

Finding Splitting Cycles in Triangulations

Vincent DESPRE

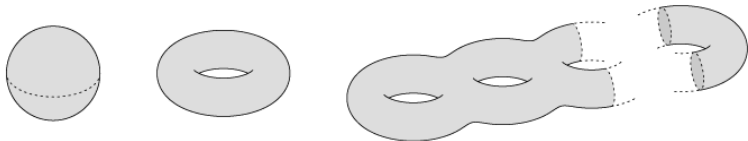
gipsa-lab, G-SCOP, Labex Persyval, Grenoble, France
vincent.despre@gipsa-lab.fr

6 November 2014

Surfaces

Definition

A surface is a compact, connected 2-manifold without boundary.



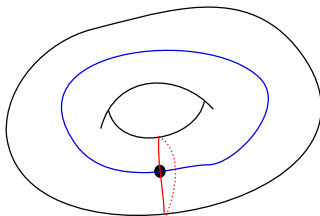
Genus

The genus of a surface is its number of handles.

Embedded graphs

Embedding

An embedding of a graph G on a surface S is a proper drawing of G on S .



2-cell embedding

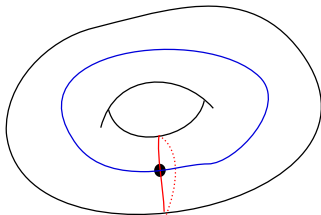
An embedding is a 2-cell embedding if the surface after cutting along the graph is an union of disks.

Euler Characteristic

Definition

let S be a surface of genus g and G a graph with v vertices, e edges and f faces, then:

$$\chi(S) = v - e + f = 2 - 2g$$

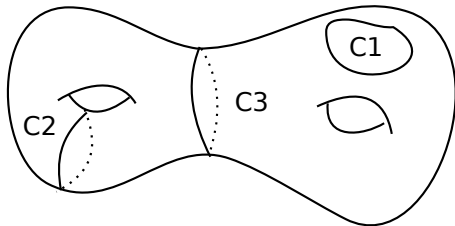


$$\chi(S) = 1 - 2 + 1 = 2 - 2 * 1 = 0$$

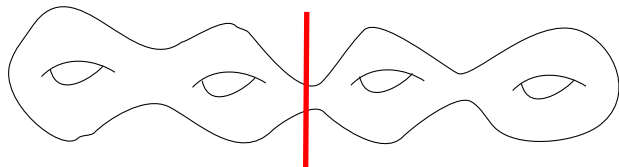
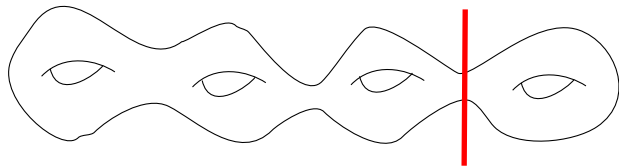
Splitting cycle

3 kinds of cycles:

- ⇒ Contractile and separating cycles (C1).
- ⇒ Non-contractile and non-separating cycles (C2).
- ⇒ Non-contractile and separating cycles also called **splitting cycles** (C3).



Different cuttings



Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete graph embeddings

Idea
Case of K_5
General case
Non-isomorphic embeddings

Algorithms and results

Naïve algorithm
Final algorithm
Results

Theoretical points

Perspectives
Example $n \equiv 7$
Face-width

Status of the problem

Complexity (Colin de Verdière et al. 2008)

Deciding if a combinatorial surface has a splitting cycle is NP-complete.

Conjecture (Barnette, 1982)

Every triangulation of a surface of genus at least 2 has a splitting cycle.

Conjecture (Mohar et Thomassen, 2001)

For all triangulation S of genus g and all $h \in \llbracket 1, g - 1 \rrbracket$, there is a splitting cycle that separates S into two pieces of genera h and $g - h$.

Example

Splitting Cycles in Triangulations

Vincent
DESPRE

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete graph embeddings

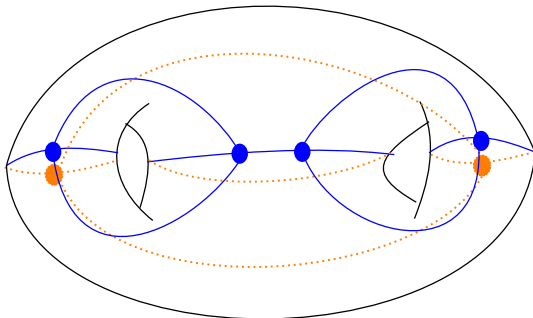
Idea
Case of K_5
General case
Non-isomorphic
embeddings

Algorithms and results

Naïve algorithm
Final algorithm
Results

Theoretical points

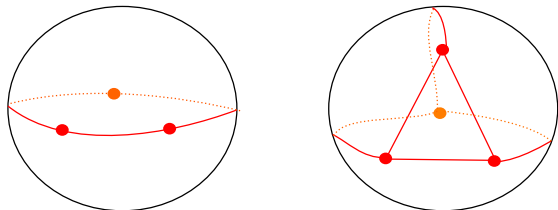
Perspectives
Example $n \equiv 7$
Face-width



Triangulations

Definition

A triangulation of a surface S is a simplicial complex C and a homeomorphism between S and C .

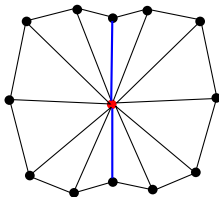
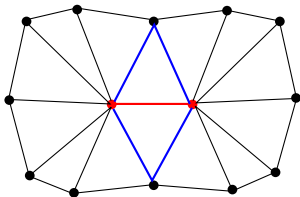


This definition excludes:

- ⇒ Loop edges.
- ⇒ 2 edges with the same end points.
- ⇒ 2 faces that share their 3 vertices.

Irreducible triangulations

A triangulation is irreducible if none of its edges can be contracted.



It is known that, for a fixed genus g , there is a finite number of irreducible triangulations with bounded number of vertices.

Consequence

The conjecture is decidable for fixed genus.

Low genus irreducible triangulations

Genus 2:

Number of maps: 396 785

Average length of the shortest splitting cycle: 5.31

Maximal length: 8 (map with 10 vertices)

11 vertices, genus 3:

Number of maps: 9 709

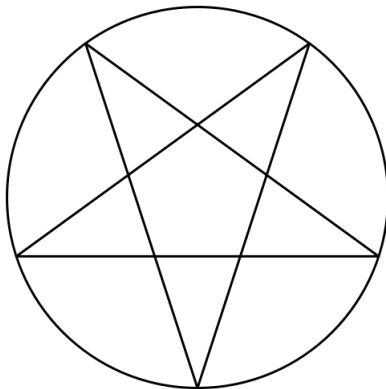
Average length of the shortest splitting cycle: 4.67

Maximal length: 6

Idea

K_n has $\binom{n}{2}$ edges.

Example of K_5 :



Vincent
DESPRE

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete
graph
embeddings

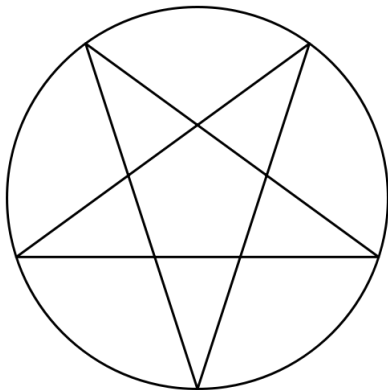
Idea
Case of K_5
General case
Non-isomorphic
embeddings

Algorithms
and results

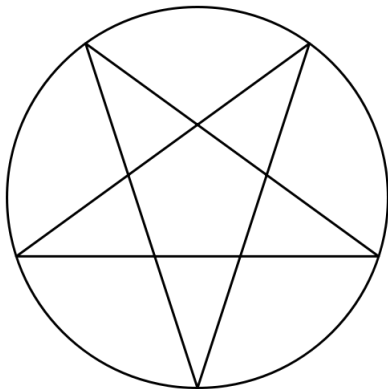
Naïve algorithm
Final algorithm
Results

Theoretical
points

Perspectives
Example $n = 7$
Face-width



Number of edges: $\binom{5}{2} = 10$



Number of edges: $\binom{5}{2} = 10$

Number of faces: $3e = 2f \Rightarrow f = \frac{2}{3}e = \frac{20}{3} \notin \mathbb{N}$

Definitions

Surfaces

Embedded graphs

The problem

Definition

Different cuttings

Preliminary results

Complete

graph

embeddings

Idea

Case of K_5

General case

Non-isomorphic
embeddings

Algorithms

and results

Naïve algorithm

Final algorithm

Results

Theoretical

points

Perspectives

Example $n \equiv 7$

Face-width

General case

Definitions

Surfaces

Embedded graphs

The problem

Definition

Different cuttings

Preliminary results

Complete graph

embeddings

Idea

Case of K_5

General case

Non-isomorphic
embeddings

Algorithms and results

Naïve algorithm

Final algorithm

Results

Theoretical points

Perspectives

Example $n \equiv 7$

Face-width

$$\chi(S) = v - e + f = n - \frac{n(n-1)}{2} + \frac{2}{3} \cdot \frac{n(n-1)}{2} = 2 - 2g$$

$$g = \frac{(n-3)(n-4)}{12}$$

$$(n-3)(n-4) \equiv 0[12] \Leftrightarrow n \equiv 0, 3, 4 \text{ or } 7[12]$$

Theorem (Ringel et Youngs, ~1970)

K_n can triangulate a surface if and only if $n \equiv 0, 3, 4 \text{ or } 7[12]$.

Vincent
DESPRE

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete
graph
embeddings

Idea
Case of K_5
General case

Non-isomorphic
embeddings

Algorithms
and results

Naïve algorithm
Final algorithm
Results

Theoretical
points

Perspectives
Example $n \equiv 7$
Face-width



" K_7 embedded on a torus," by sarah-marie belcastro (Hadley, MA)

K_7

Splitting Cycles in Triangulations

Vincent
DESPRE

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete graph embeddings

Idea
Case of K_5
General case

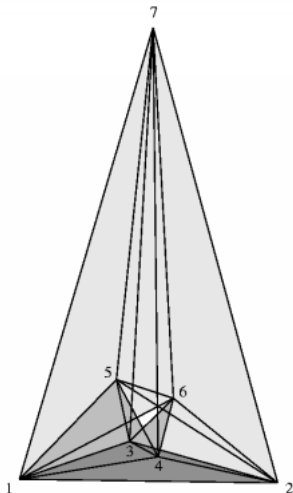
Non-isomorphic
embeddings

Algorithms and results

Naïve algorithm
Final algorithm
Results

Theoretical points

Perspectives
Example $n \equiv 7$
Face-width



Known results

We talk about K_n with $n = 12 \cdot s + i$.

- ⇒ $i = 0, 3, 4$ and 7 , there is an embedding (Ringel and Youngs 1972).
- ⇒ $i = 3, 4$ and 7 , there are $O(4^s)$ non-isomorphic embeddings (Korzhik and Voss 2001).
- ⇒ $n = 12$ (NO), there are exactly 182 200 non-isomorphic embeddings (Ellingham and Stephen 2003).
- ⇒ $n = 13$ (NO), there are exactly 243 088 286 non-isomorphic embeddings (Ellingham and Stephen 2003).
- ⇒ $i = 3$ and 7 and for an infinite numbers of values of n , there are at least $n^{c \cdot n^2}$ 2-colorable non-isomorphic embeddings (Granell and Knor 2012).

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete graph embeddings

Idea
Case of K_5
General case
Non-isomorphic embeddings

Algorithms and results

Naïve algorithm
Final algorithm
Results

Theoretical points

Perspectives
Example $n \equiv 7$
Face-width

Naïve algorithm

Splitting
Cycles in
Triangulations

Vincent
DESPRE

Definitions

Surfaces

Embedded graphs

The problem

Definition

Different cuttings

Preliminary results

Complete

graph
embeddings

Idea

Case of K_5

General case

Non-isomorphic
embeddings

Algorithms

and results

Naïve algorithm

Final algorithm

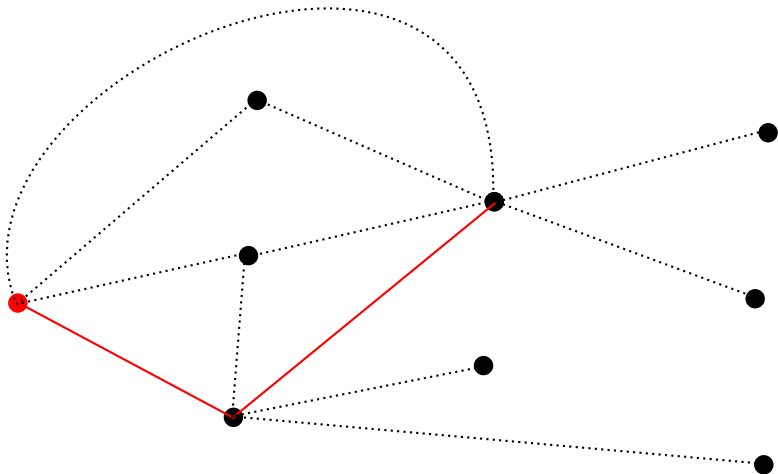
Results

Theoretical
points

Perspectives

Example $n \equiv 7$

Face-width



Naïve algorithm

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete graph embeddings

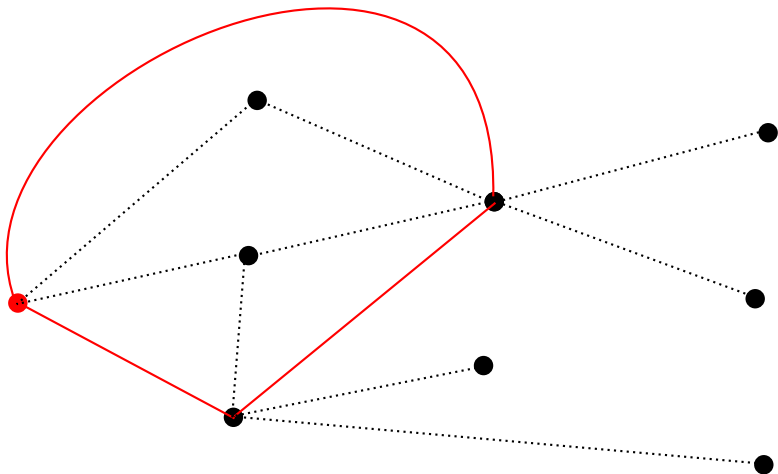
Idea
Case of K_5
General case
Non-isomorphic embeddings

Algorithms and results

Naïve algorithm
Final algorithm
Results

Theoretical points

Perspectives
Example $n \equiv 7$
Face-width



Computation time

Definitions

Surfaces

Embedded graphs

The problem

Definition

Different cuttings

Preliminary results

Complete graph embeddings

Idea

Case of K_5

General case

Non-isomorphic
embeddings

Algorithms and results

Naïve algorithm

Final algorithm

Results

Theoretical points

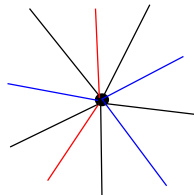
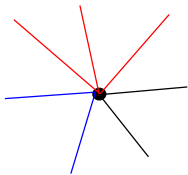
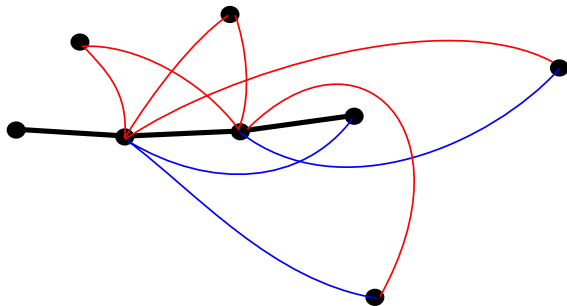
Perspectives

Example $n \equiv 7$

Face-width

n	12	15	16	19
Computation time	2 sec.	1 h.	12 h.	~10 years

The algorithm



Splitting
Cycles in
Triangulations

Vincent
DESPRE

Definitions

Surfaces

Embedded graphs

The problem

Definition

Different cuttings

Preliminary results

Complete

graph

embeddings

Idea

Case of K_5

General case

Non-isomorphic
embeddings

Algorithms

and results

Naïve algorithm

Final algorithm

Results

Theoretical
points

Perspectives

Example $n \equiv 7$

Face-width

Computation time

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete graph embeddings

Idea
Case of K_5
General case
Non-isomorphic embeddings

Algorithms and results

Naïve algorithm
Final algorithm
Results

Theoretical points

Perspectives
Example $n \equiv 7$
Face-width

n	12	15	16	19	31	43
duration	10 s.	20 s.	25 sec.	1 m.	25 m.	10 h.

Results, case of K_{19}

Genera	Shortest splitting cycle	#cycles
$1 \rightarrow 19$	10	2080
$2 \rightarrow 18$	14	1374
$3 \rightarrow 17$	18	278
$4 \rightarrow 16$	19	38
$5 \rightarrow 15$	\perp	0
$6 \rightarrow 14$	\perp	0
$7 \rightarrow 13$	\perp	0
$8 \rightarrow 12$	\perp	0
$9 \rightarrow 11$	\perp	0
$10 \rightarrow 10$	\perp	0

Definitions

Surfaces

Embedded graphs

The problem

Definition

Different cuttings

Preliminary results

Complete
graph

embeddings

Idea

Case of K_5

General case

Non-isomorphic
embeddings

Algorithms
and results

Naïve algorithm

Final algorithm

Results

Theoretical
points

Perspectives

Example $n \equiv 7$

Face-width

Splitting Cycles in Triangulations

Vincent
DESPRE

Definitions

Surfaces

Embedded graphs

The problem

Definition

Different cuttings

Preliminary results

Complete

graph
embeddings

Idea

Case of K_5

General case

Non-isomorphic
embeddings

Algorithms
and results

Naïve algorithm

Final algorithm

Results

Theoretical
points

Perspectives

Example $\tau_7 \equiv 7$

Face-width

•	15	19	27	28	31	39	40	43	36 NO
1	8	10	12	12	8	12	10	8	7
2	11	14	16	17	13	15	15	11	8
3	12	18	19	18	17	20	18	12	13
4	13	19	20	⊥	19	24	19	15	14
5	14	⊥	27	⊥	20	26	24	18	17
6		⊥	⊥	⊥	25	30	26	23	19
7		⊥	⊥	⊥	26	32	28	26	21
8		⊥	⊥	⊥	26	⊥	30	26	22
9		⊥	⊥	⊥	⊥	⊥	33	28	24
10		⊥	⊥	⊥	⊥	⊥	35	34	27
11			⊥	⊥	⊥	⊥	36	34	28
12			⊥	⊥	⊥	⊥	38	35	29
13			⊥	⊥	⊥	⊥	40	35	⊥
14			⊥	⊥	⊥	⊥	⊥	⊥	⊥

K_{31} : only 78 of the 130 980 splitting cycles are separating the surface into 2 pieces of genera 8 and 55.

Perspectives

Definitions

Surfaces

Embedded graphs

The problem

Definition

Different cuttings

Preliminary results

Complete graph

embeddings

Idea

Case of K_5

General case

Non-isomorphic
embeddings

Algorithms and results

Naïve algorithm

Final algorithm

Results

Theoretical points

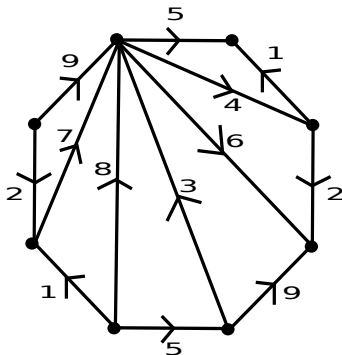
Perspectives

Example $n \equiv 7$

Face-width

- ⇒ Prove that the embedding of K_{19} do not have a splitting cycle that separates the surfaces into 2 pieces of genera 10.
- ⇒ Give a complexity for the algorithm in the case of complete graphs.

Example $n \equiv 7$



Rotation scheme of 0:

$$0 : (9, 7, 8, 3, 13, 15, 14, 11, 18, 4, 17, 10, 16, 5, 1, 12, 2, 6)$$

The other rotation scheme comes from the addition in $\mathbb{Z}/19\mathbb{Z}$:

$$1 : (10, 8, 9, 4, 14, 16, 15, 12, 0, 5, 18, 11, 17, 6, 2, 13, 3, 7)$$

Face-width

Let G be a graph with a 2-cell embedding on a surface S .

Definition (Face-width)

The **face-width** of G is the minimum number of intersection between any non-contractile cycle of S and the graph G .

Theorem

Let f a face, $k \in \mathbb{N}^*$ and $E_{f,k} = \{f', d(f, f') \leq k\}$. If $fw(G) > 2k + 1$ then there is a disk D such that:

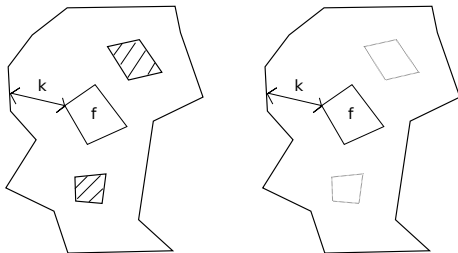
$$E_{f,k} \subset D \text{ and } \partial D \subset \partial E_{f,k}$$

Face-width

Theorem

Let f a face, $k \in \mathbb{N}^*$ and $E_{f,k} = \{f', d(f, f') \leq k\}$. If $fw(G) > 2k + 1$ then there is a disk D such that:

$$E_{f,k} \subset D \text{ and } \partial D \subset \partial E_{f,k}$$



Conjectures

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete graph embeddings

Idea
Case of K_5
General case
Non-isomorphic
embeddings

Algorithms and results

Naïve algorithm
Final algorithm
Results

Theoretical points

Perspectives
Example $n \equiv 7$
Face-width

Conjecture (Zha, 1991)

Every combinatorial map of genus at least 2 and face-width at least 3 has a splitting cycle.

Conjecture (Zha, 1991)

Every combinatorial map of genus $g \geq 2$ and face-width at least 3 has a splitting cycle that split the surface into one surface of genus h and one of genus $g - h$, for all $0 < h < g$.

Optimality

Splitting
Cycles in
Triangulations

Vincent
DESPRE

Definitions

Surfaces

Embedded graphs

The problem

Definition

Different cuttings

Preliminary results

Complete

graph
embeddings

Idea

Case of K_5

General case

Non-isomorphic
embeddings

Algorithms
and results

Naïve algorithm

Final algorithm

Results

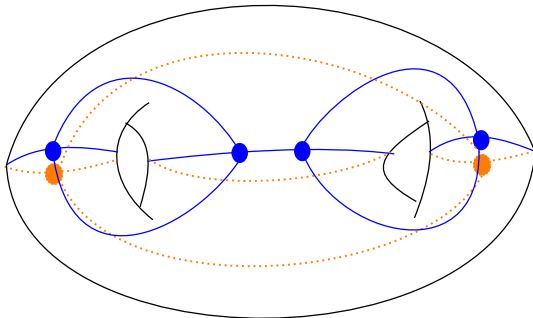
Theoretical
points

Perspectives

Example $n \equiv 7$

Face-width

A combinatorial map of genus 2 and face-width 2 without splitting cycle:



Results

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete graph embeddings

Idea
Case of K_5
General case
Non-isomorphic
embeddings

Algorithms and results

Naïve algorithm
Final algorithm
Results

Theoretical points

Perspectives
Example $n \equiv 7$
Face-width

In the orientable case:

- ➔ The conjecture is true for face-width at least 6 (Zha and Zhao 1993).
- ➔ The conjecture is true for genus 2 and face-width at least 4 (Ellingham and Zha 2003).

In the non-orientable case:

- ➔ The conjecture is true for face-width at least 5 (Zha and Zhao 1993).
- ➔ The conjecture is true for genus 2 (Robertson and Thomas 1991).

Definitions

Surfaces
Embedded graphs

The problem

Definition
Different cuttings
Preliminary results

Complete
graph
embeddings

Idea
Case of K_5
General case
Non-isomorphic
embeddings

Algorithms
and results

Naïve algorithm
Final algorithm
Results

Theoretical
points

Perspectives
Example $n \equiv 7$
Face-width

This is
the end