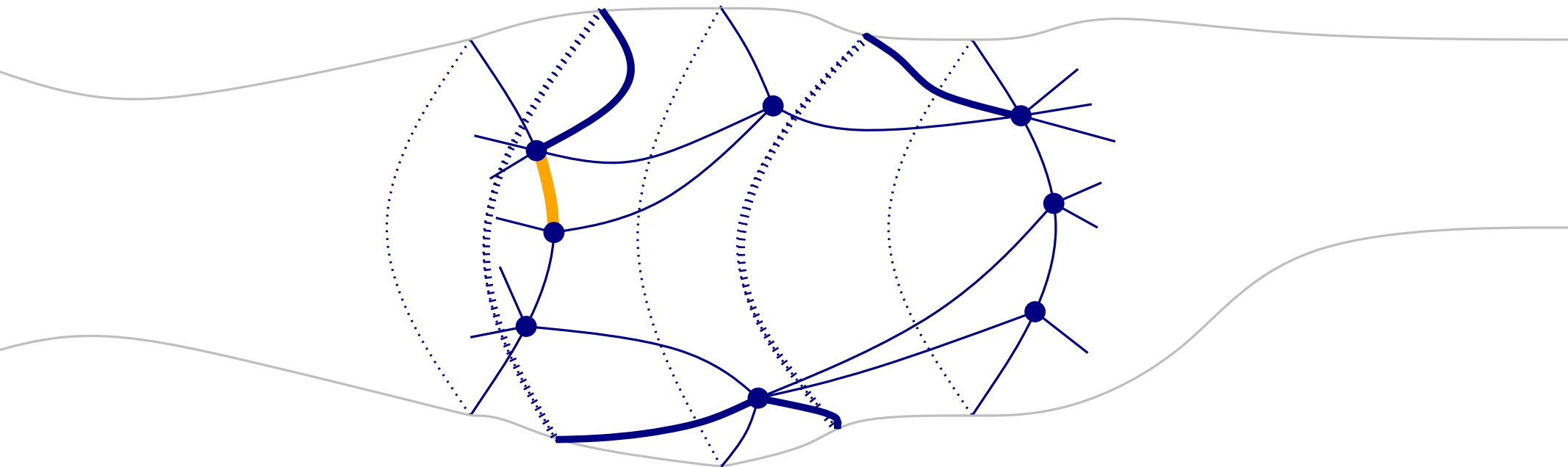
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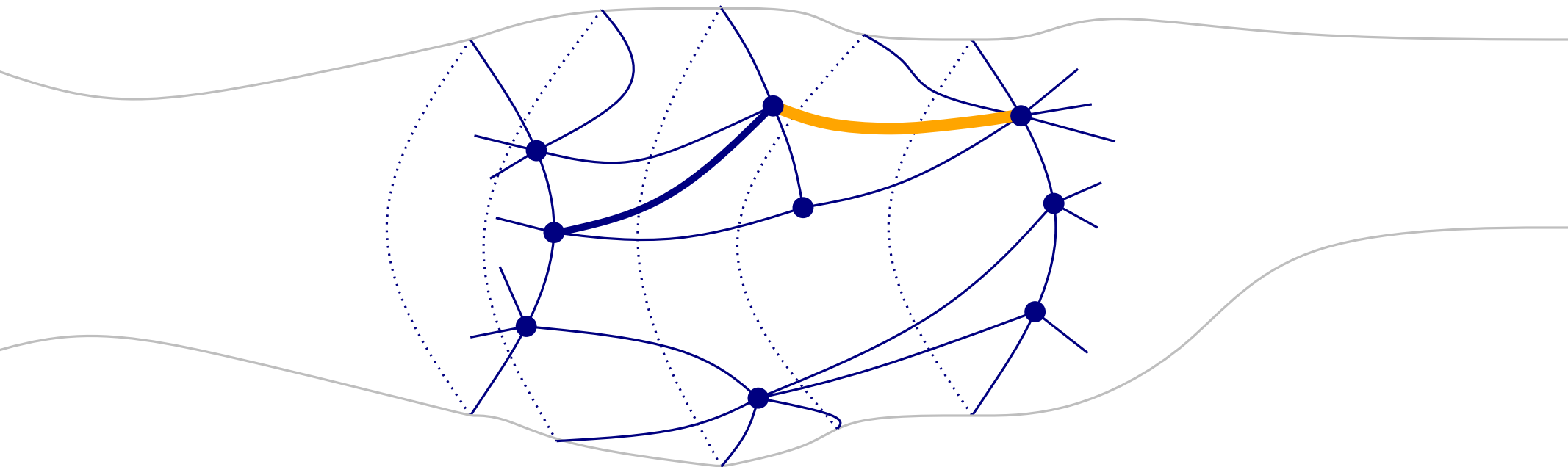
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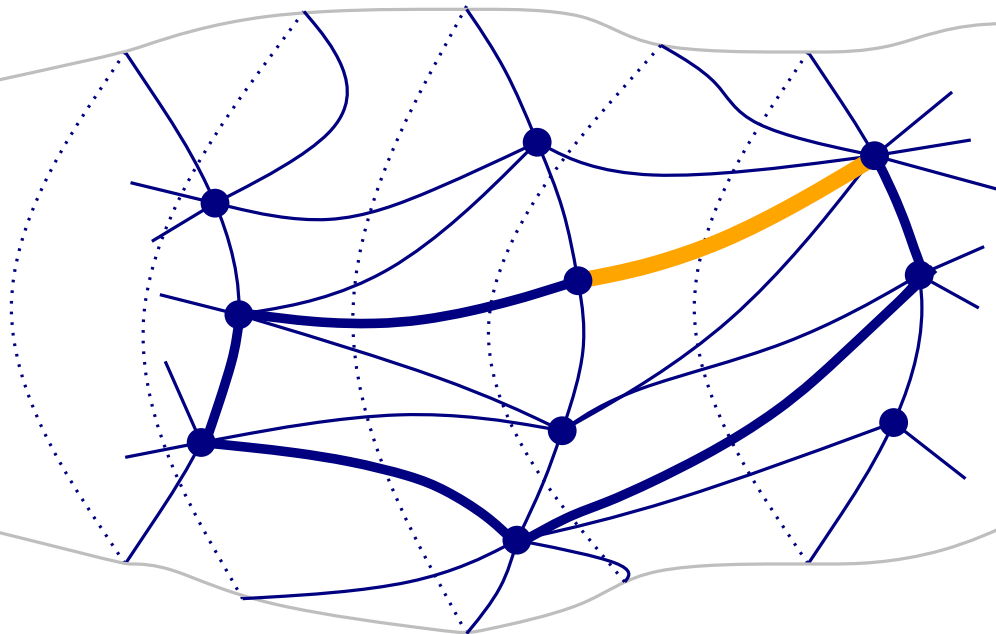
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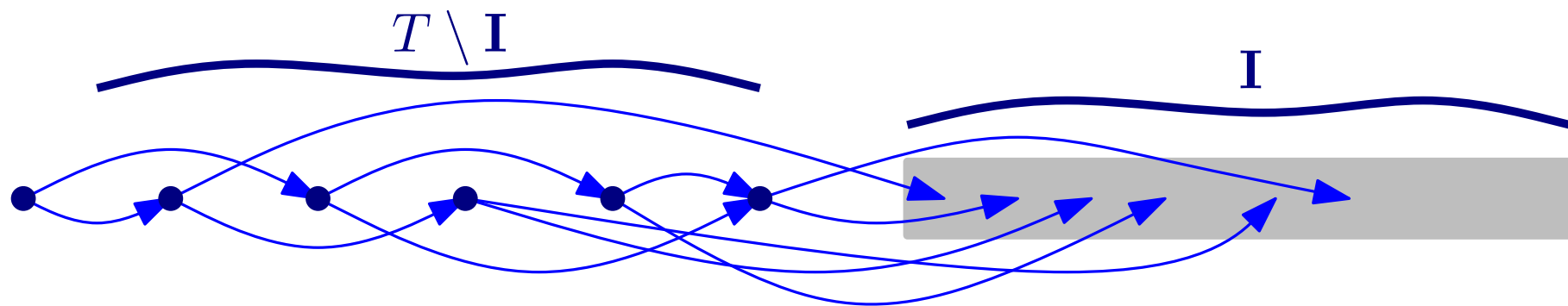
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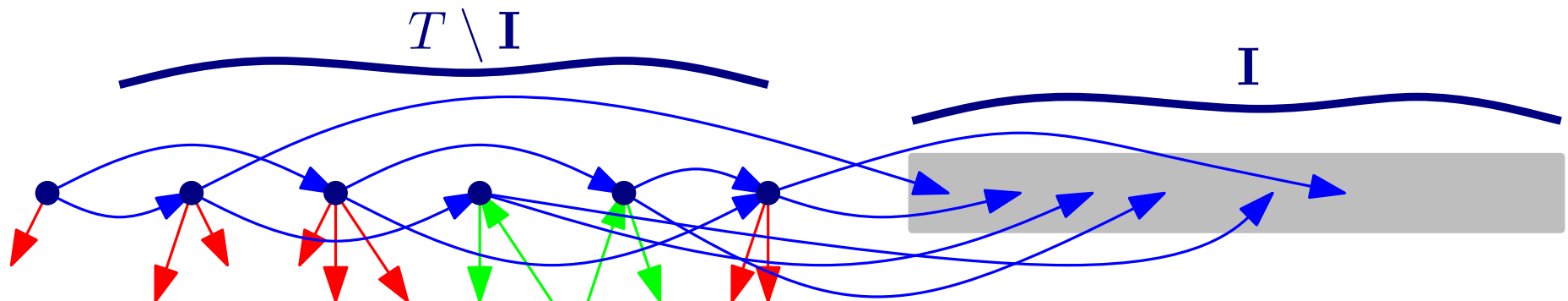
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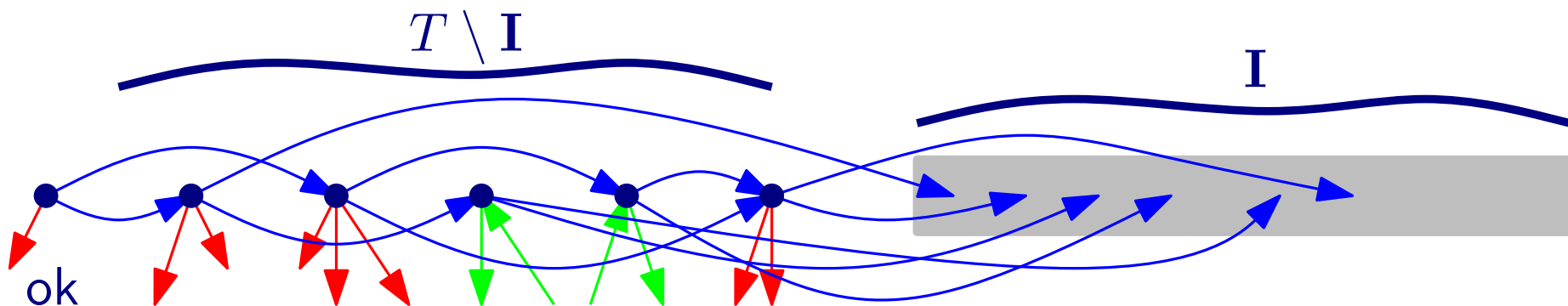
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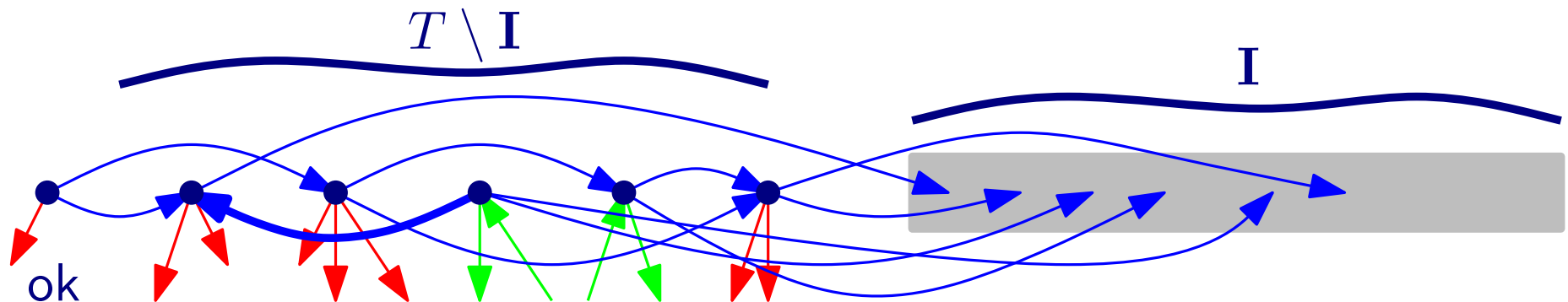
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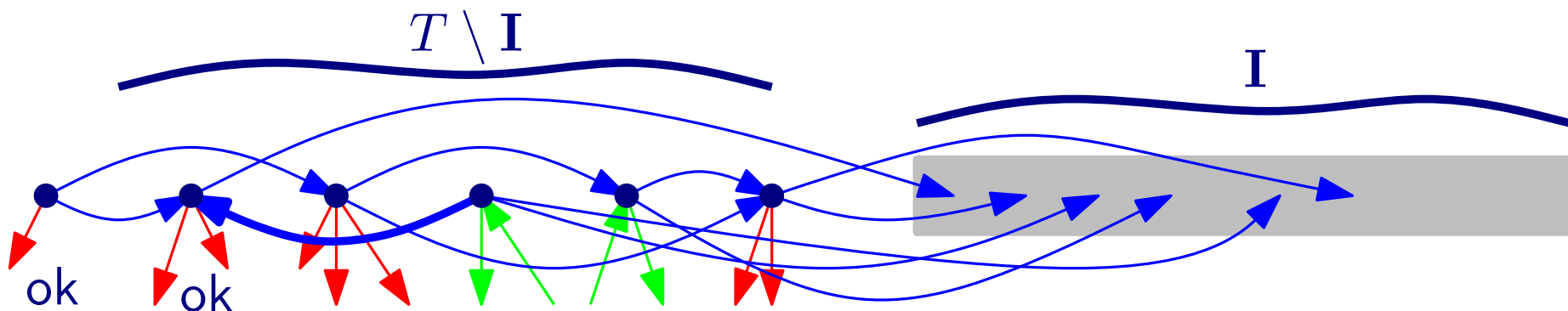
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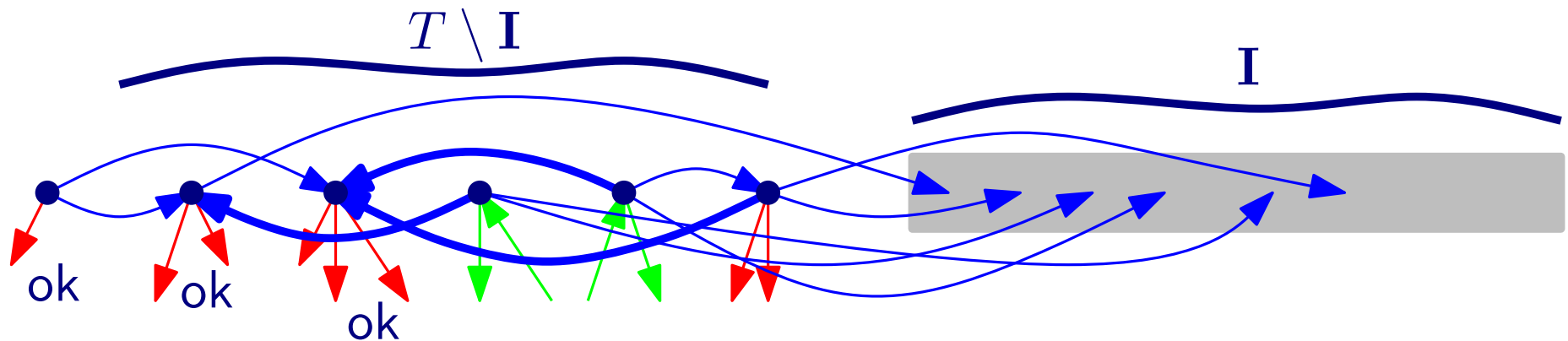
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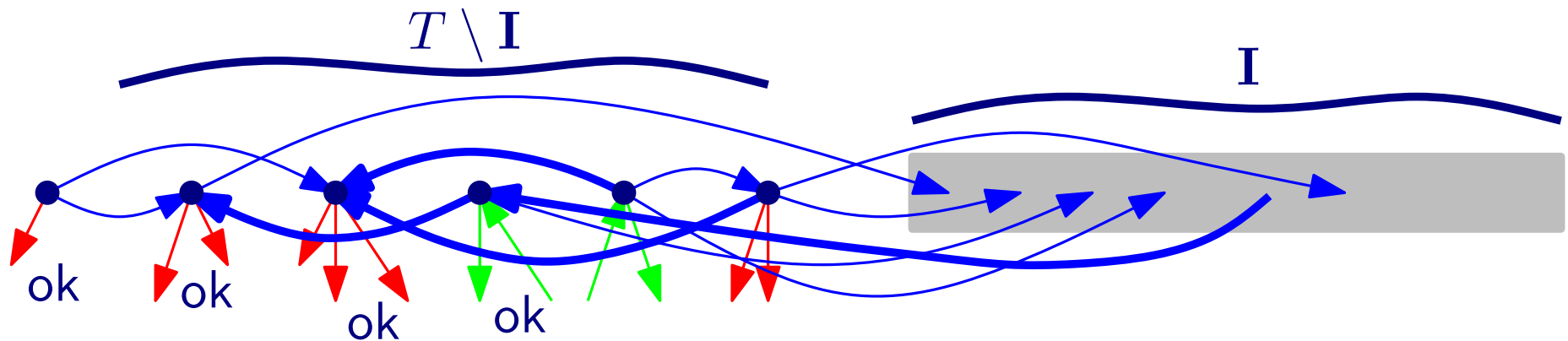
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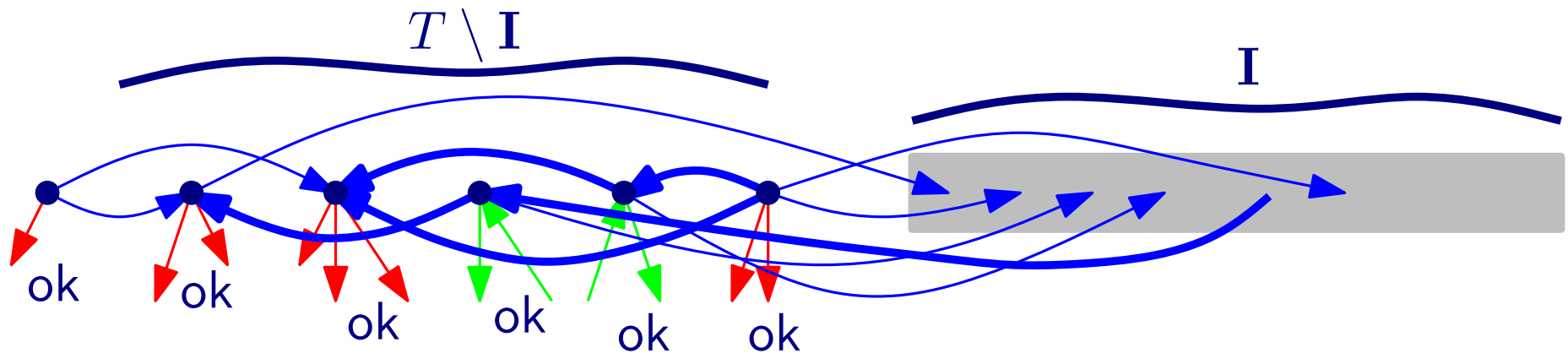
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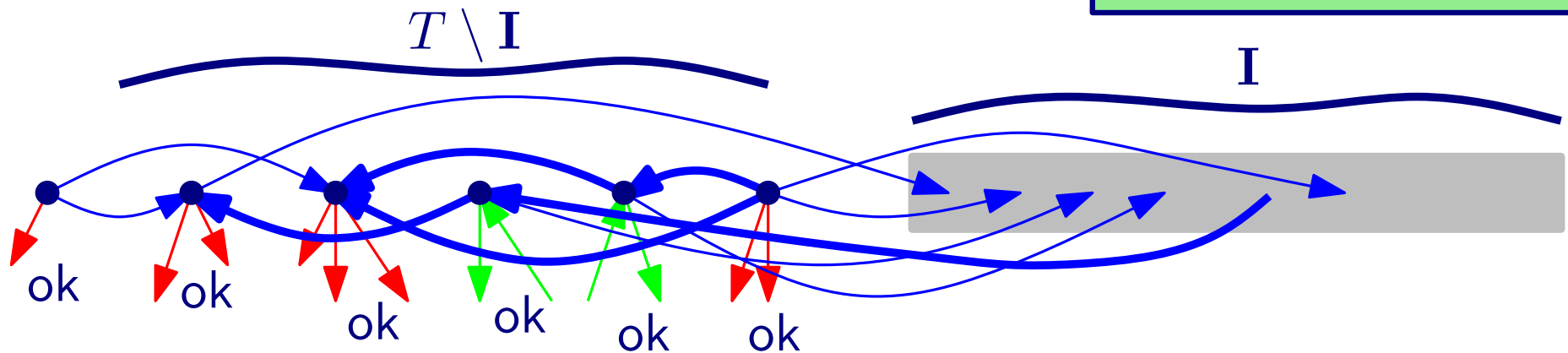
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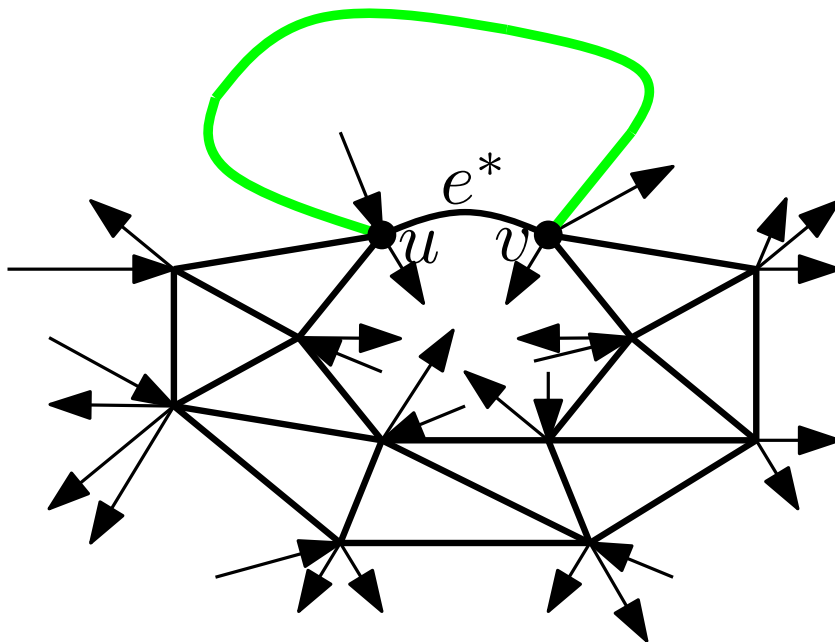
- order $T \setminus \mathbf{I}$ sth everybody has fw-degree 2 wrt T .

- process left-to-right

- use out-edges to correct outdegree to $0 \pmod 3$

- will be part of final orientation
- ensures non-zero outdegree

I has to deal with problems of others



partition T into:

- *initial graph* **I**: starting configuration ... later
- *correction graph* **B**: acyclic orientation $T \setminus \mathbf{I}$ outdegree 2
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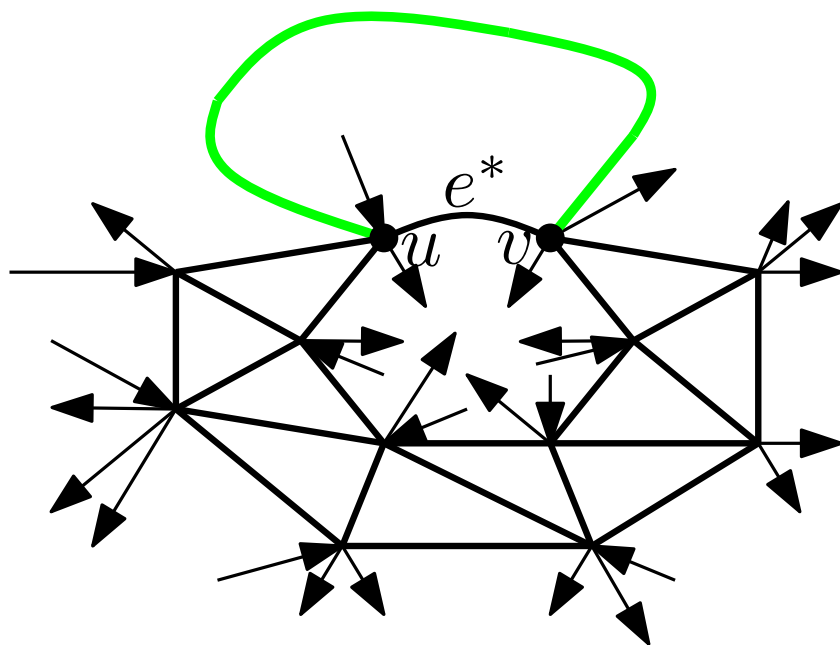
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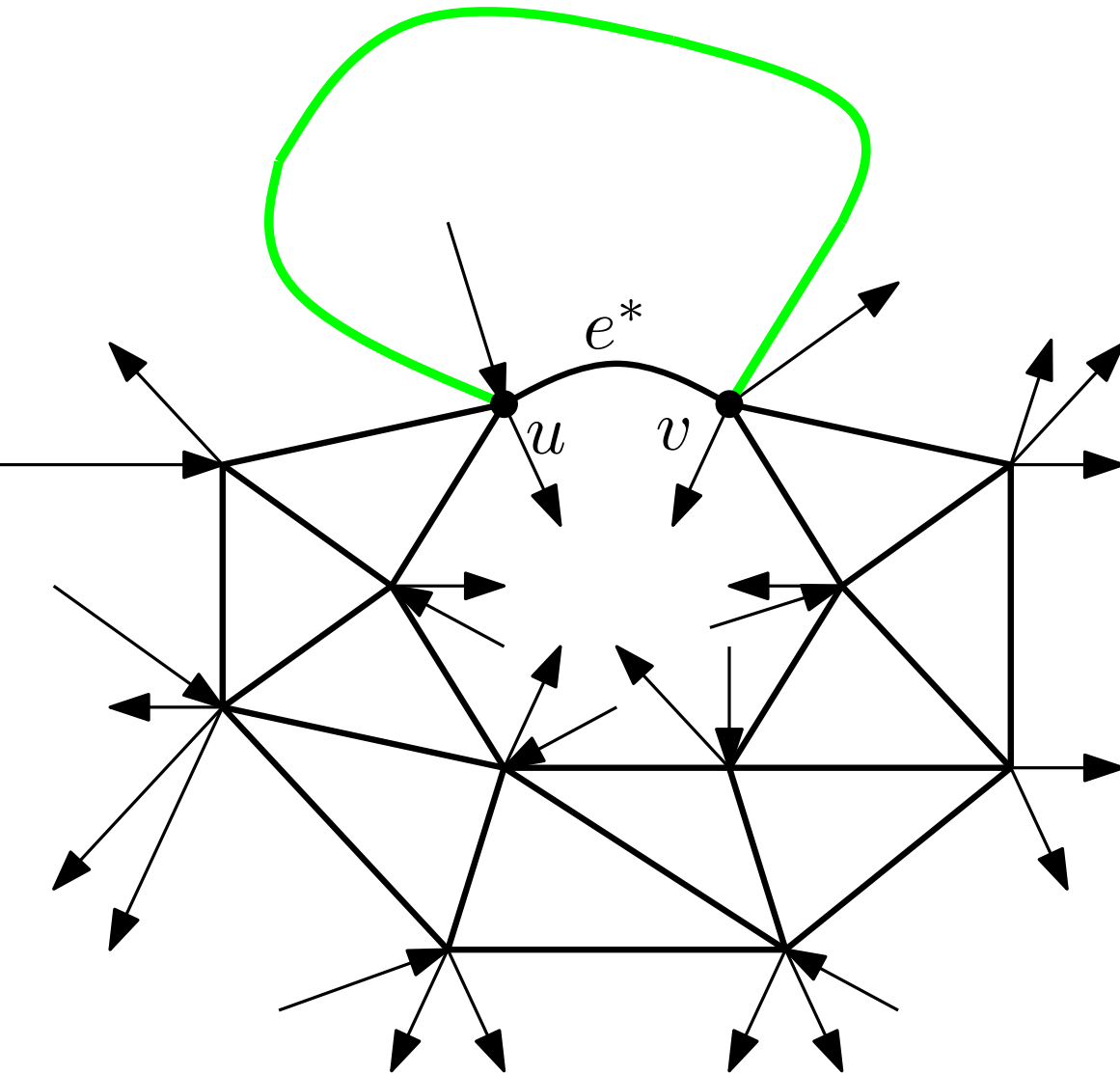
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- maximal outerplanar graph $D + e^* = uv$
- u, v are deg 2 in D
- u, v are ends of **G**
- some more stuff later

complete the orientation on \mathbf{I} with:

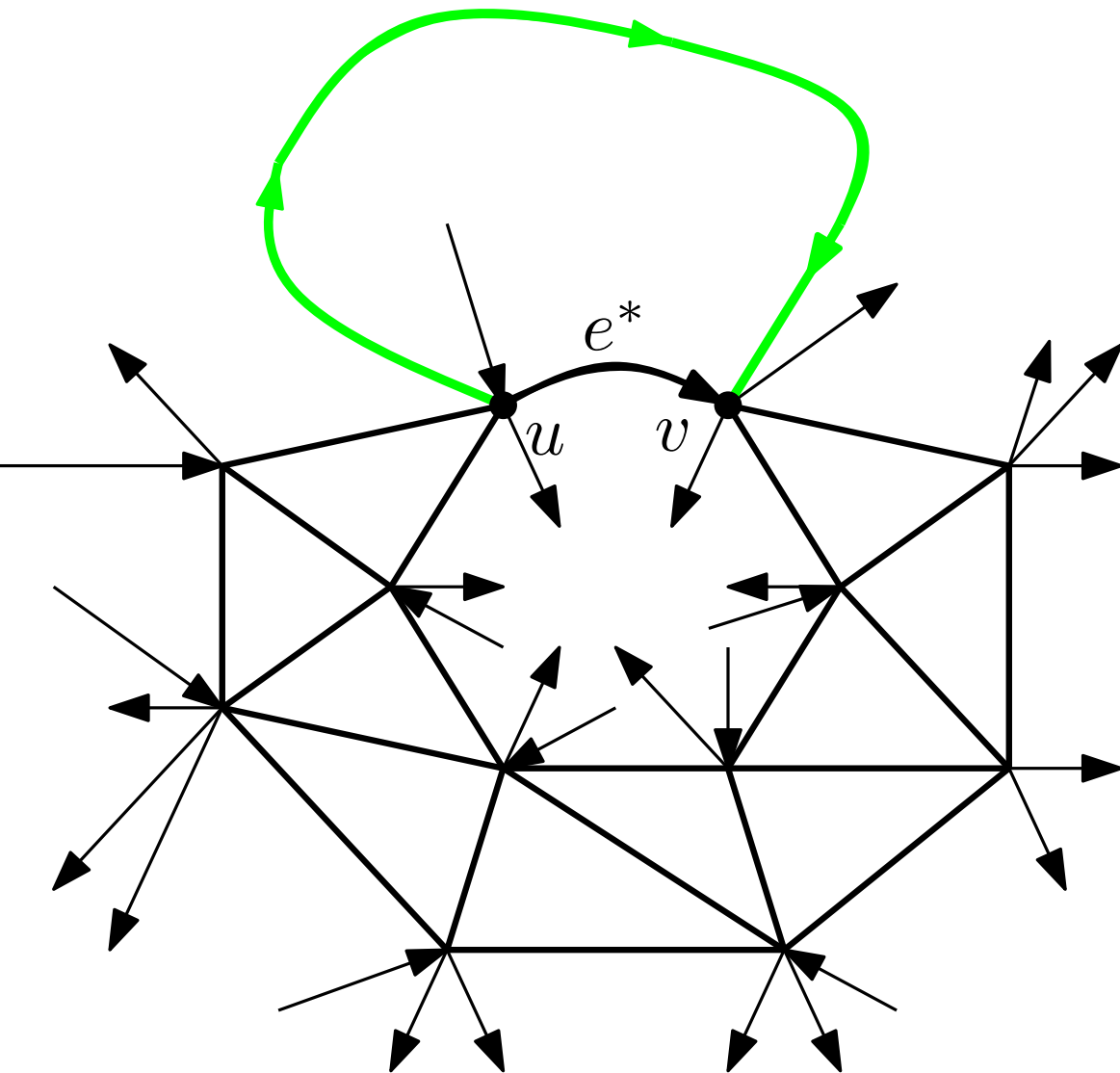
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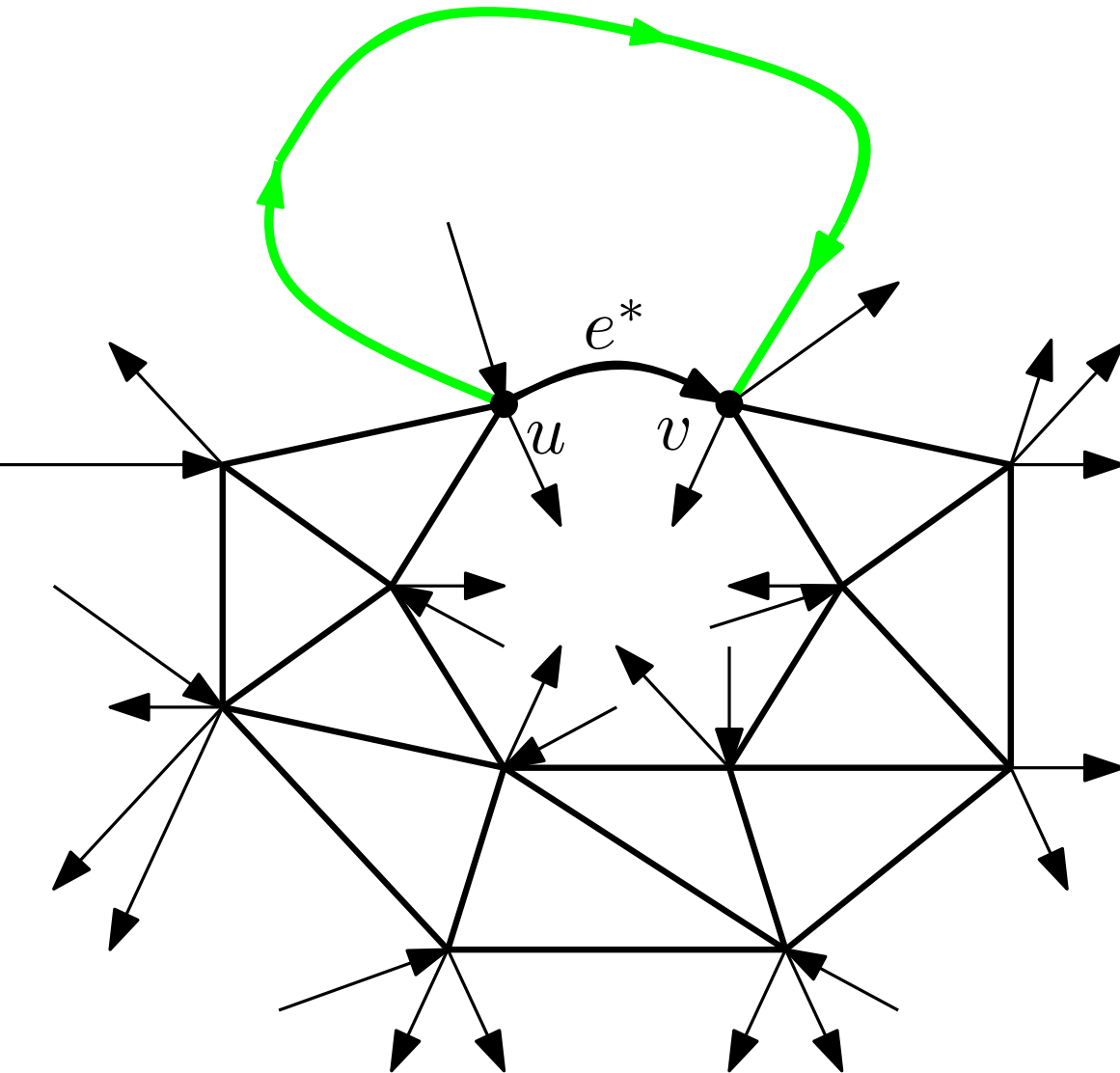
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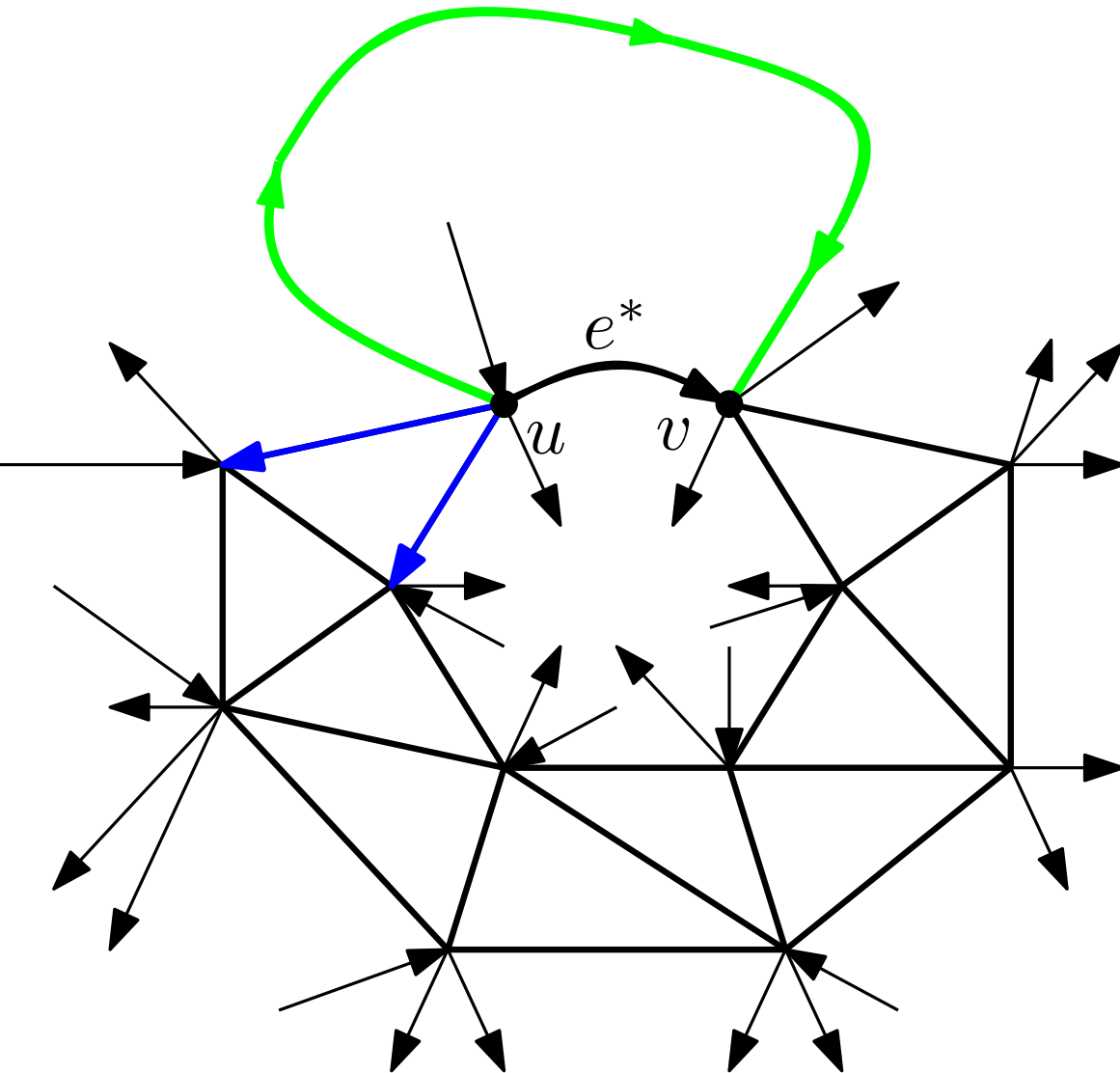


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starting from u get
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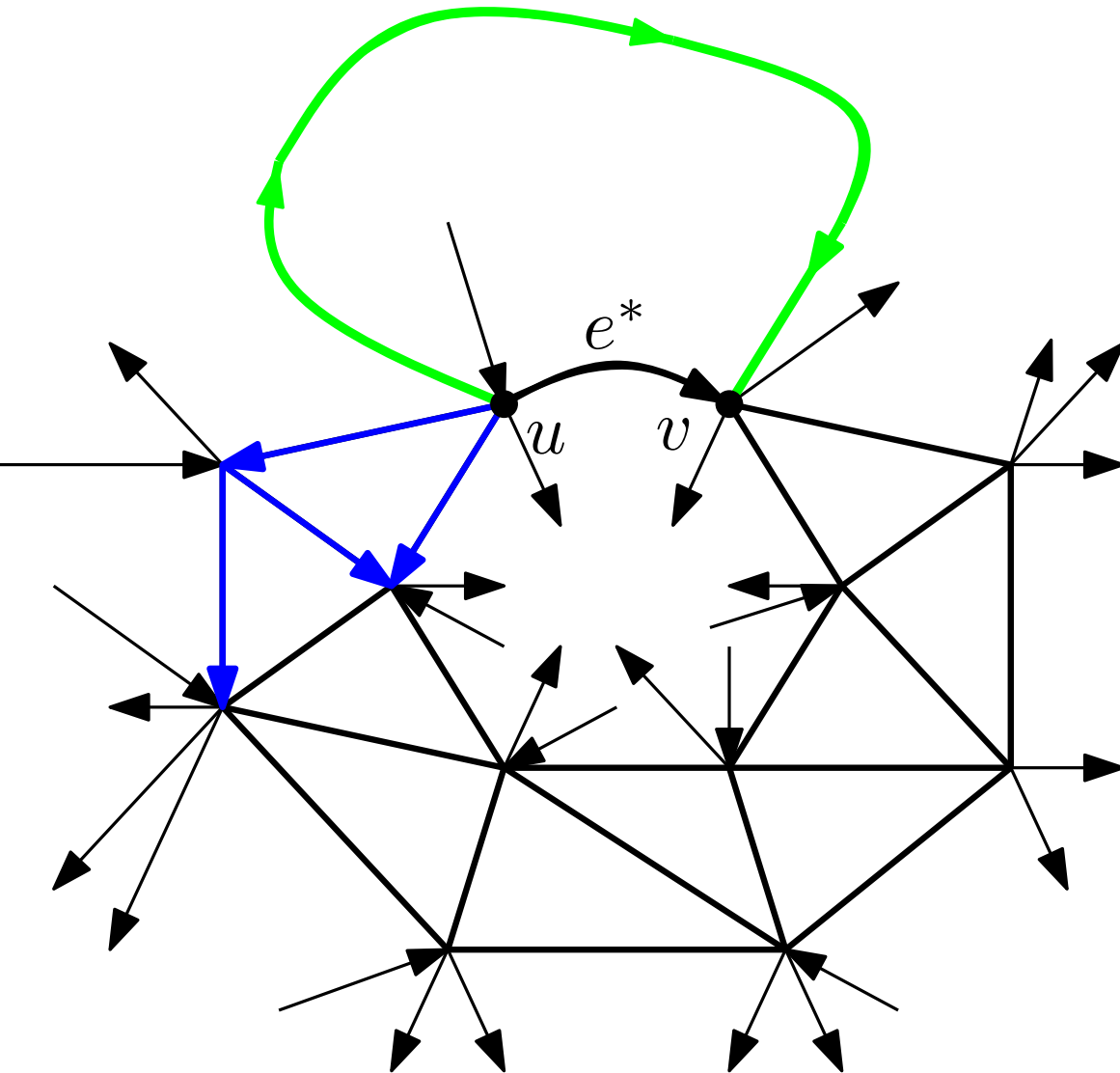


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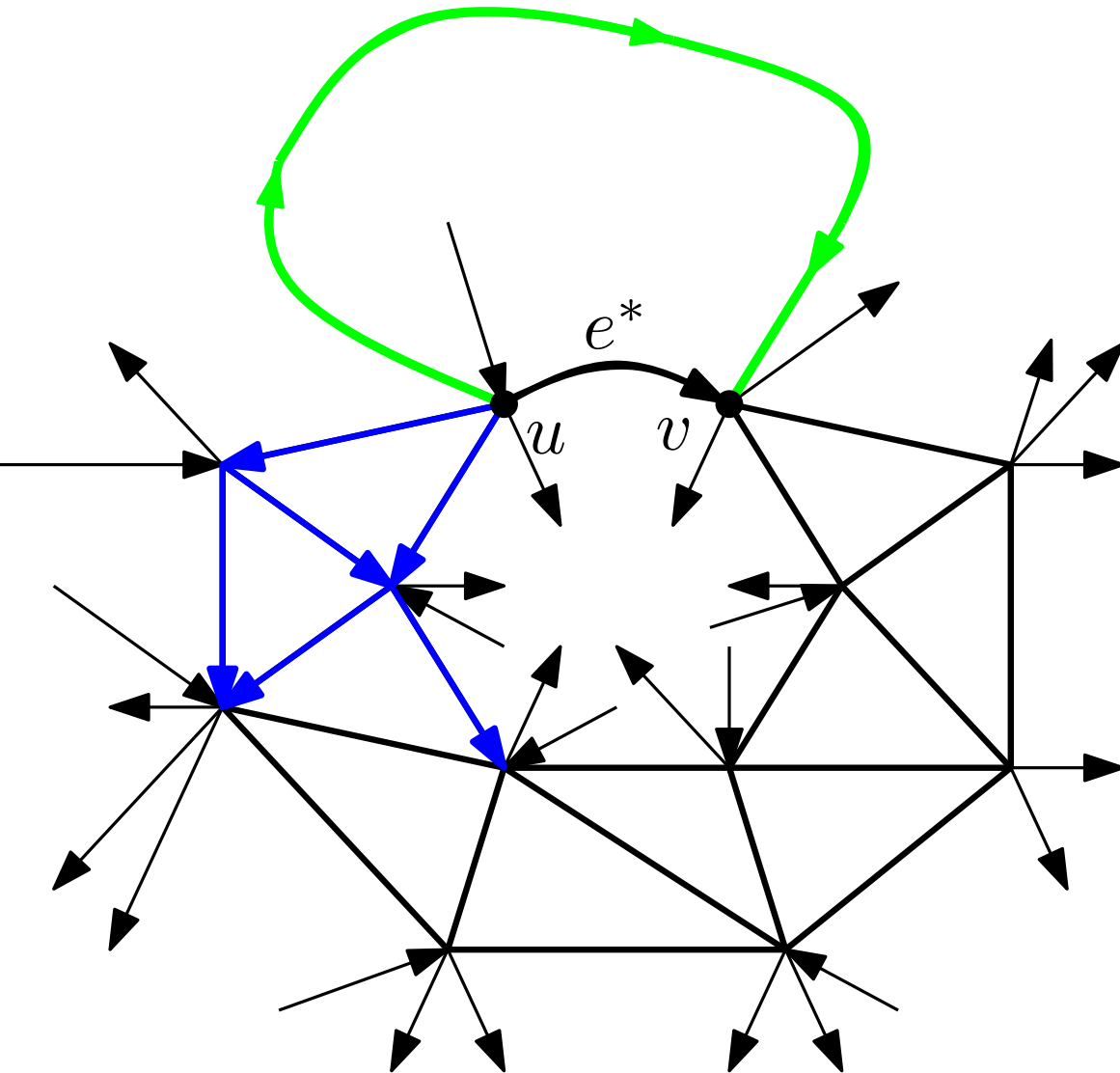


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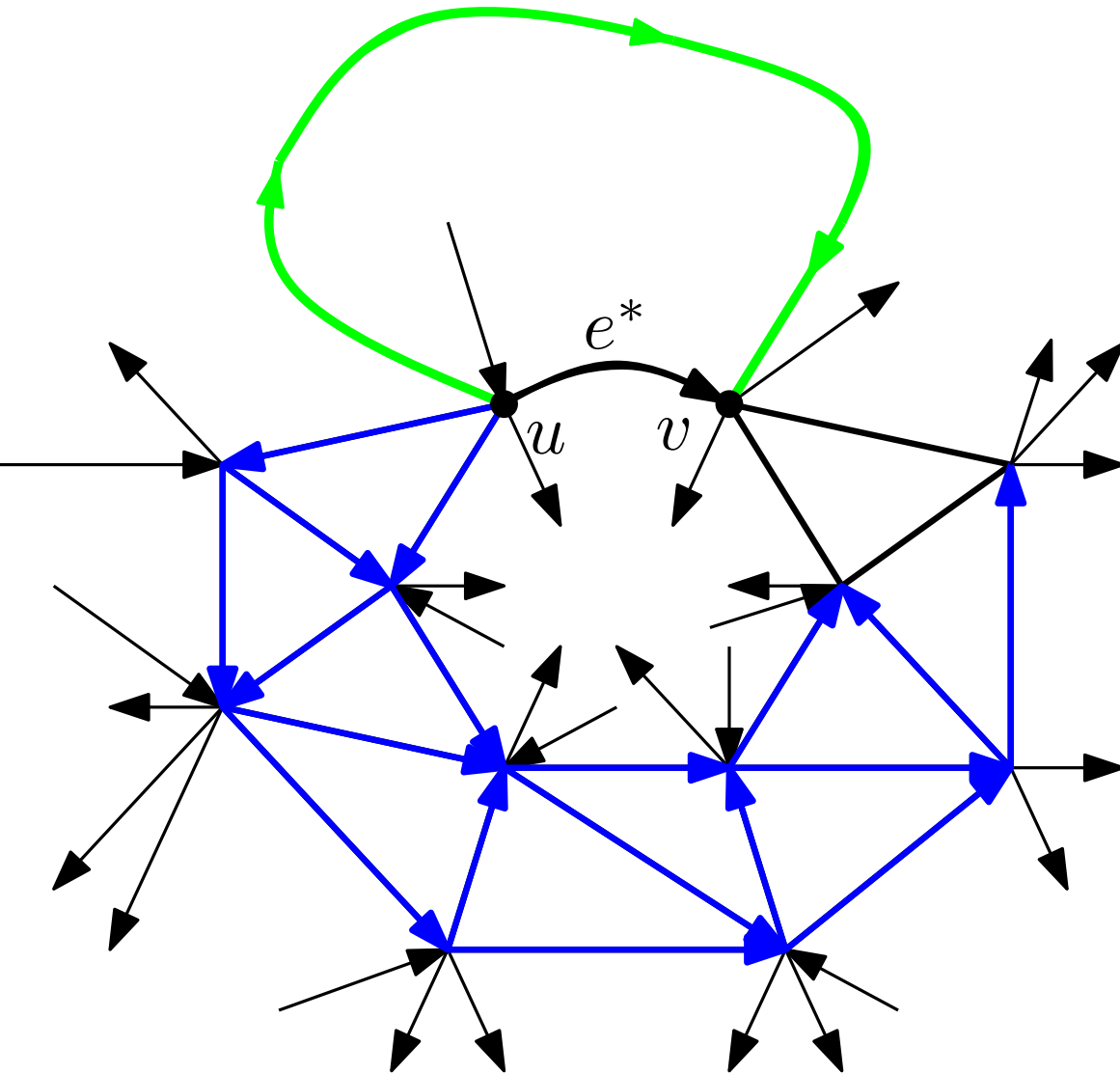


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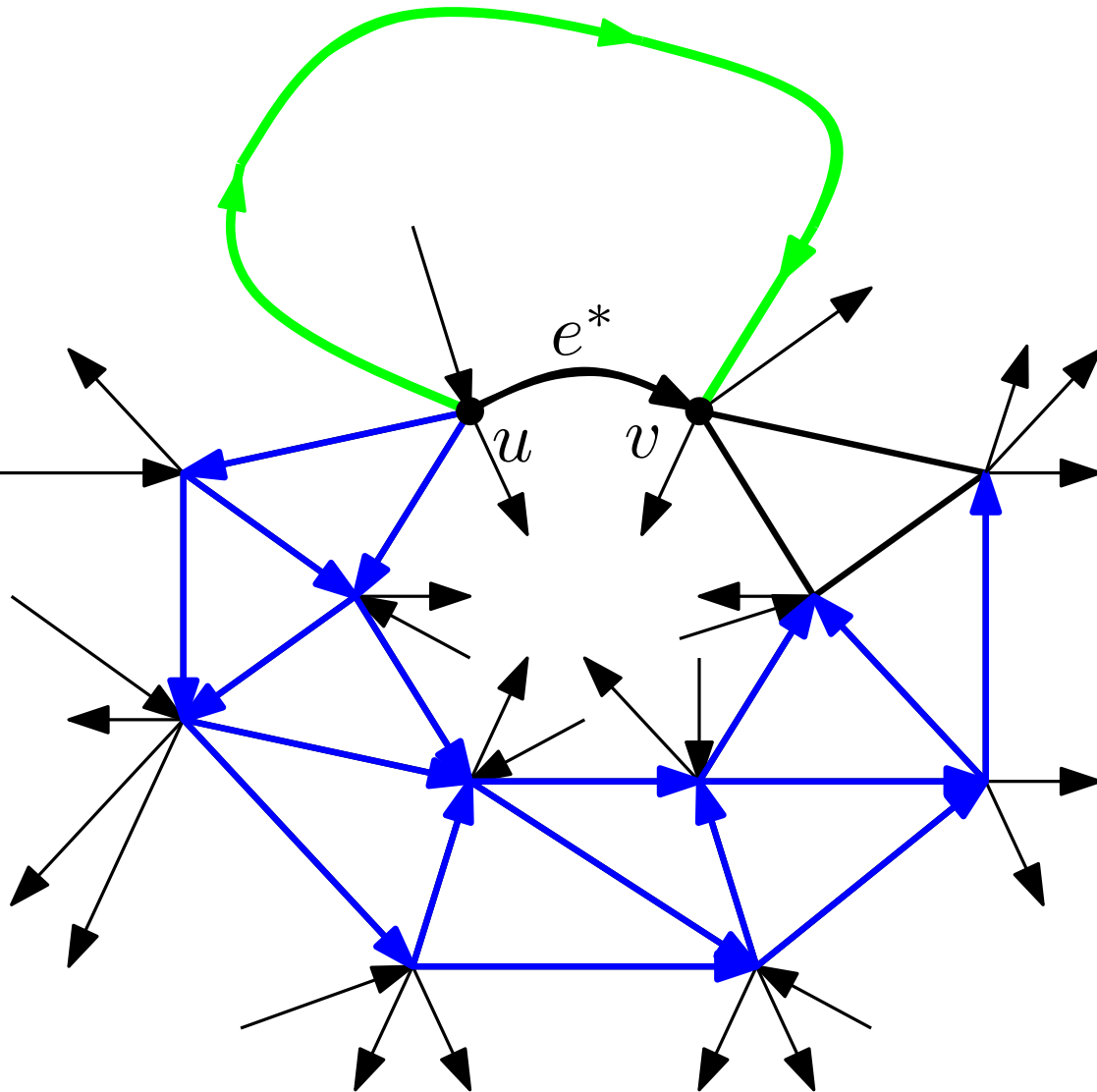


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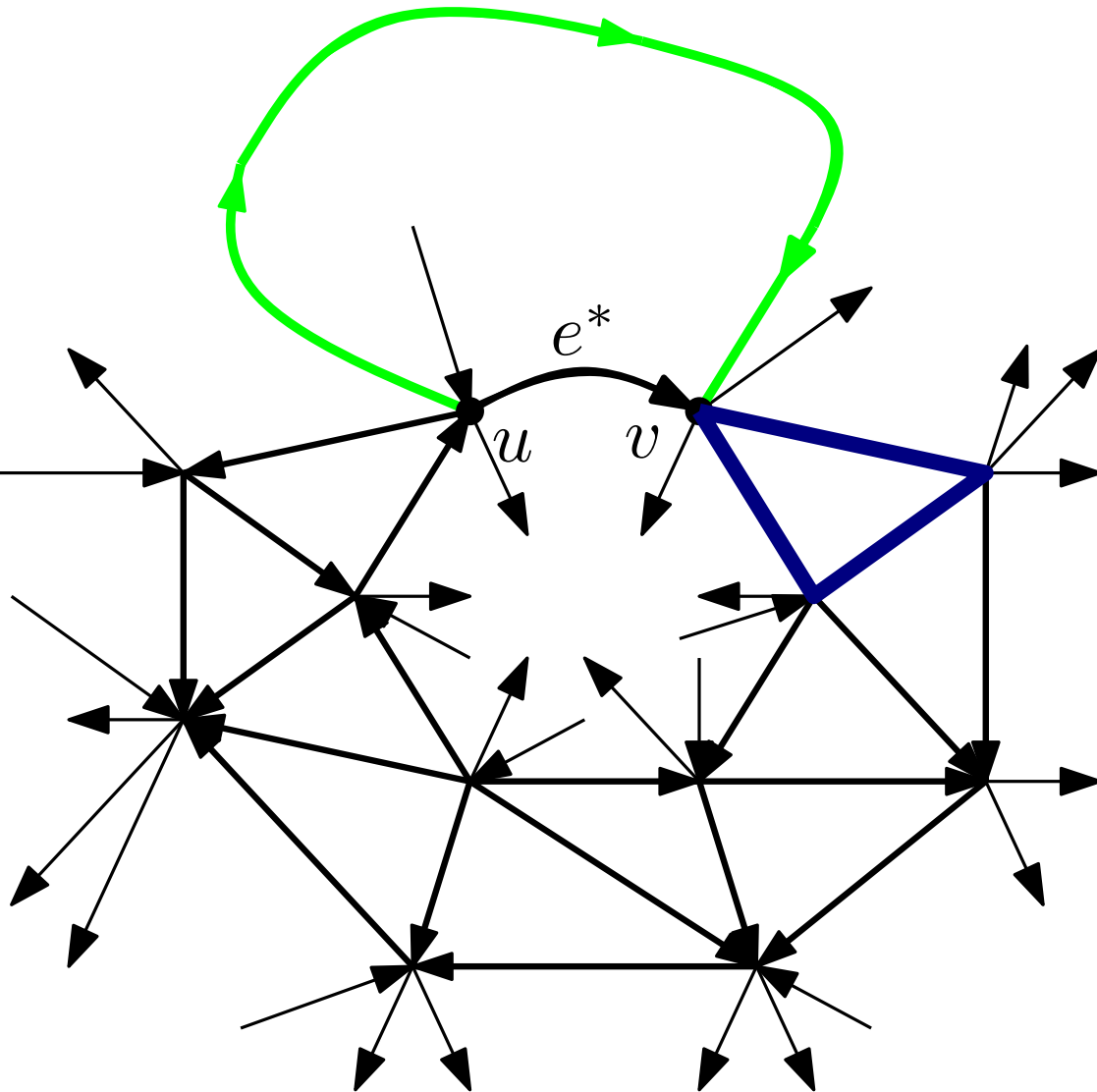
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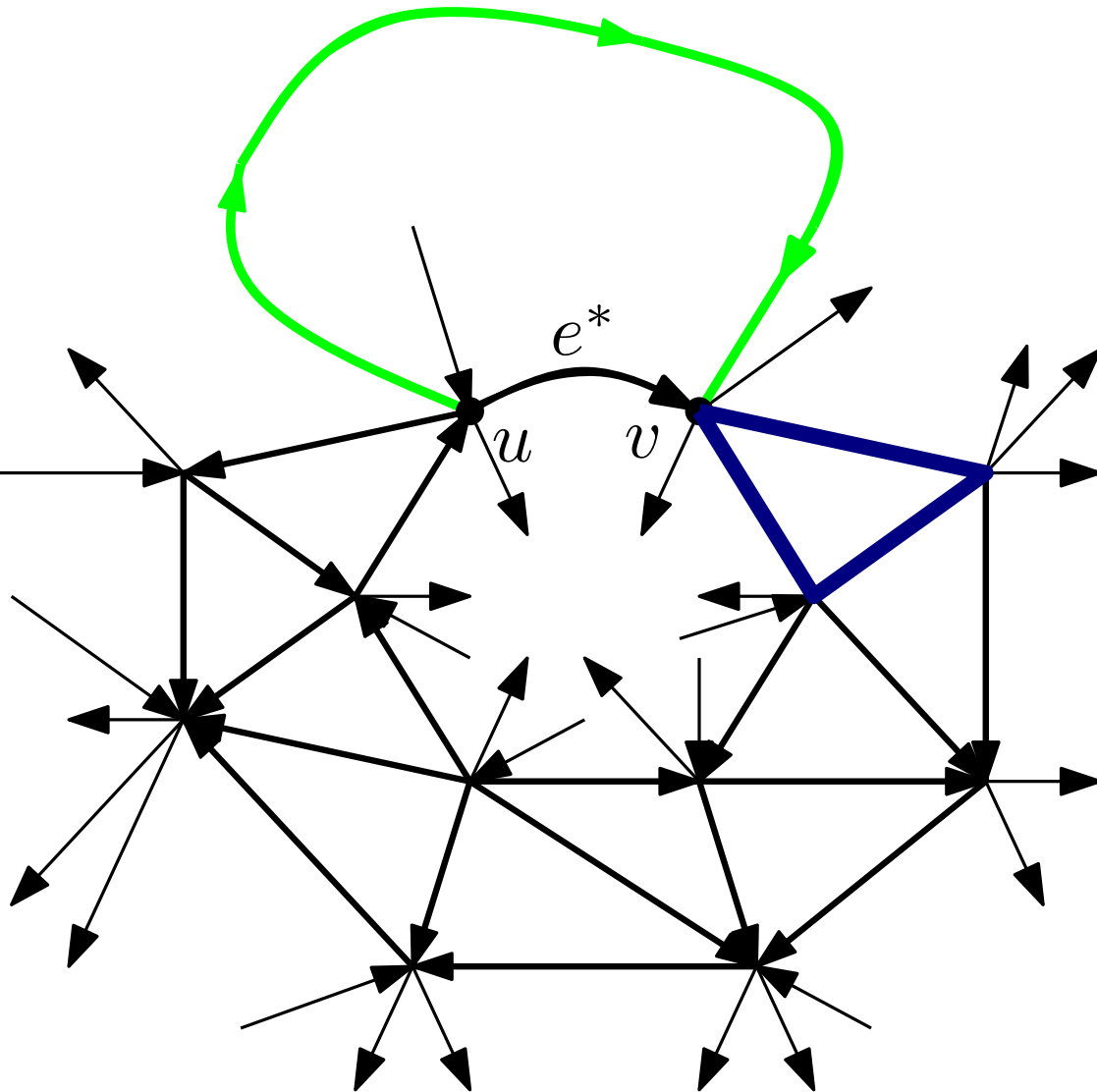
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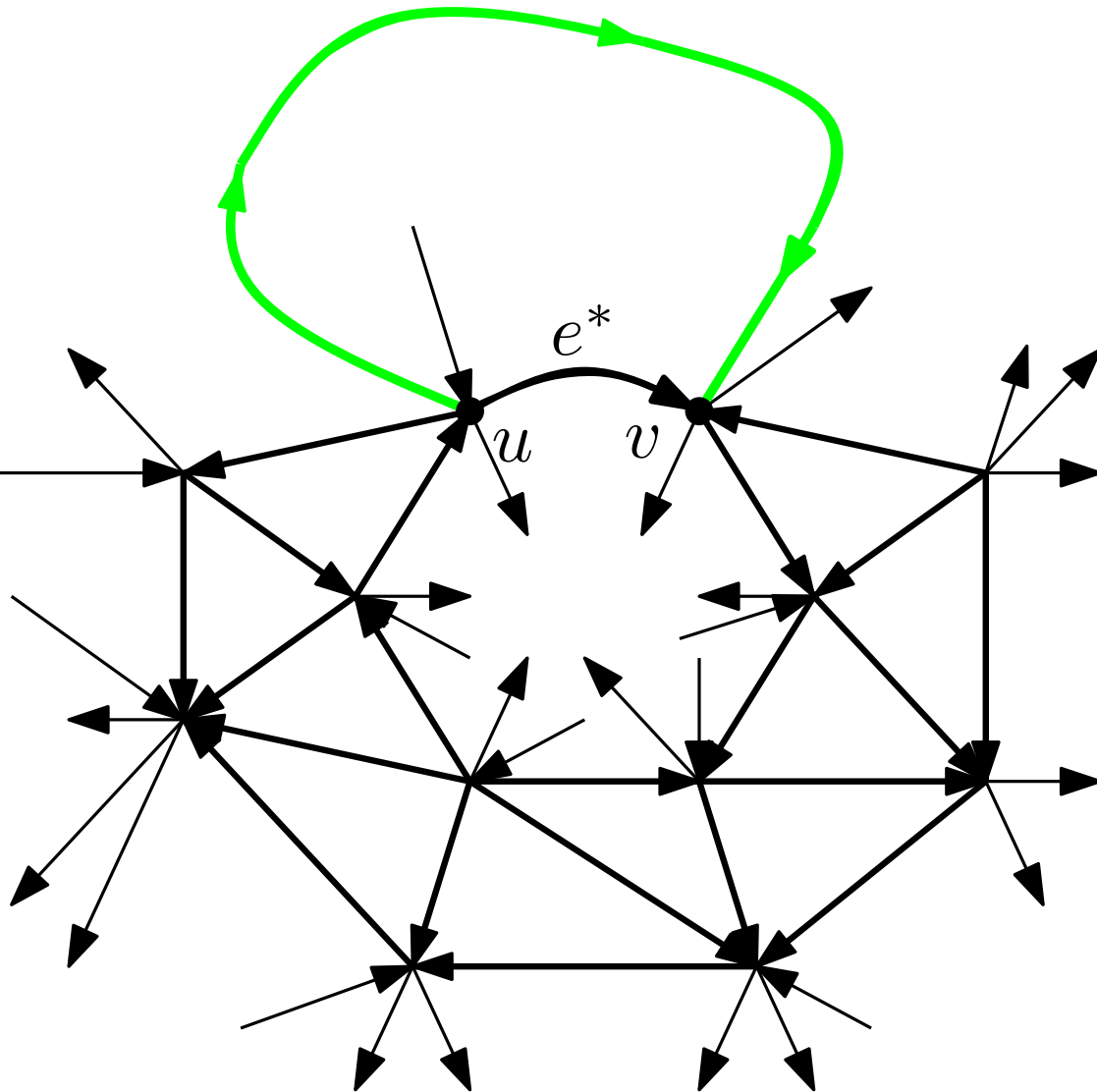
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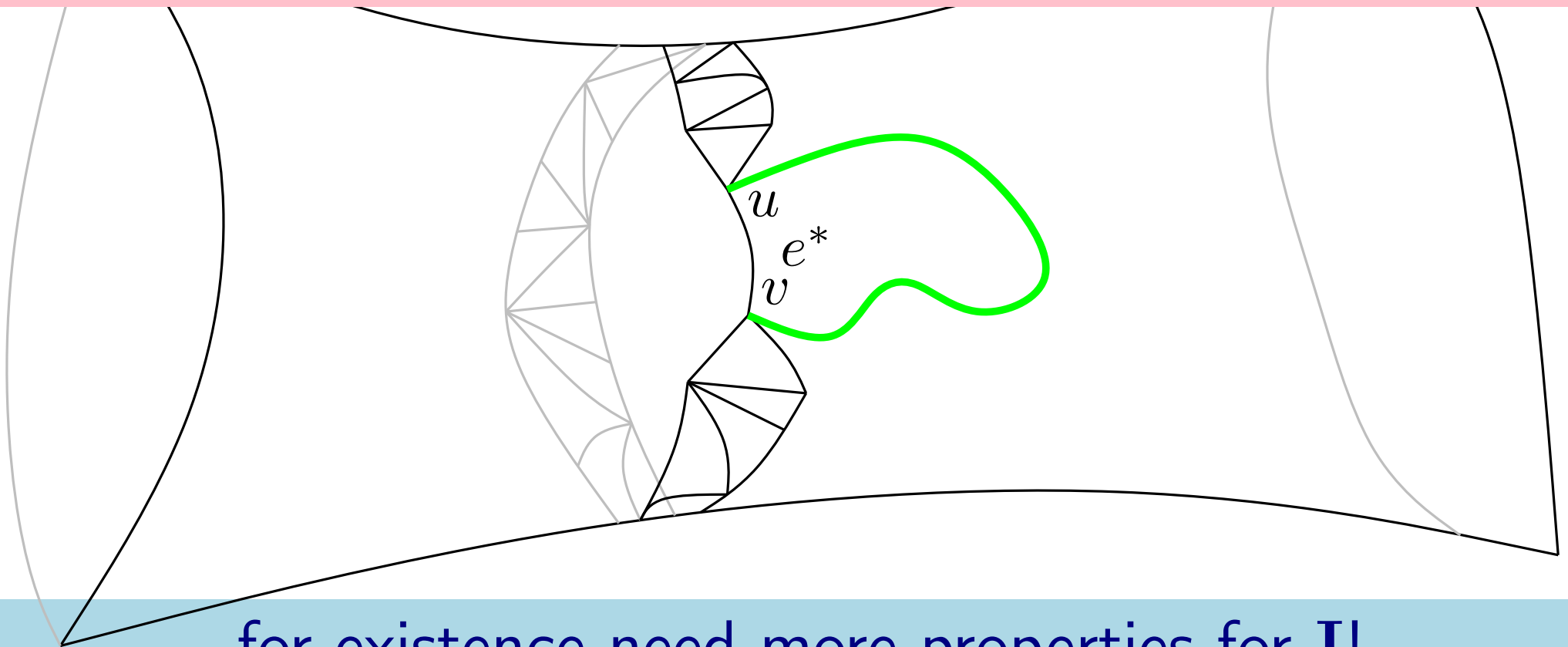
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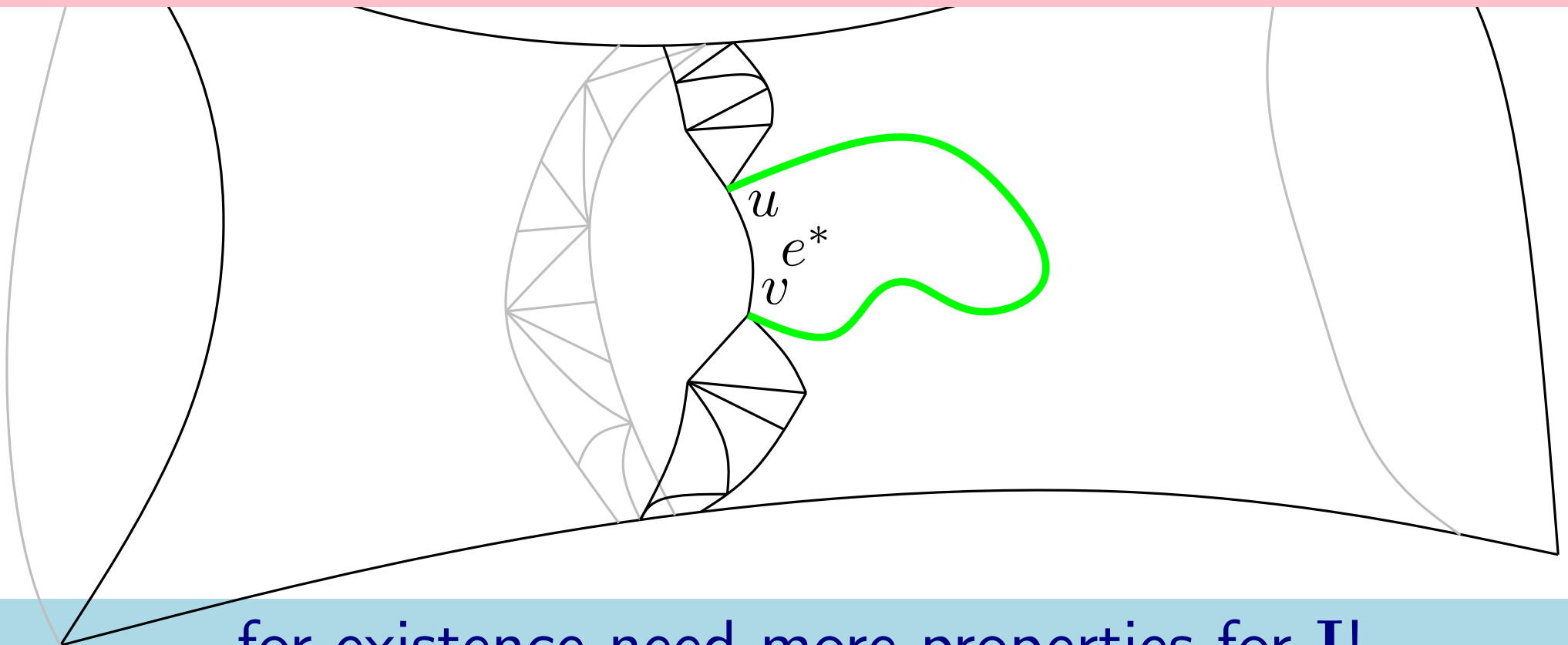


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Preprocessing:

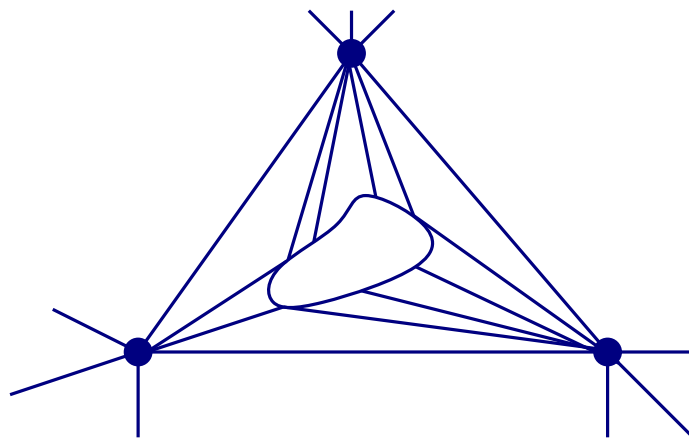
- remove interiors of 3-sided disks
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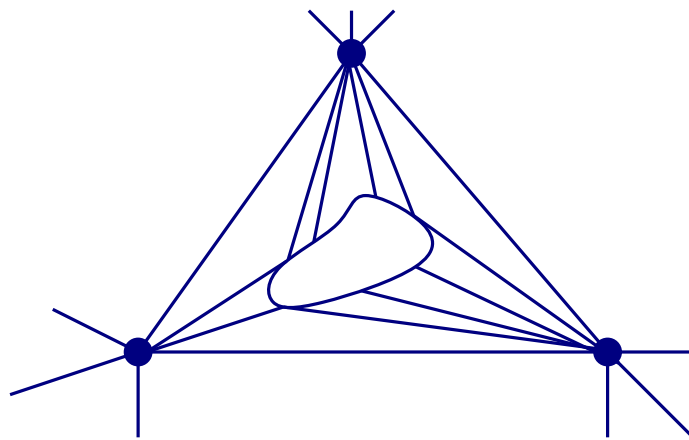


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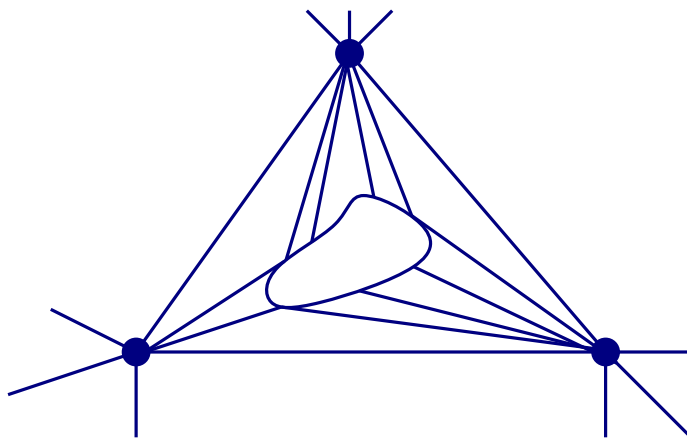
- planar triangulation
- find Schnyder Wood
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- boundary not oriented

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Explore T by stacking vertices to edges maintaining **Invariants**

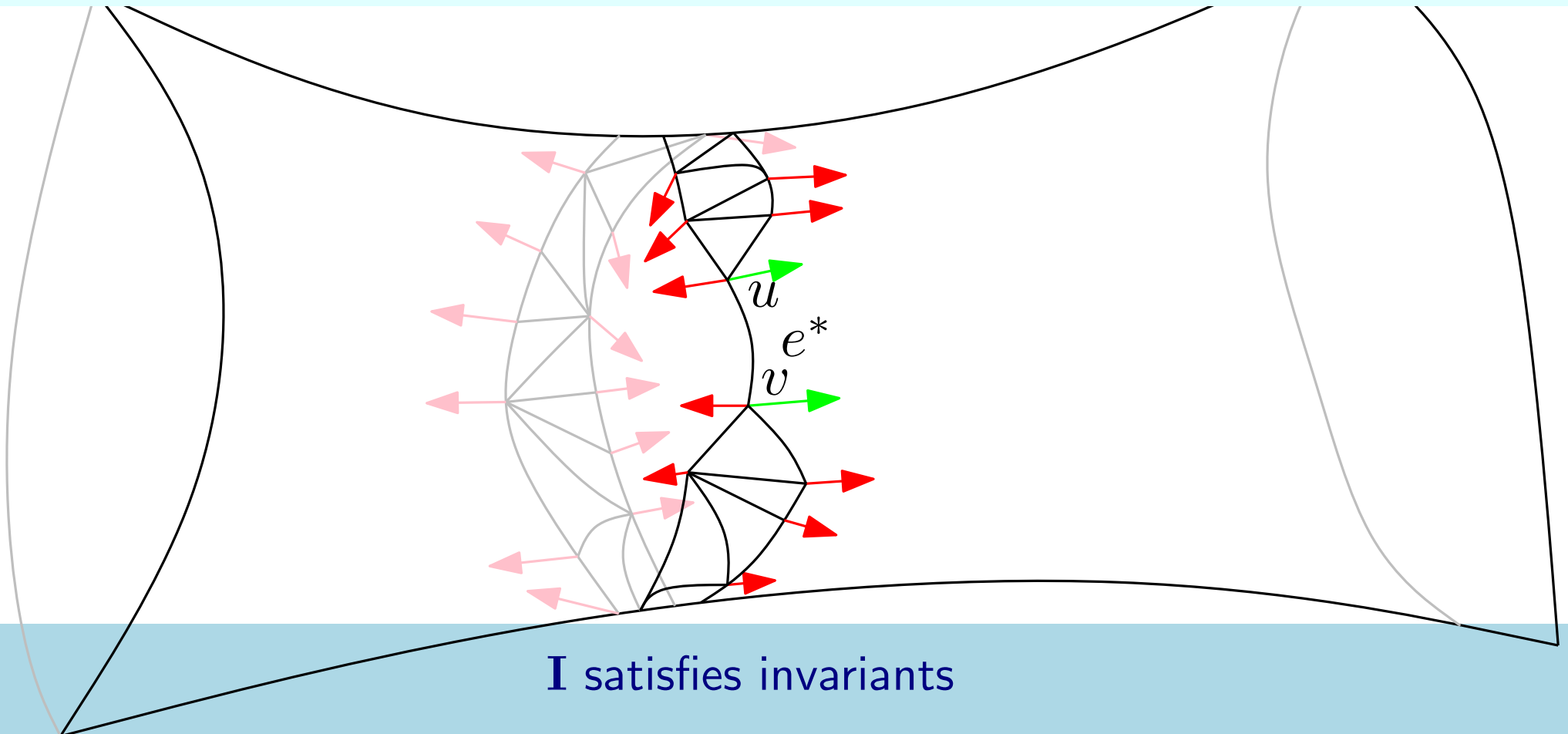
- interior vertices happy wrt partition
- boundary angles
 - happy wrt **B**
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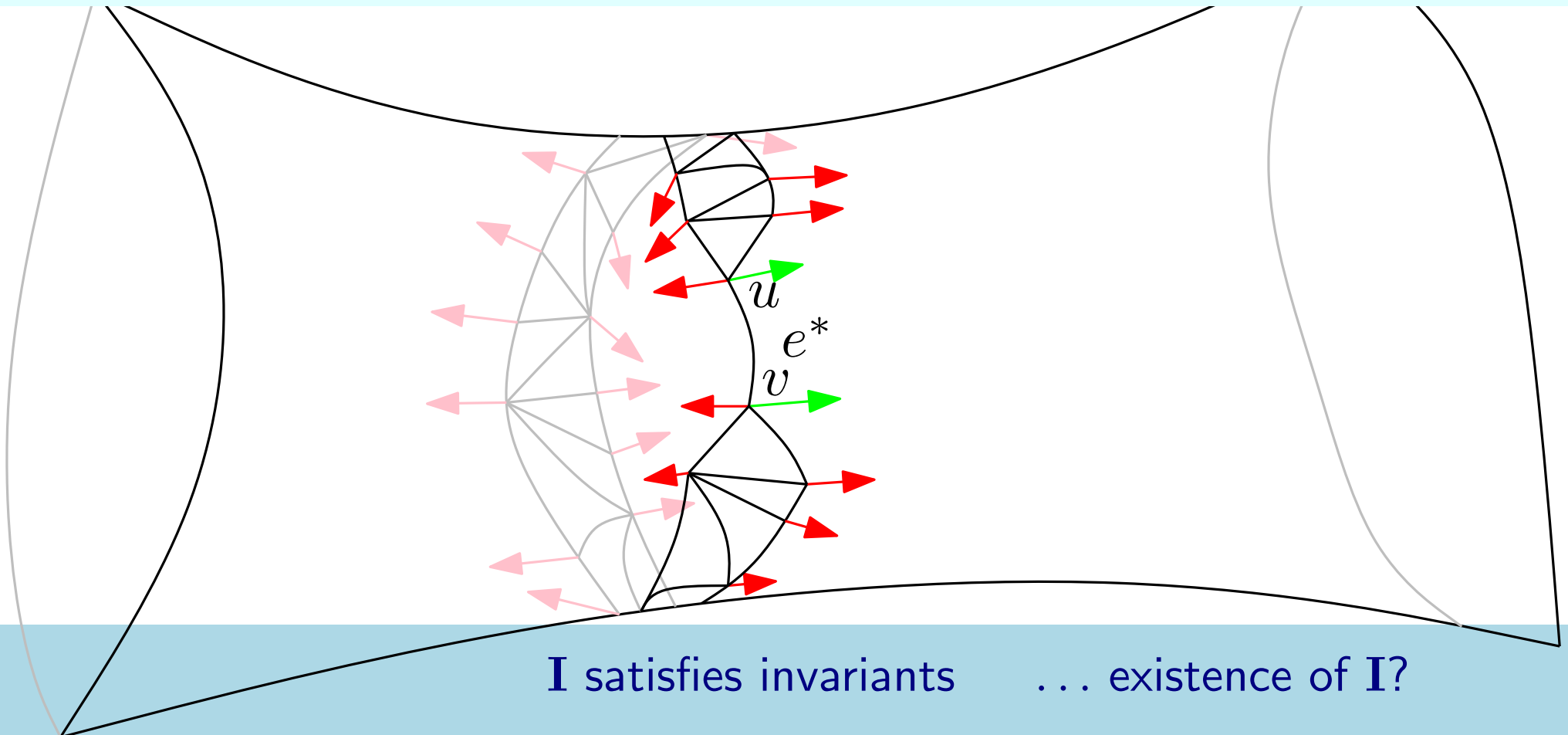
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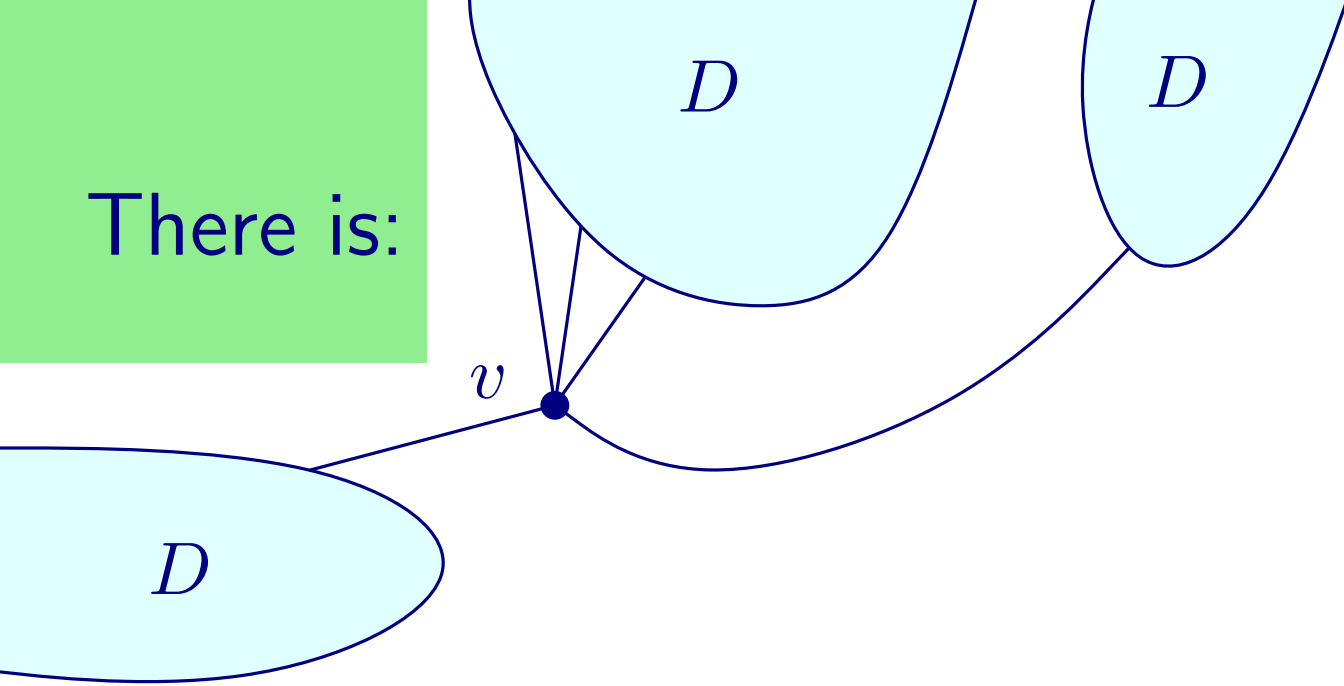
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I satisfies invariants ... existence of **I**?

There is:



D induced, $D + v$ induced, no cycle through v and D is contractible

There is:



D



D

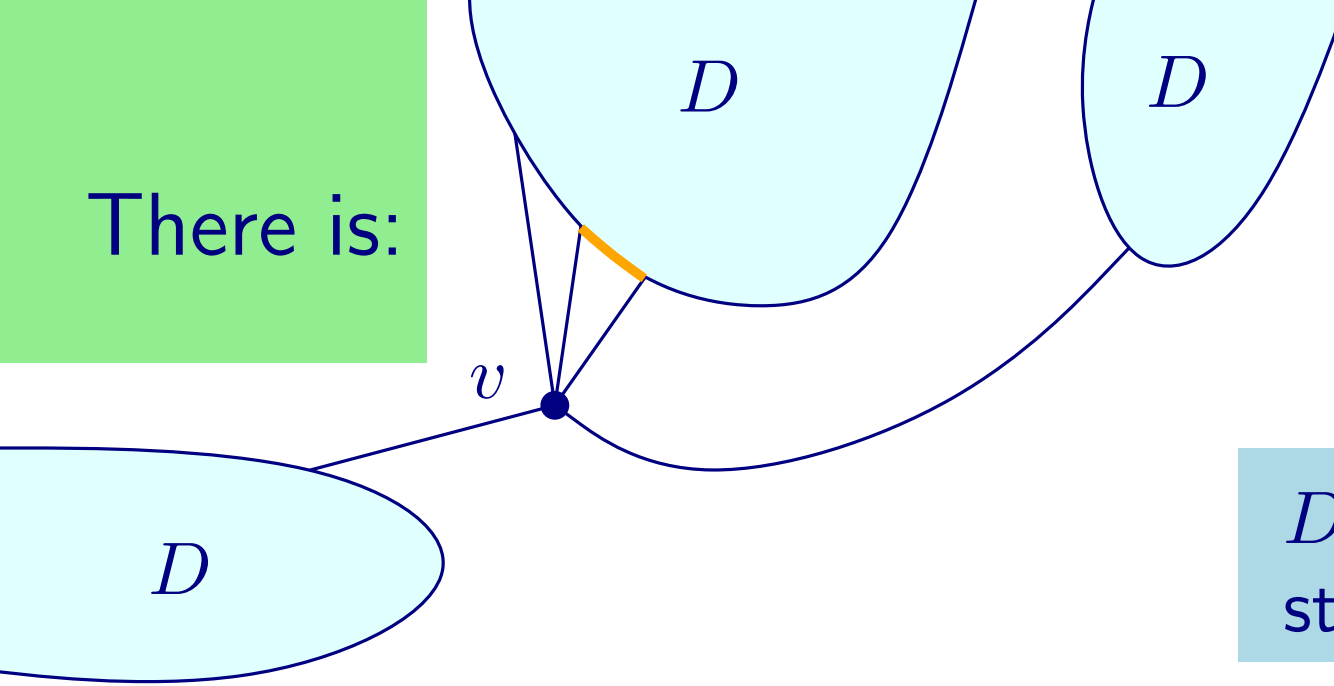
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D

D maximal induced disk stack v to any edge

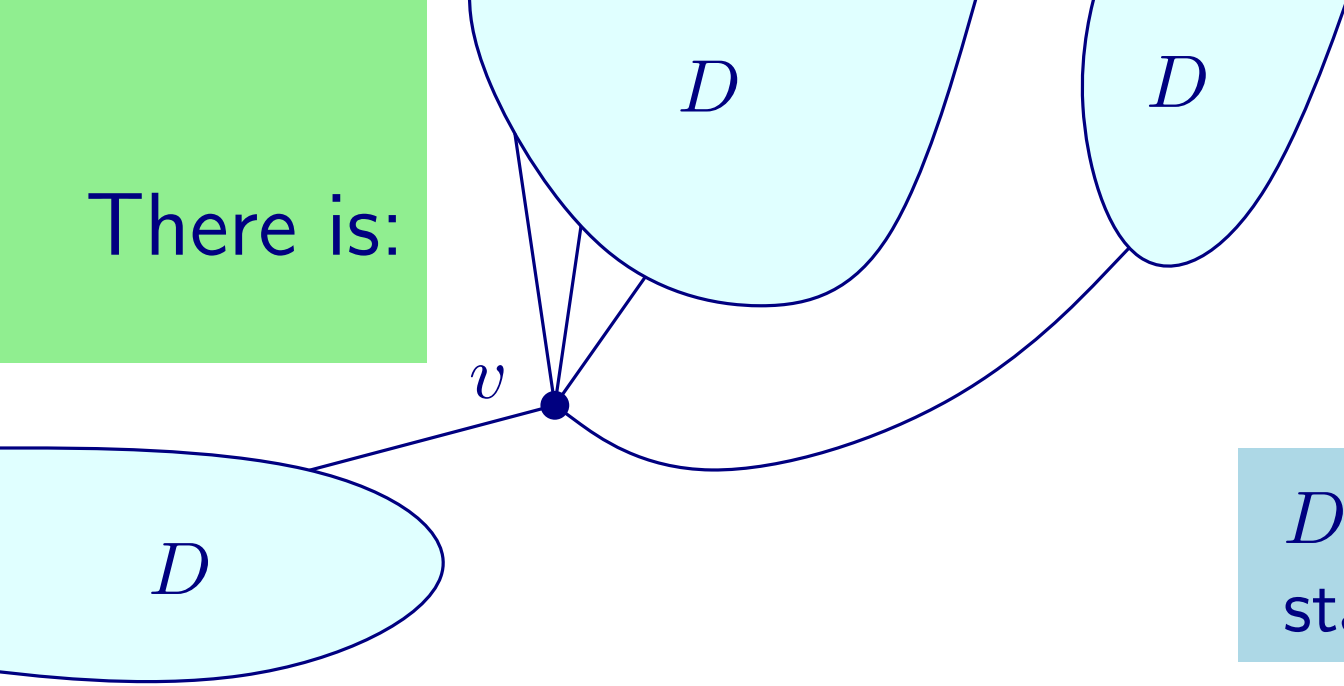
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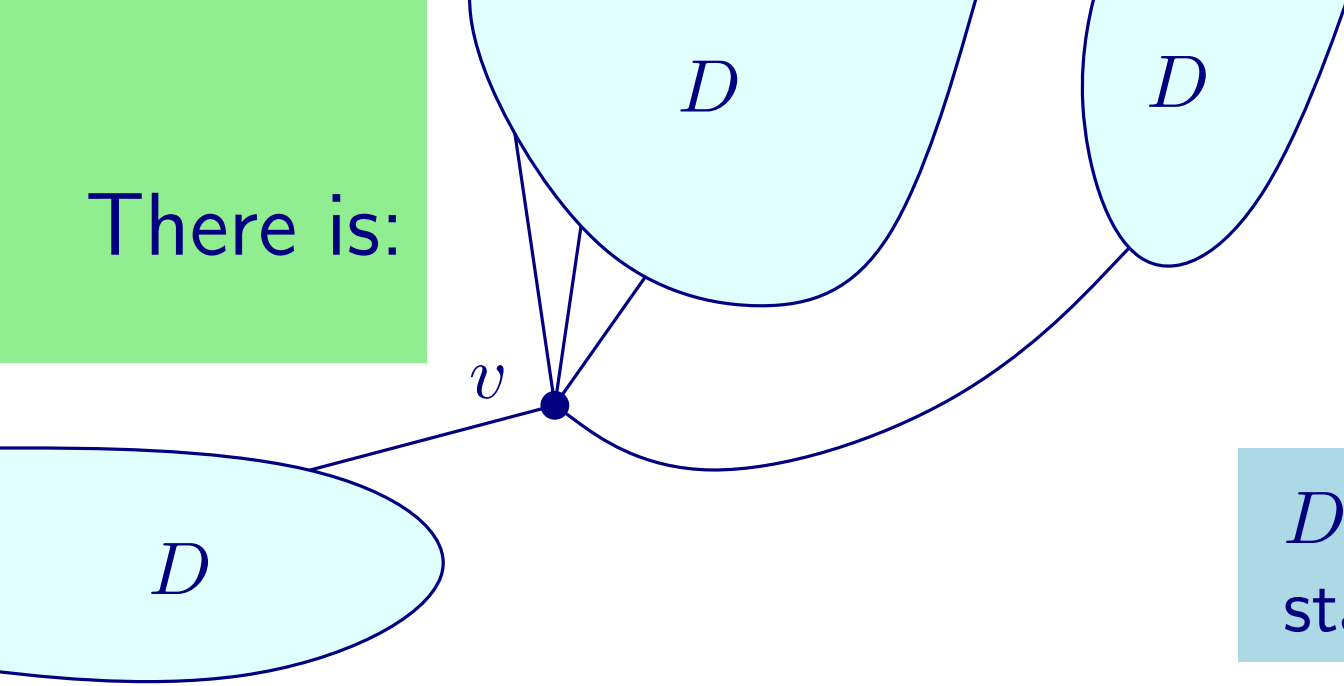


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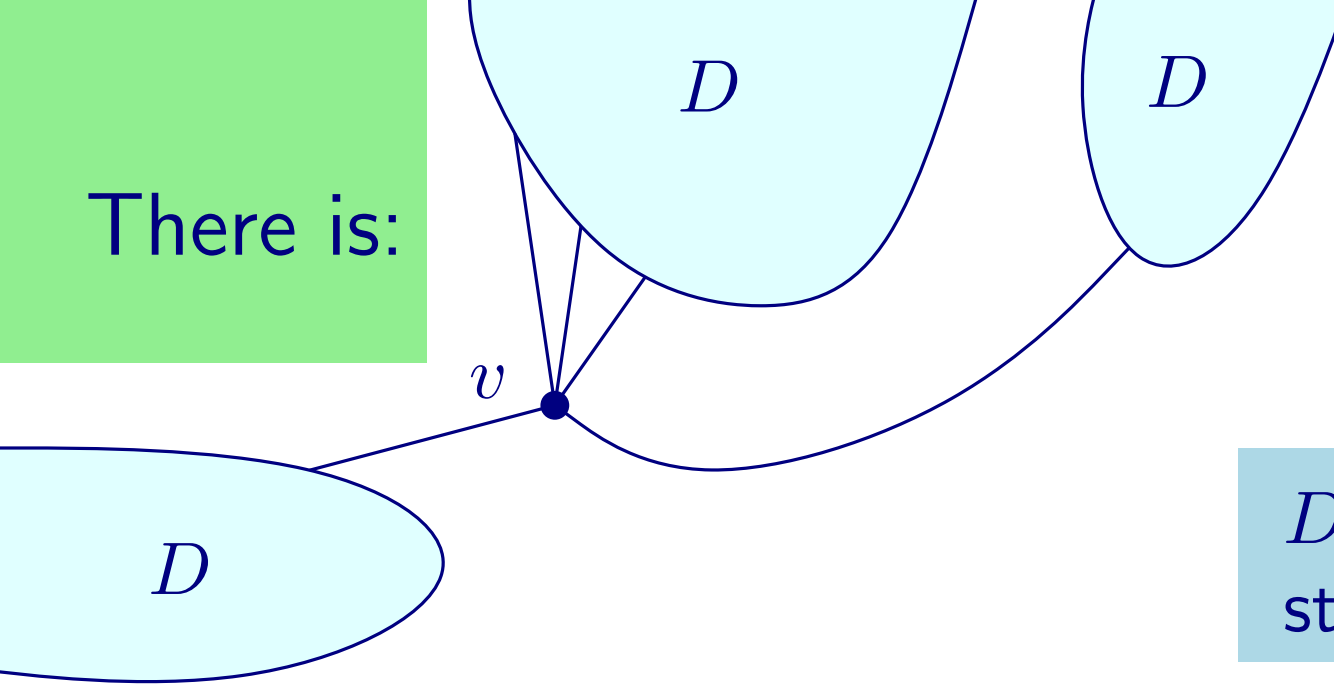
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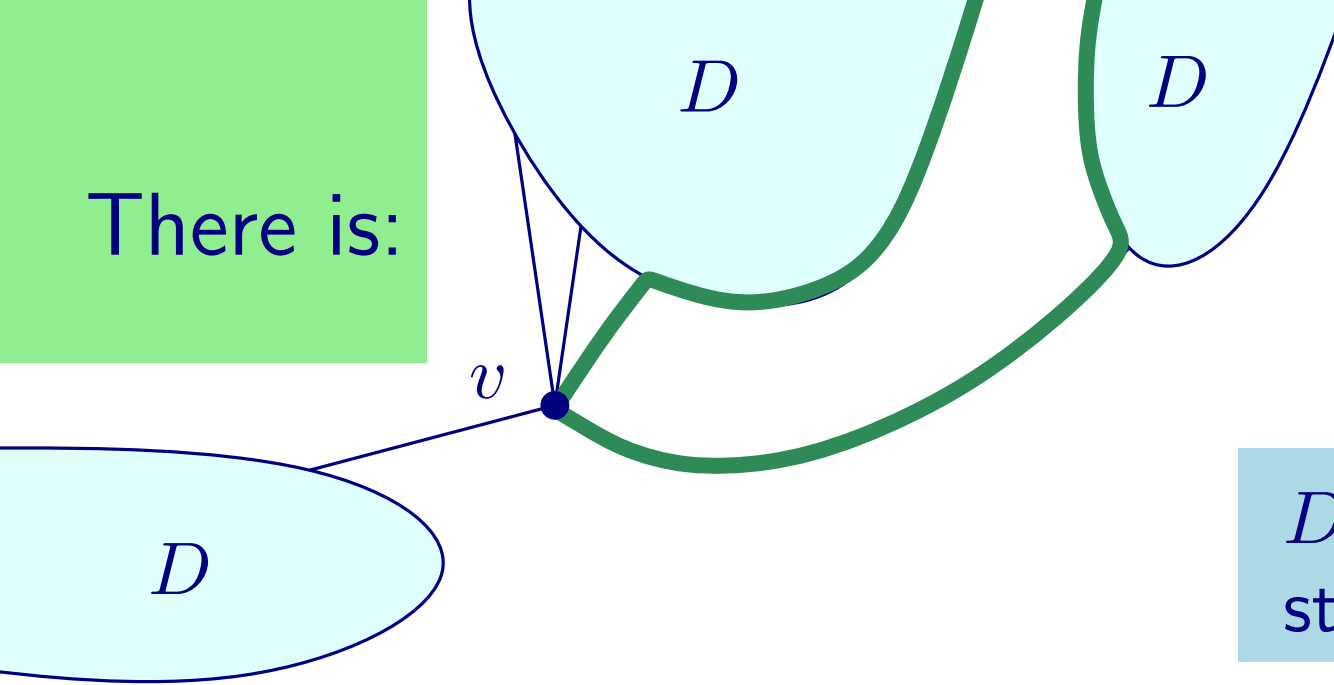
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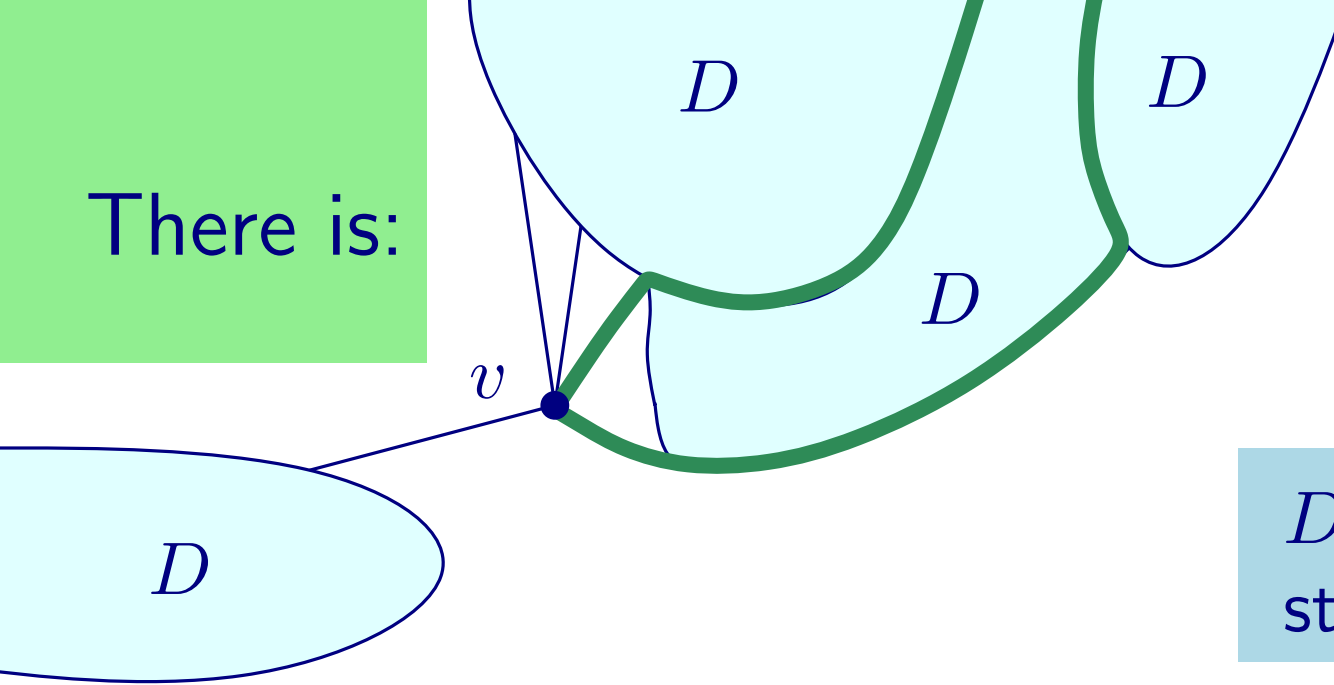
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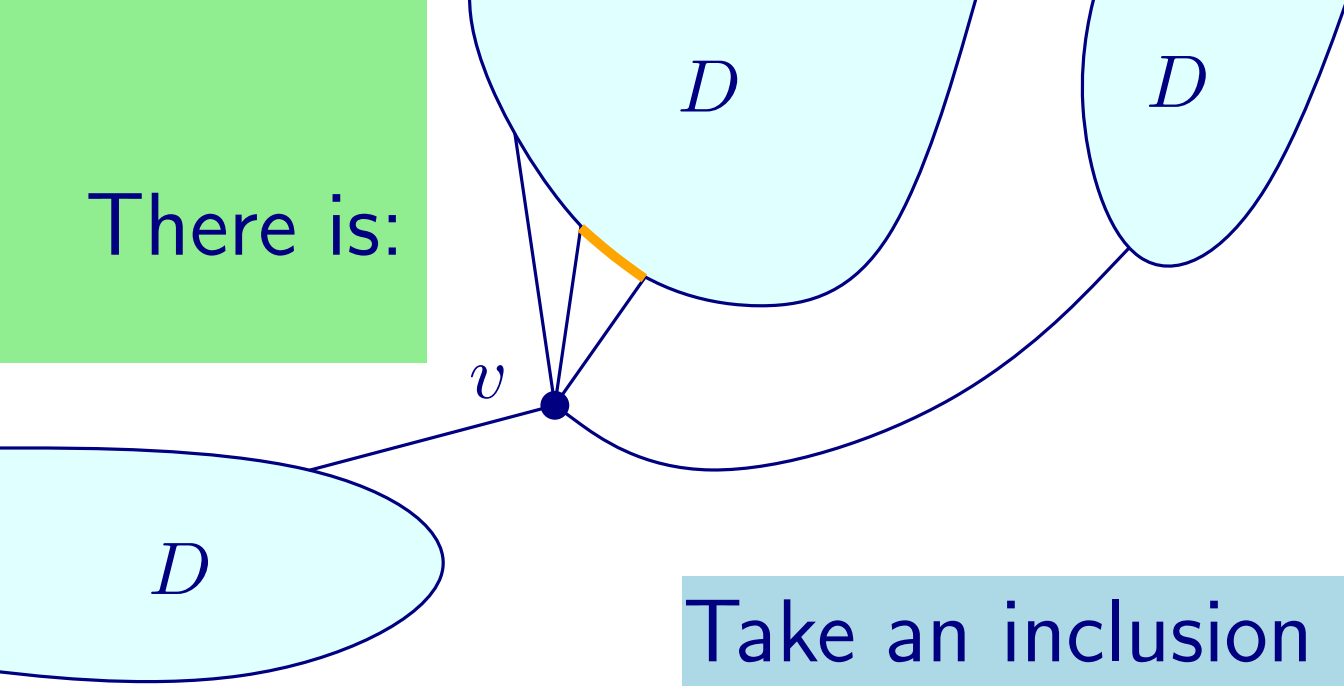
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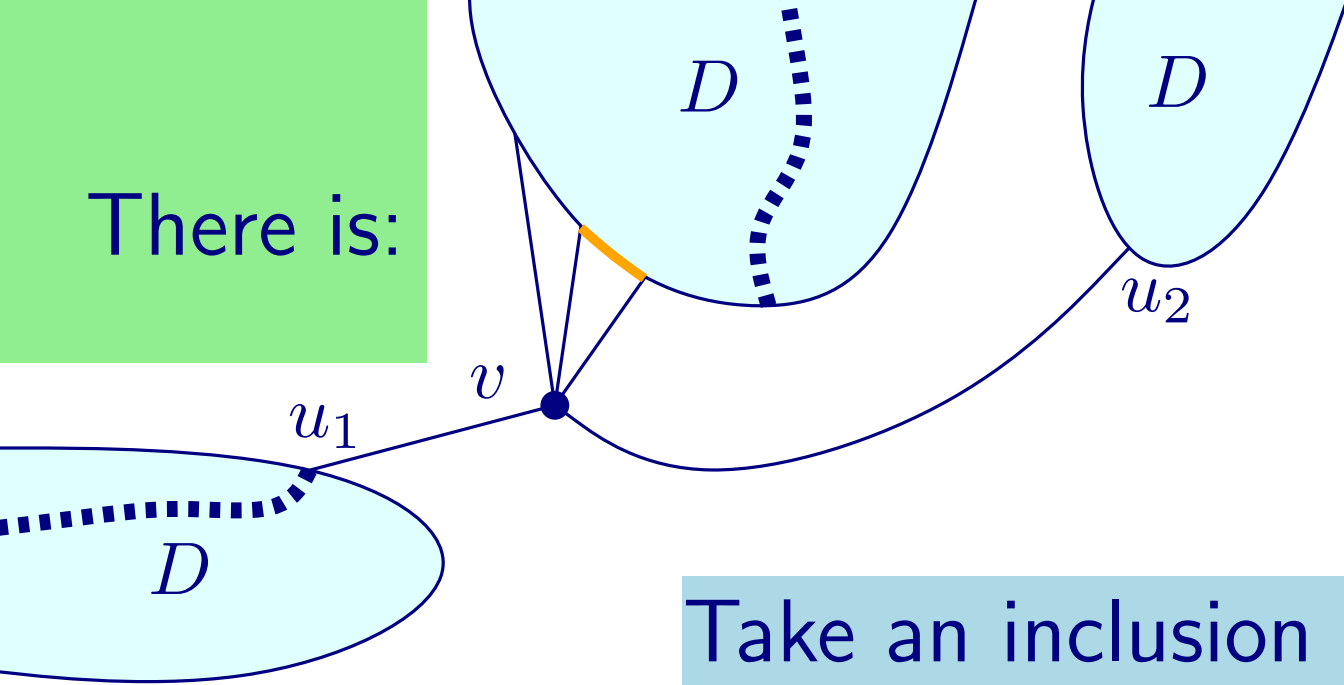
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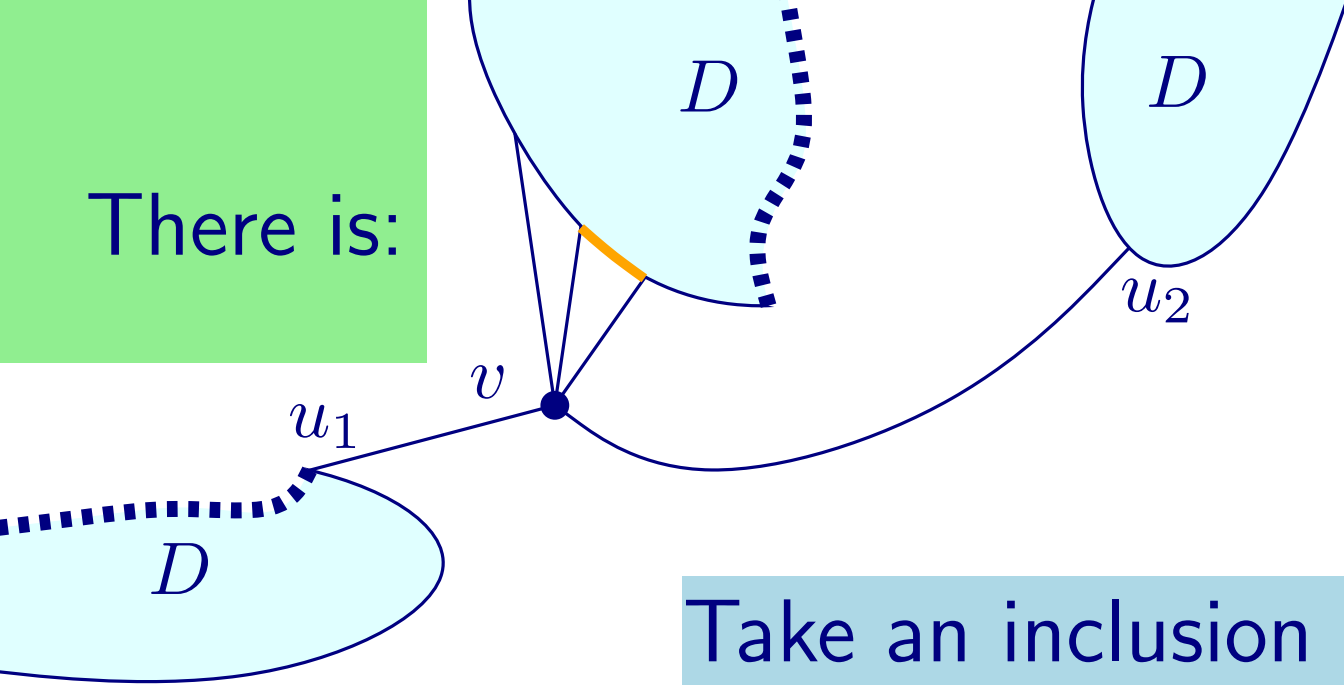


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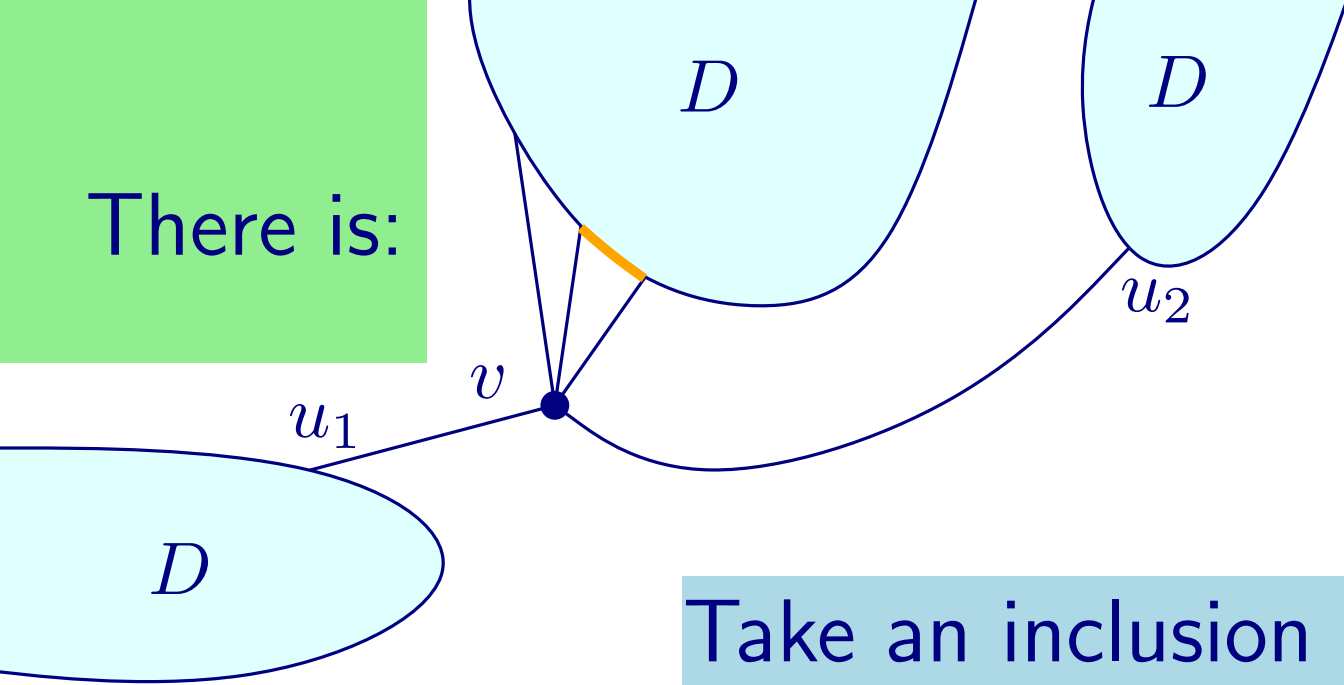


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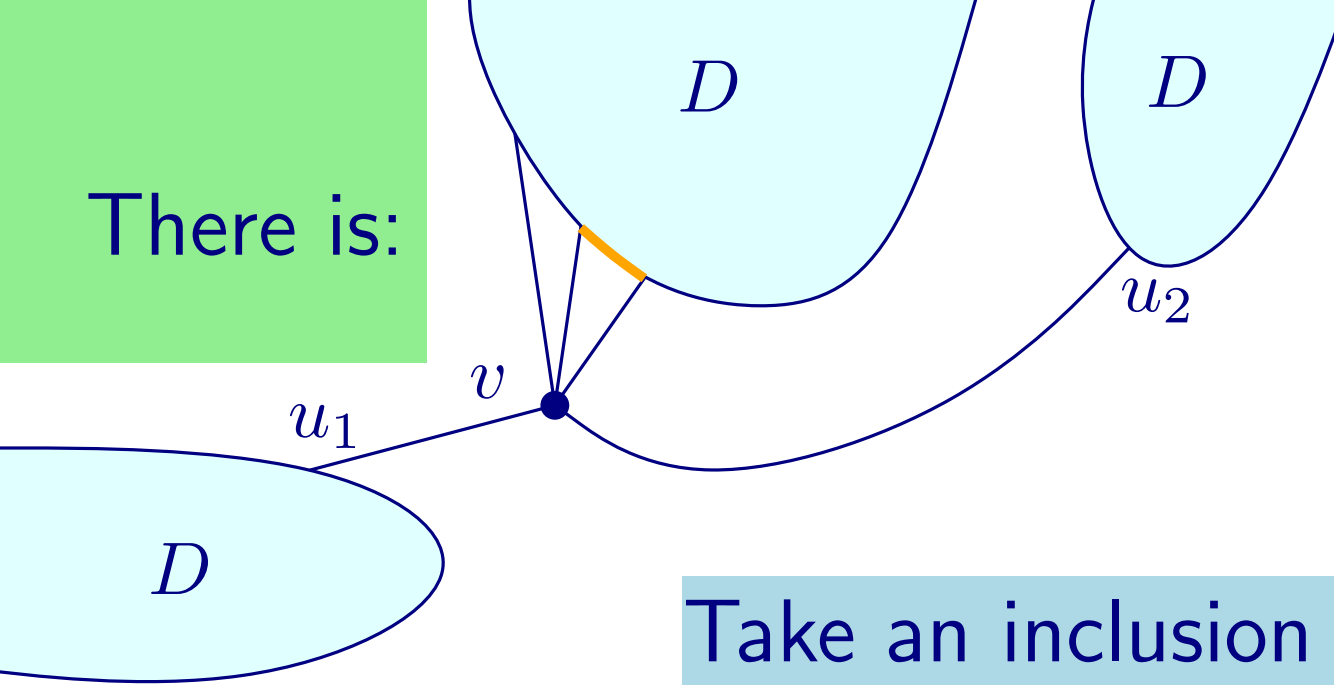


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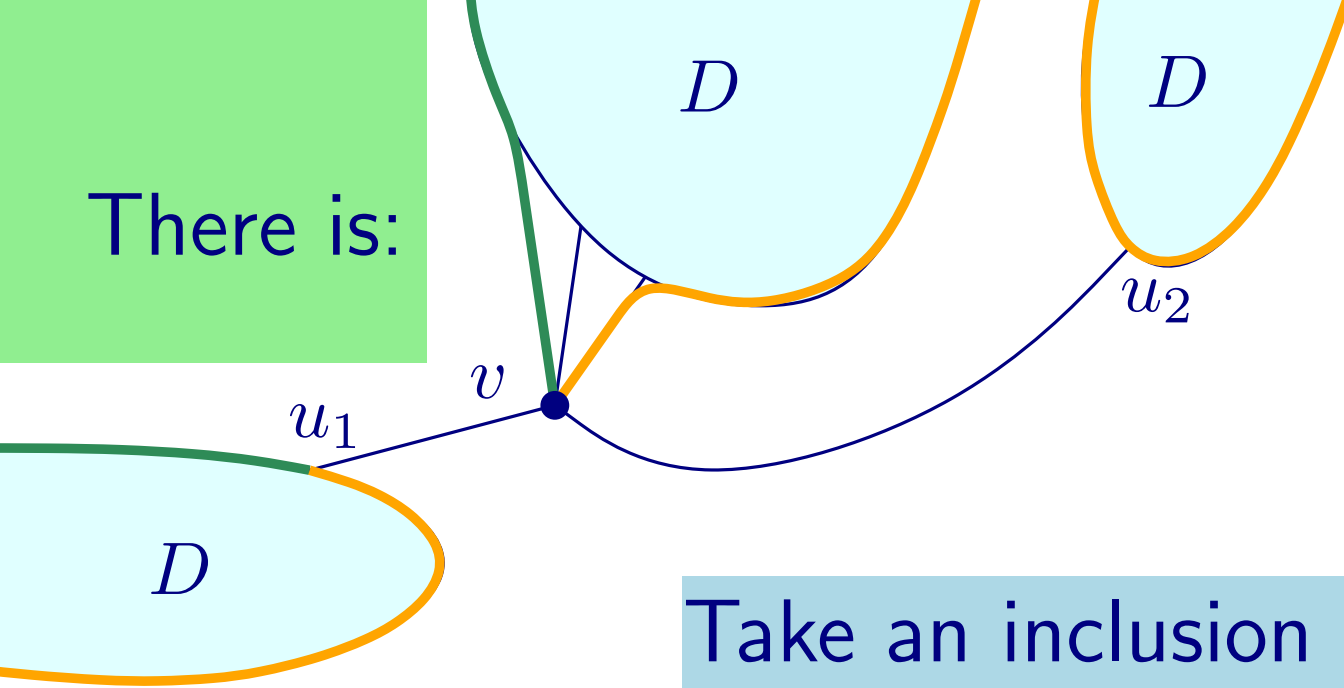
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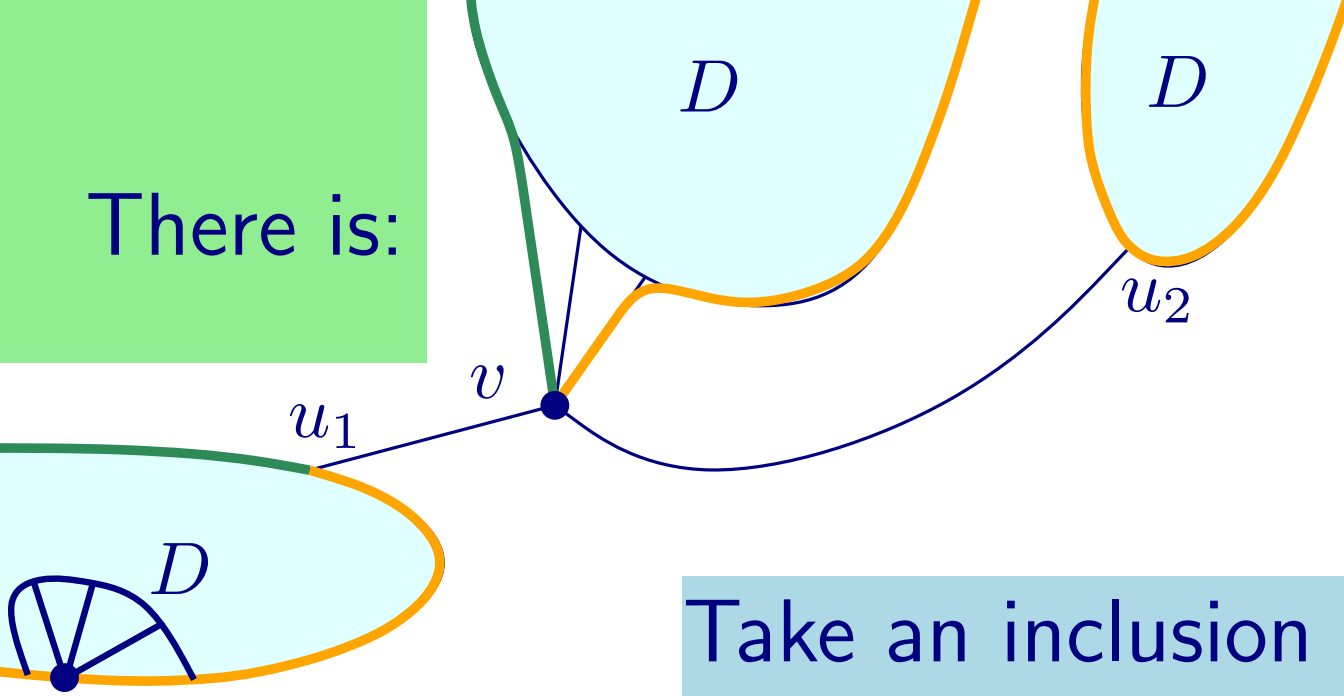
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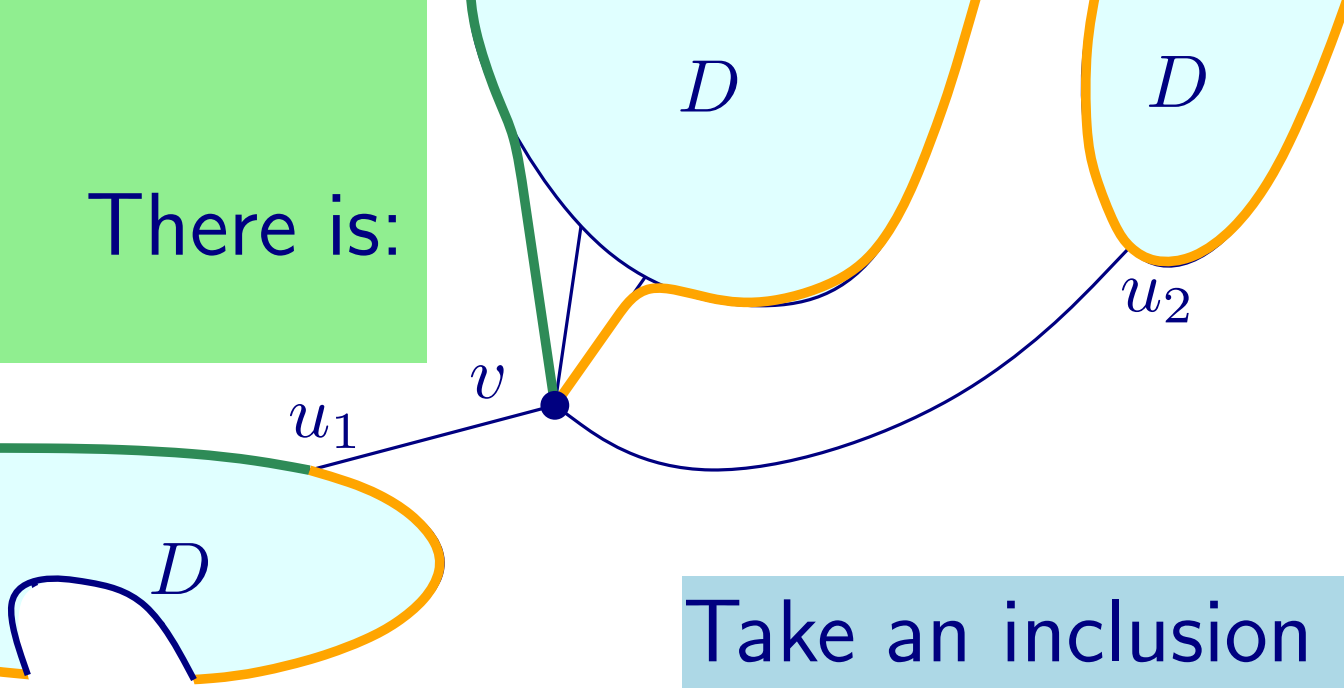
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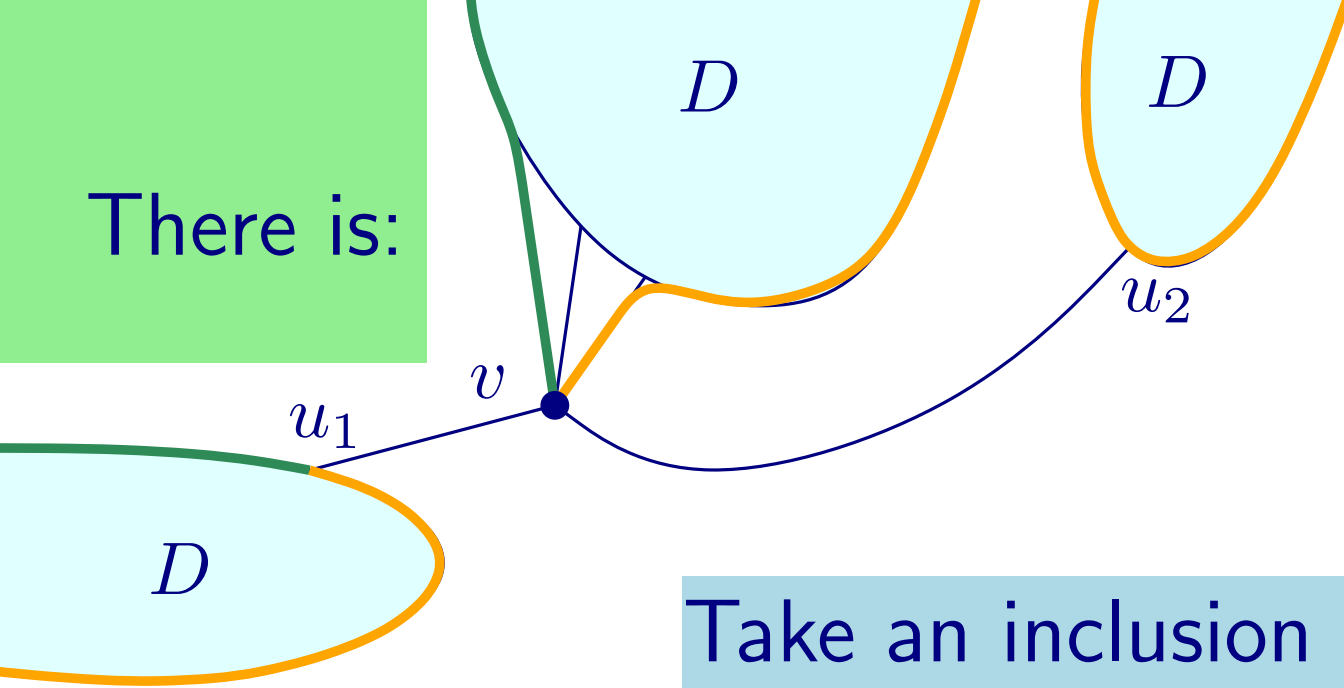
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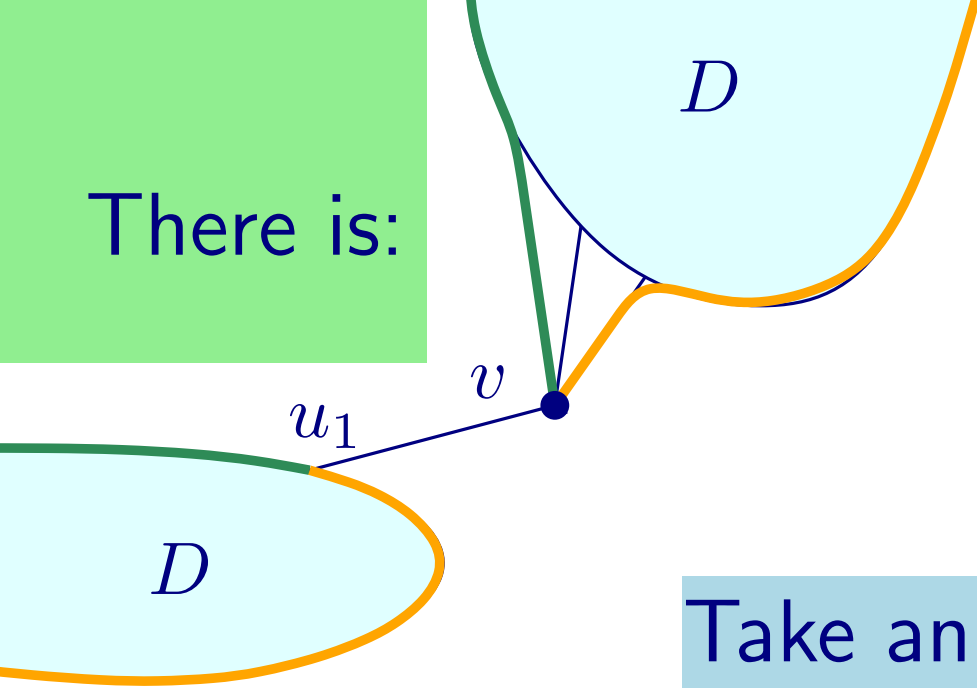
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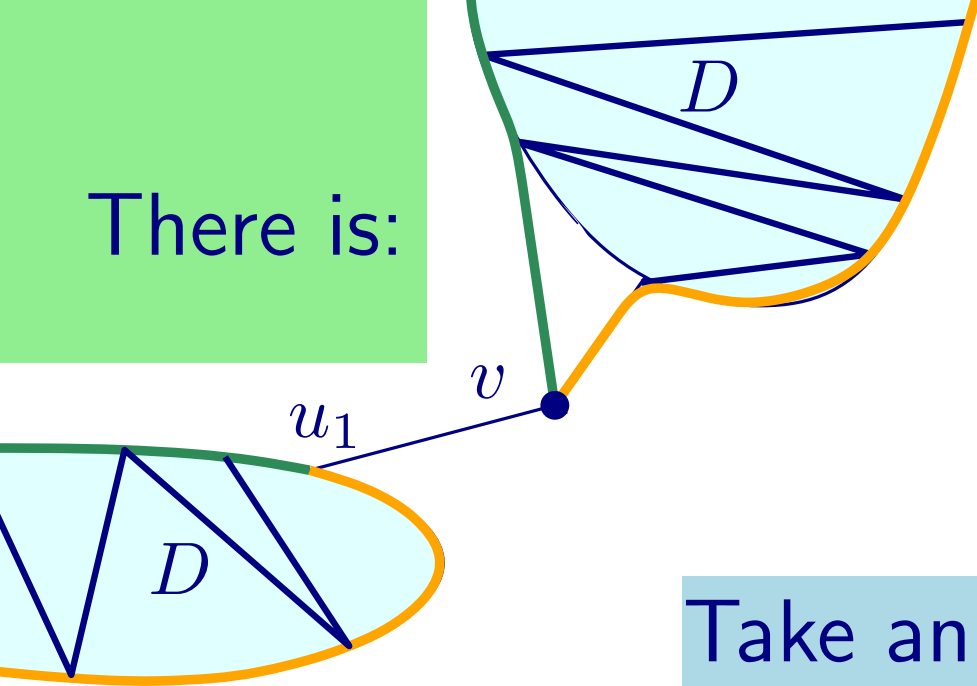
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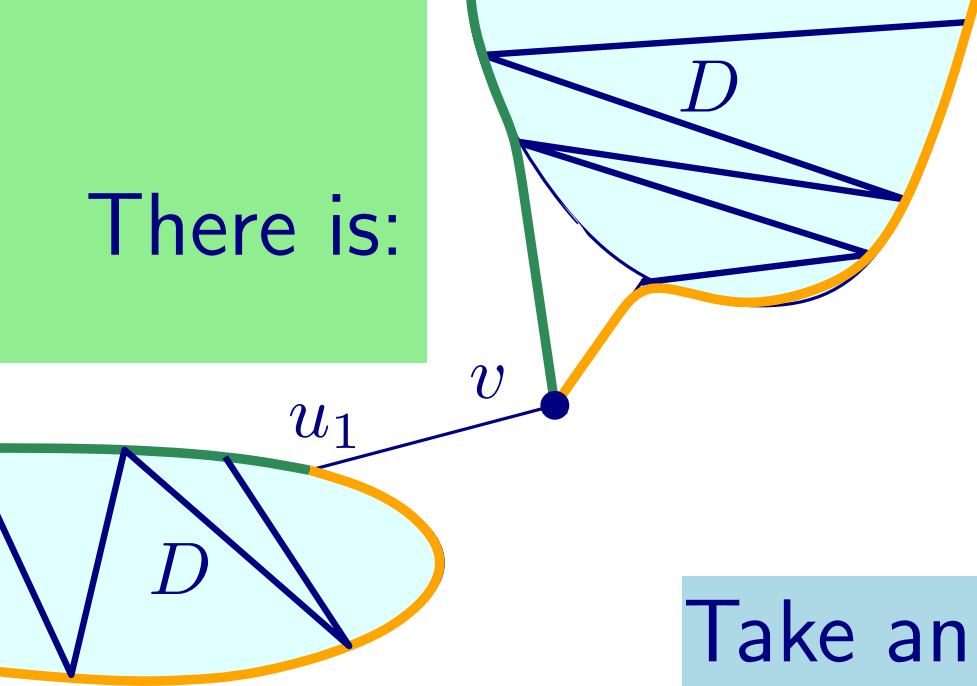
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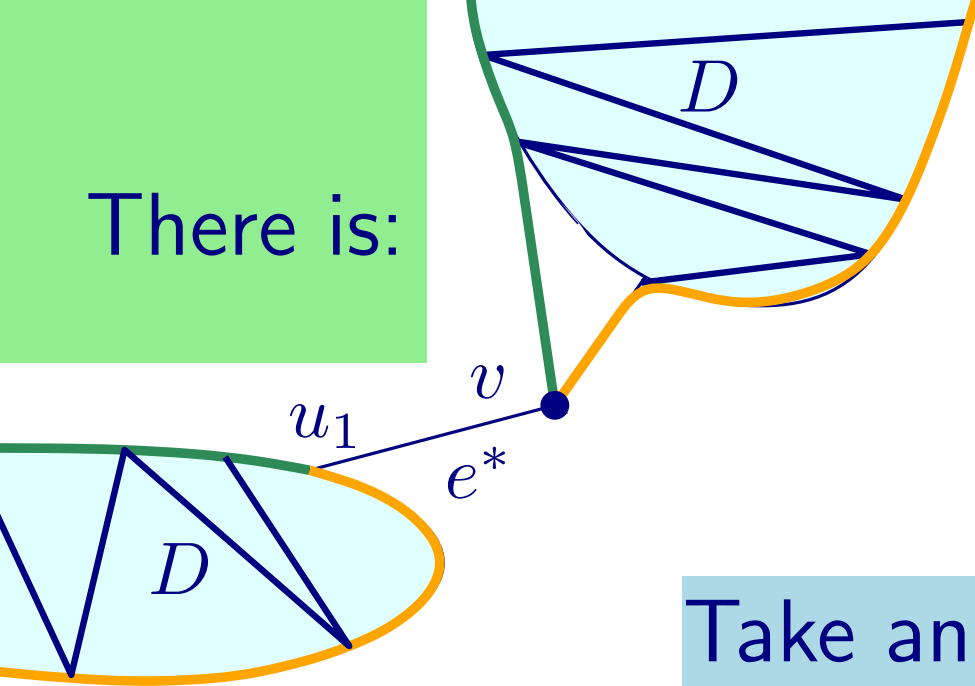
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no separating 3 and 4-disks
 \implies no vertices inside D

There is:



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Take an inclusion minimal such $D + v$

minimality \implies no chords at u_i

minimality \implies in $D - (vu_i)_i$ everybody neighbor on other side
 \implies there is only one u_i

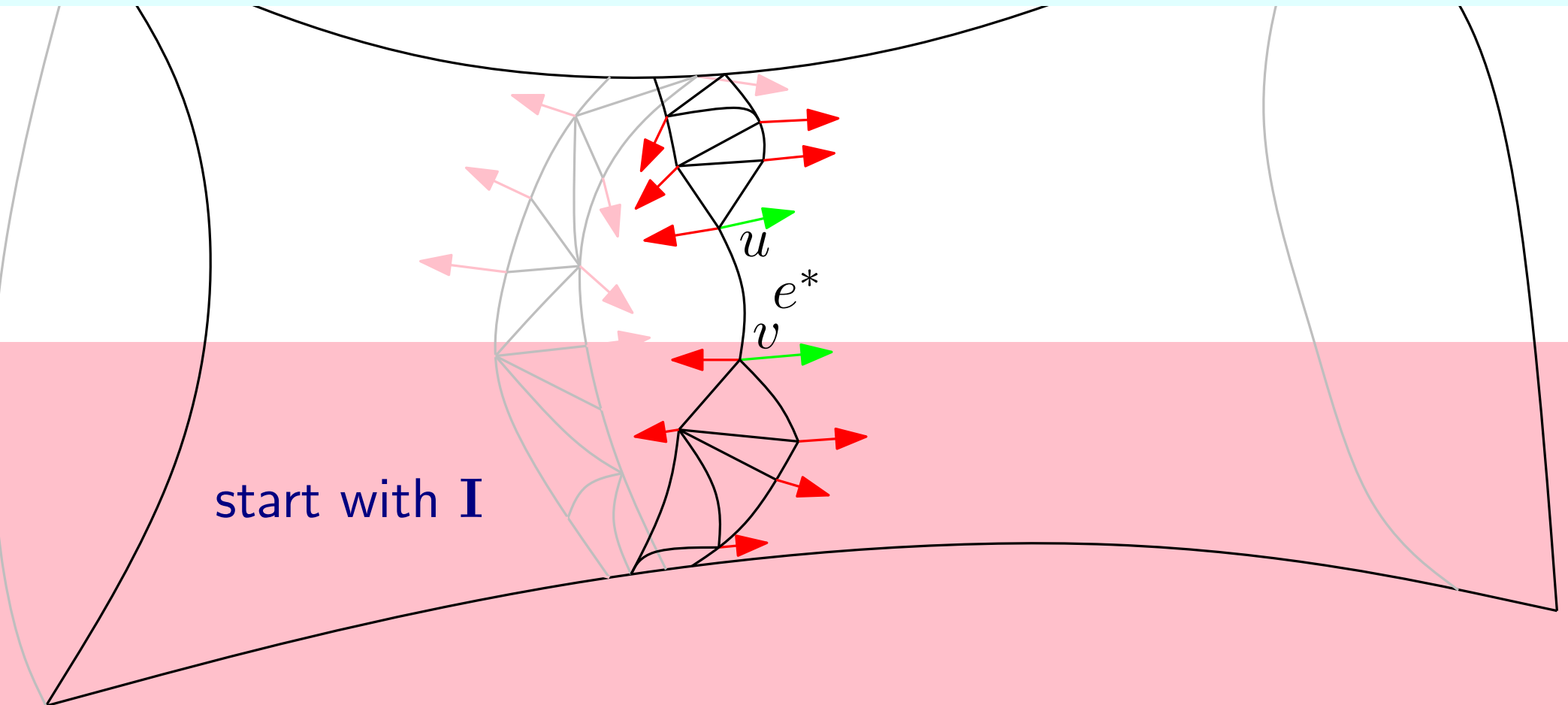
no separating 3 and 4-disks
 \implies no vertices inside D

initial graph I:

- maximal outerplanar graph
 $D + e^* = u_1v$
- induced
- non-contractible

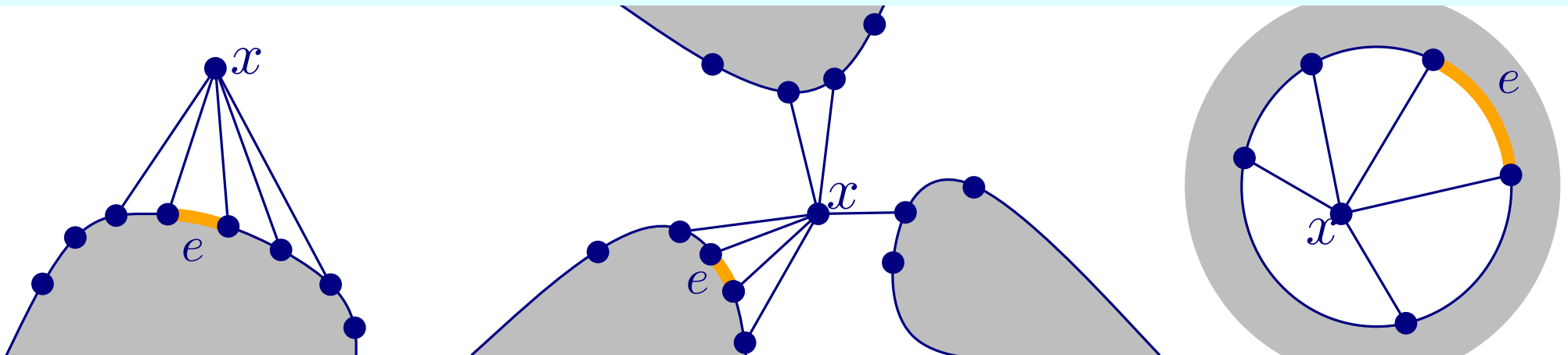
Explore T by stacking vertices to edges maintaining **Invariants**

- interior vertices happy wrt partition
- boundary angles
 - happy wrt **B**
 - have either **R** or **G** request or none
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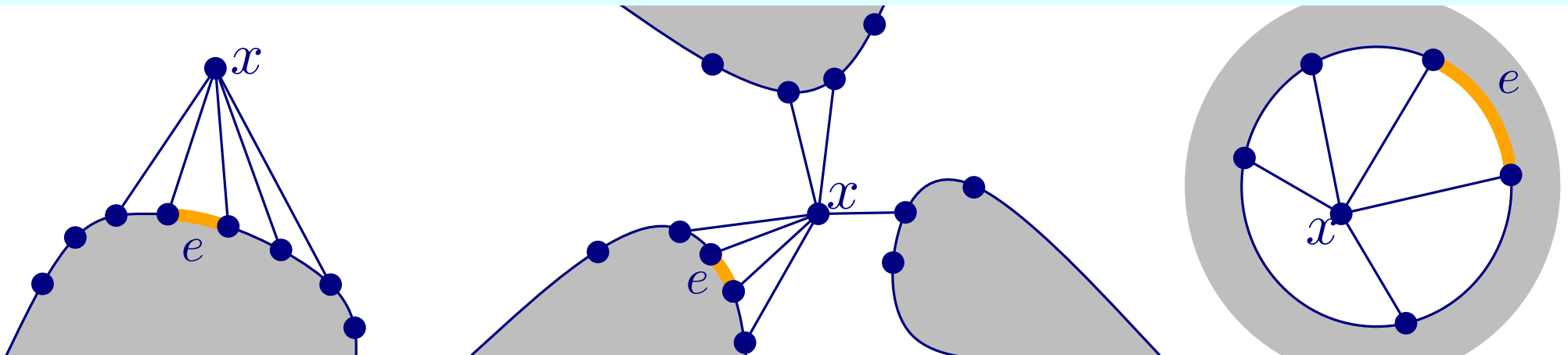


start with **I**

distinguish different situations of stacking

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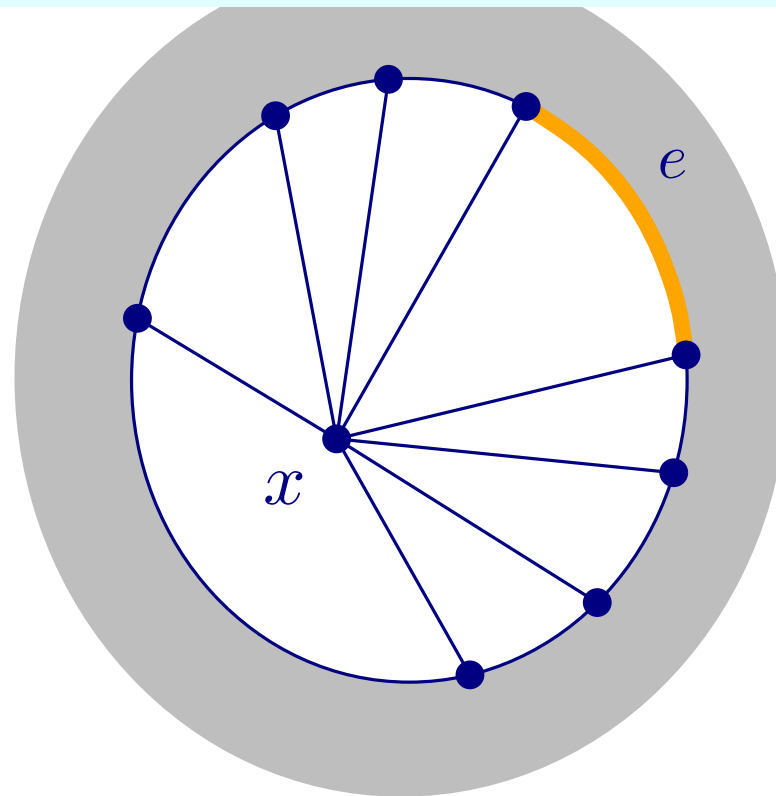


always stack some x to some e such that

- not both ends of e have **G** request
- if x is in unexplored disk, then either adjacent to all boundary or has exactly one neighboring path not covering all free angles
- create new unexplored disks as late as possible

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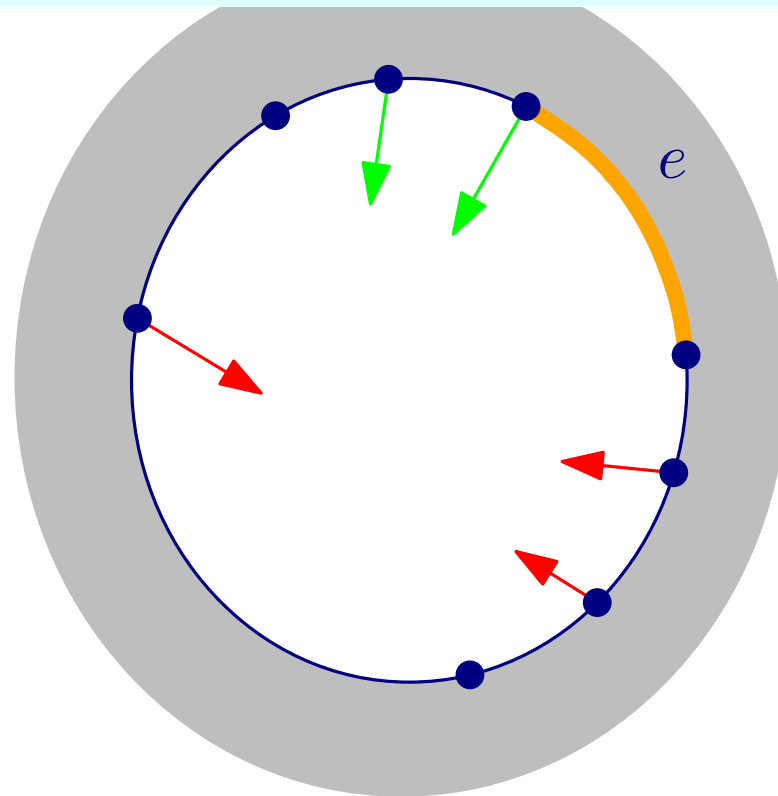


case A:

x in unexplored disk and has neighboring cycle

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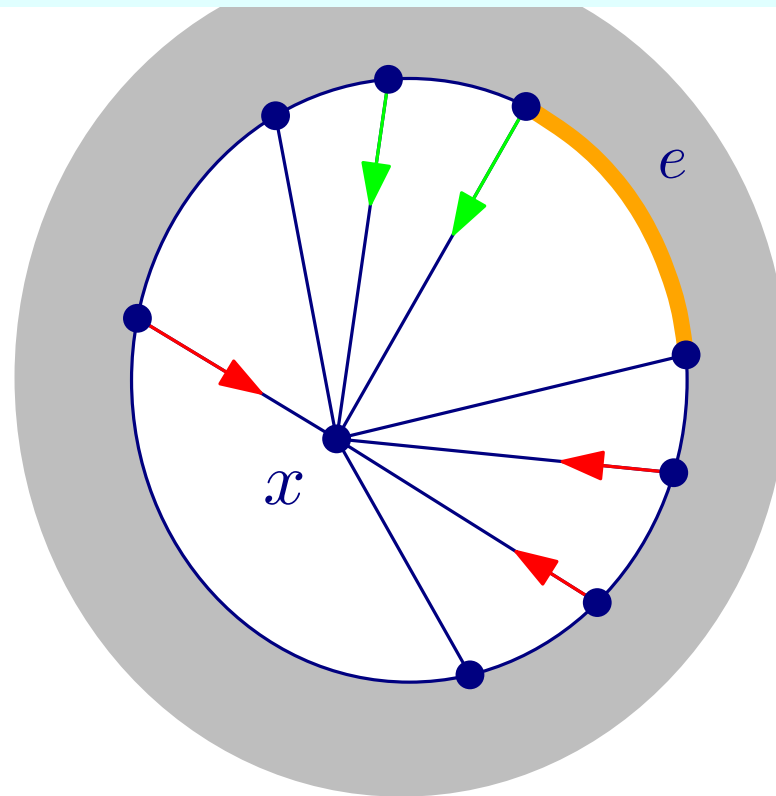


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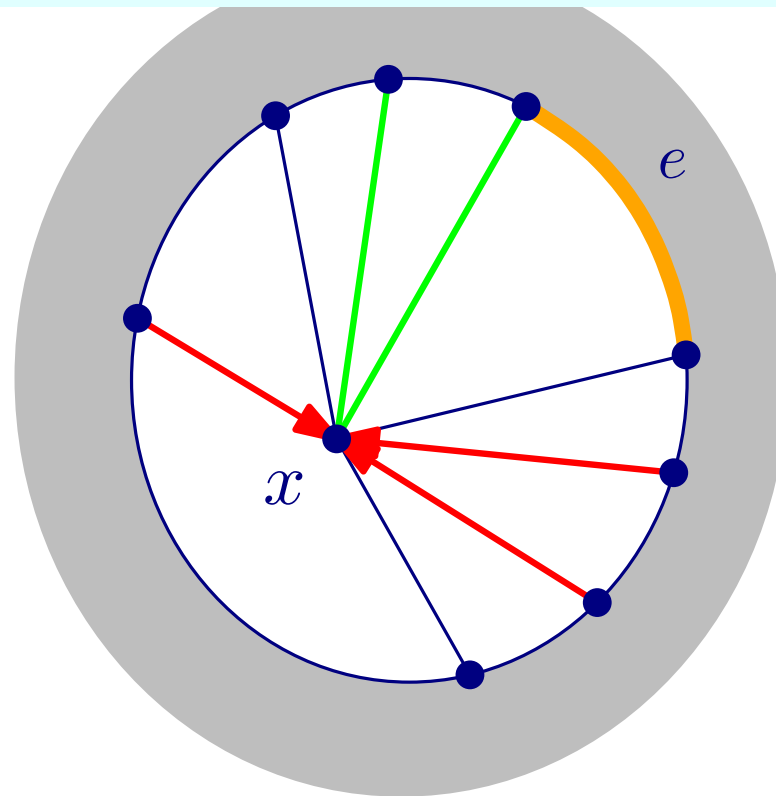


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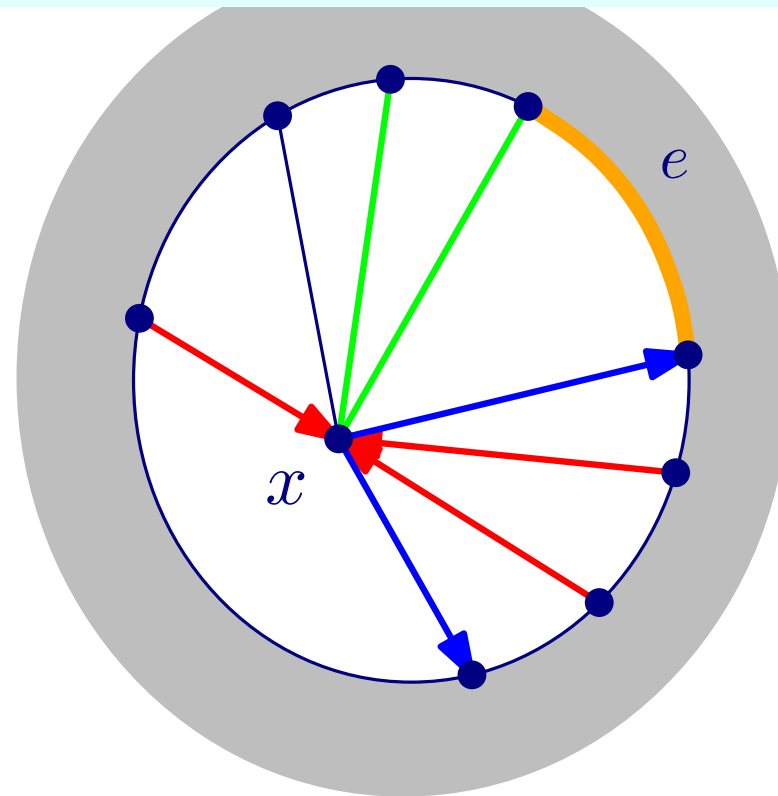


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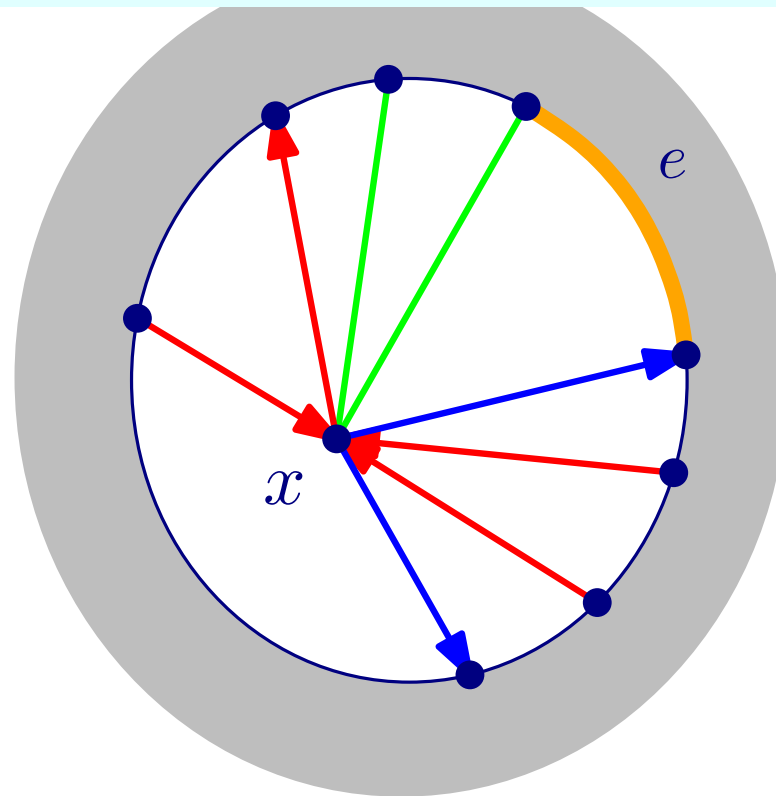


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case B:

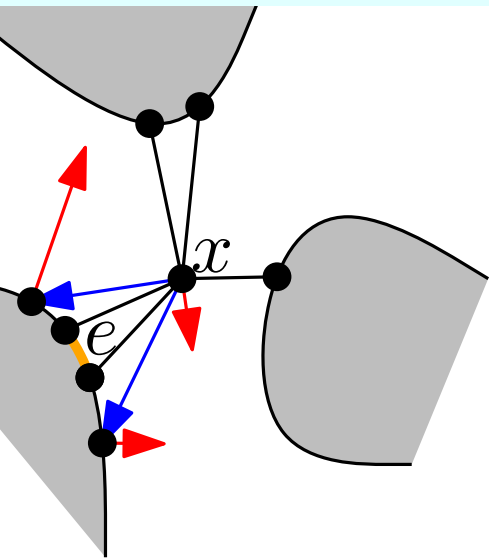
x not in unexplored disk

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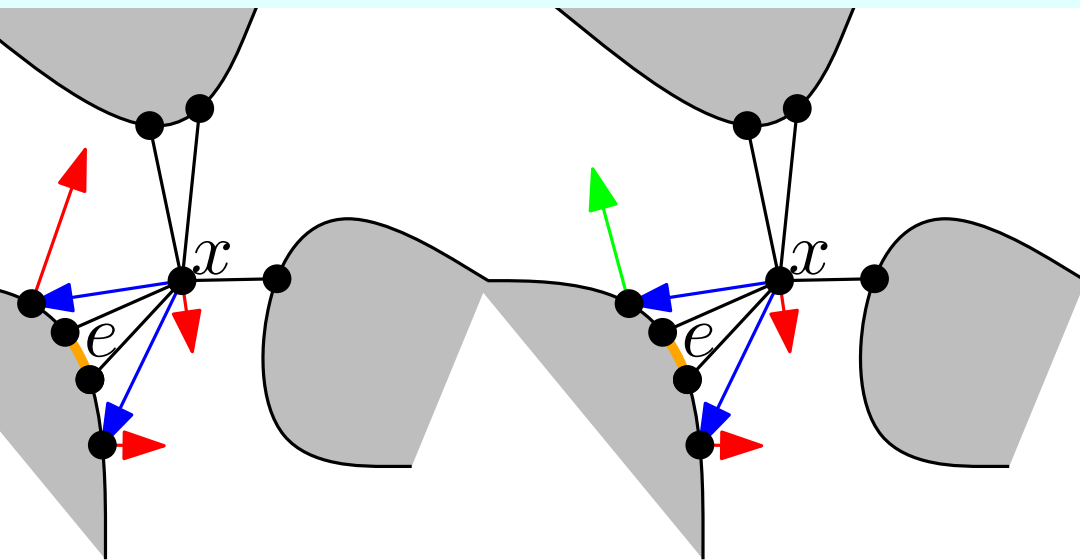
- several neighboring paths
- wlog no free angles
- satisfy all requests on inner angles

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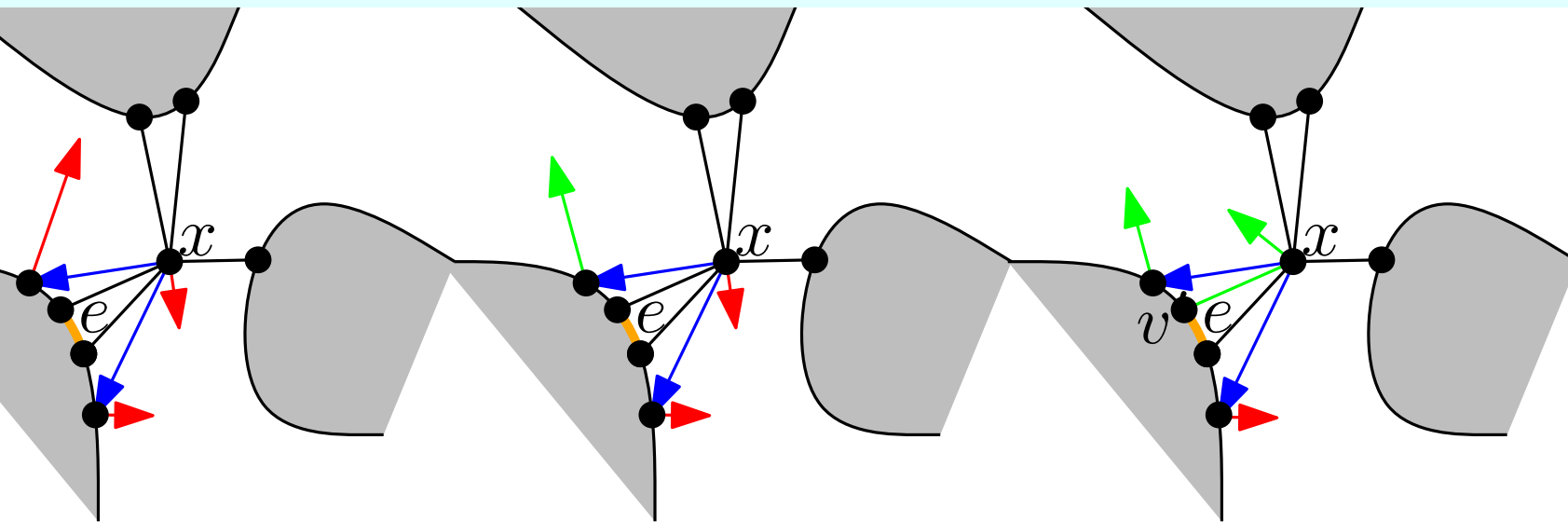
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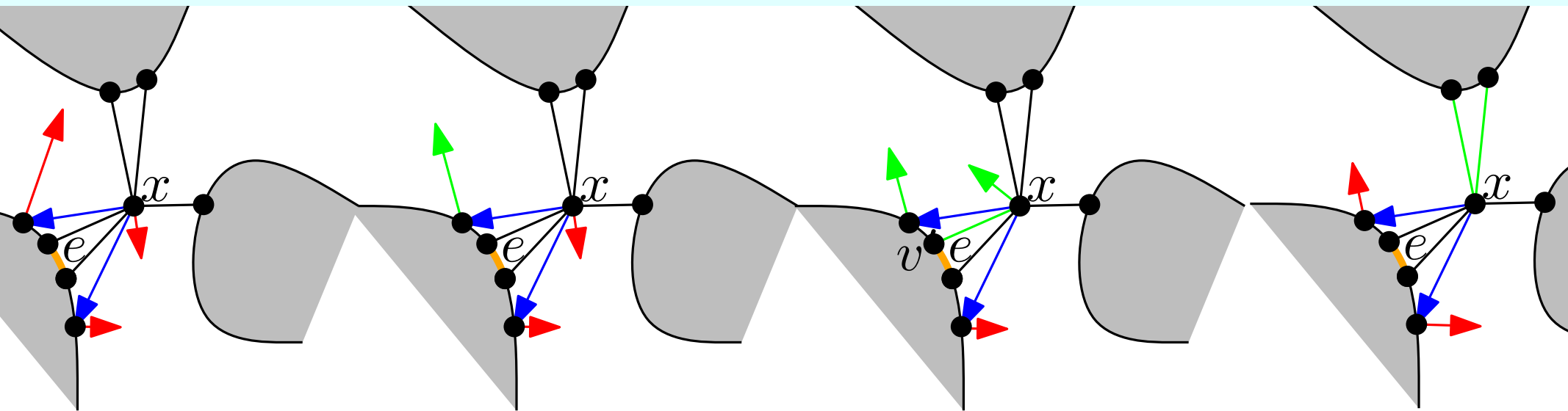
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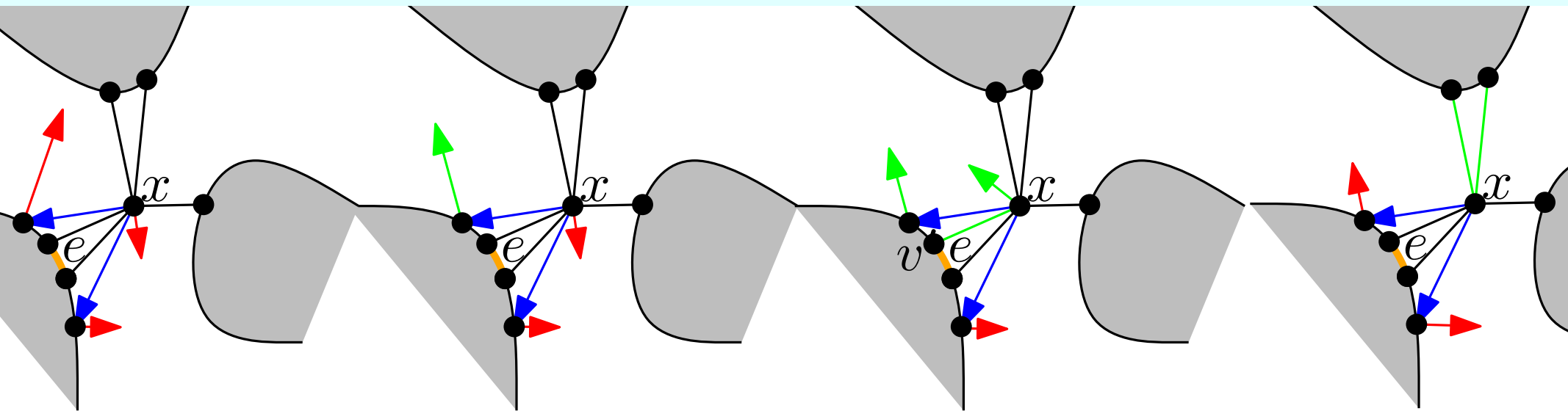
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case B:

x not in unexplored disk



if x closes unexplored disk it appears several times on boundary, otherwise there would have been x' closing no unexplored disk

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case C:

x in unexplored disk and
has neighboring path

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case **C**:

x in unexplored disk and
has neighboring path

one **G** request

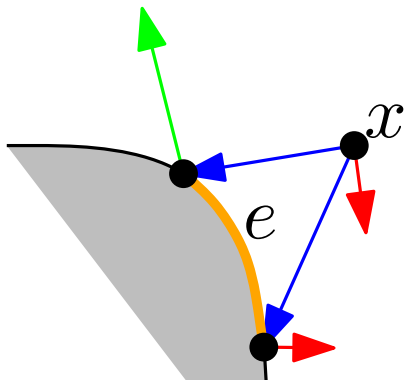
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no free angles

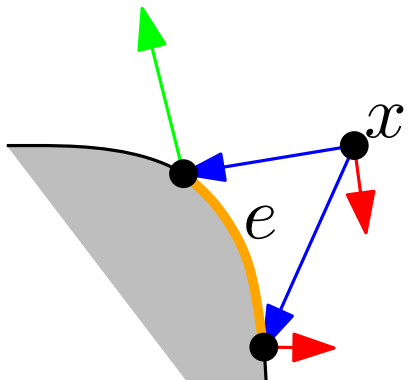
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case C:

x in unexplored disk and has neighboring path

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no free angles

- only one neighboring path
- neighboring path contains ≤ 2 free angles
- satisfy x 's requests using free and outer
- satisfy all requests on inner angles

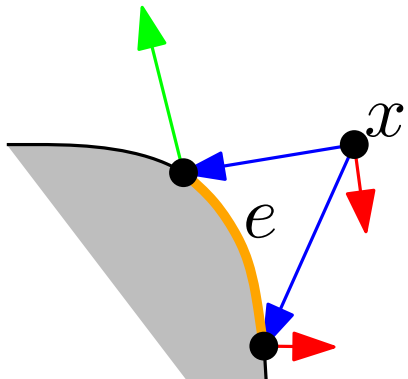
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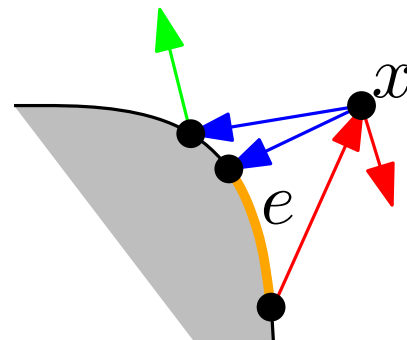
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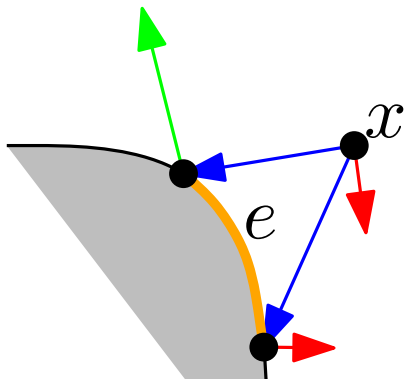
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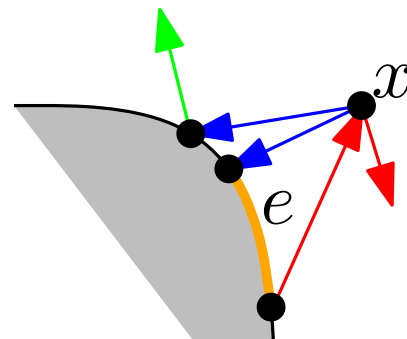
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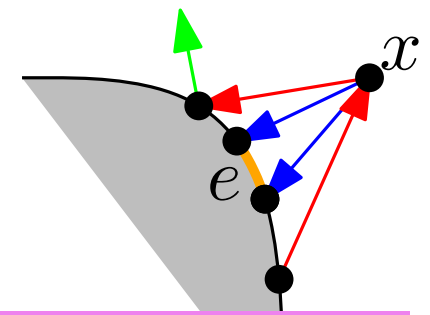
one **G** request



no free angles



one free angle



two free angles

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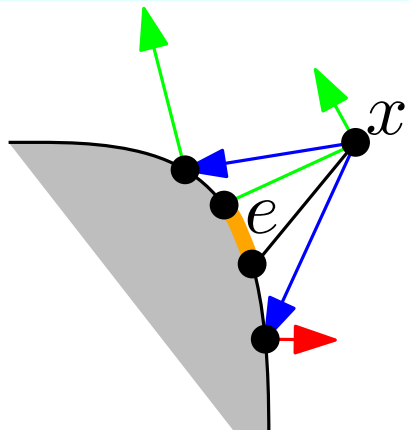
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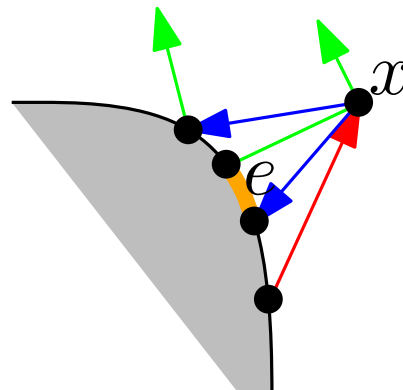
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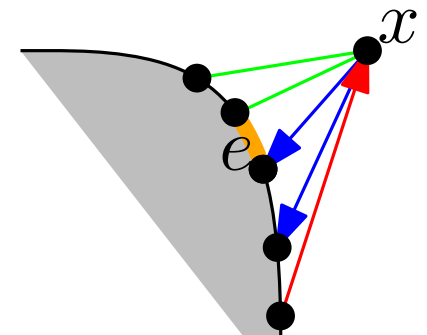
two **G** requests



no free angles



one free angle



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Theorem[Albar, Gonçalves, K '14]:

Every triangulation of a closed surface *different from sphere and projective plane* admits an orientation with all vertices of *non-zero outdegree divisible by three*.

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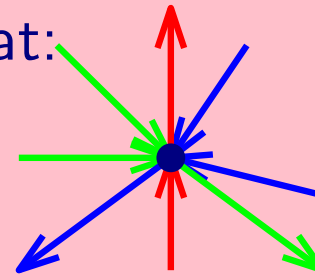
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Orientation and red-green-blue-coloring of inner edges such that:

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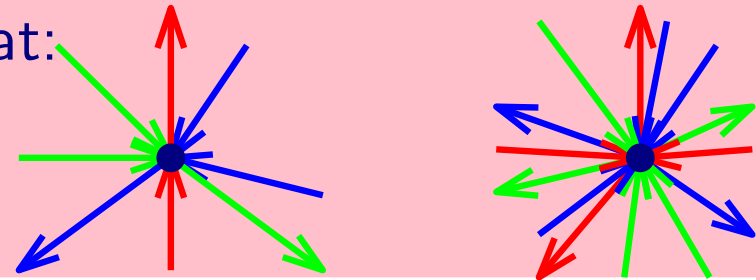
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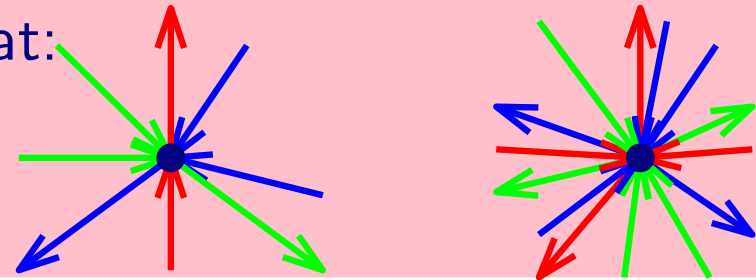
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Outdegree 3 orientations of planar graph carry distributive lattice structure. How about higher genus surfaces?

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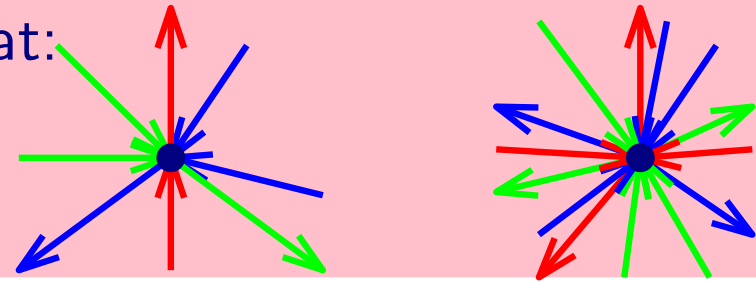
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Conjecture [Barát&Thomassen '06]:

Every planar 4-edge-connected graph with number of edges divisible by 3 has an orientation with outdegrees divisible by 3.