

# Drawing Graphs with Few Slopes: State of the Art & New Results

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Joint work with Emilio Di Giacomo and Giuseppe Liotta

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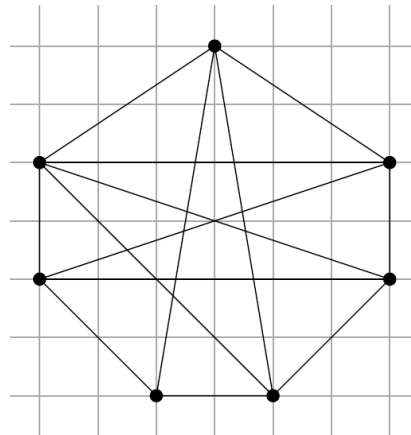


**ANR EGOS - Université Bordeaux, 5-7 November 2014**

# Preliminaries

**(planar) straight-line drawing of a graph** = (crossing-free) drawing on the *plane* such that vertices are depicted as points and edges as *straight-line segments* that connect the corresponding endpoints

**(planar) straight-line grid drawing** = (crossing-free) straight-line drawing such that every point representing a vertex has integer coordinates



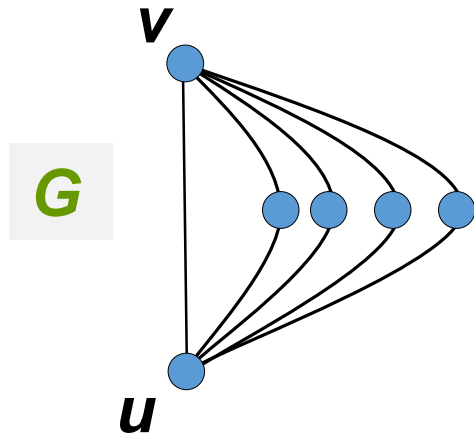
Problem definition,  
motivation, and state of the art

# The (Planar) Slope Number of a Graph $G$

minimum number of **edge slopes** to compute a (planar) **straight-line** drawing of  $G$

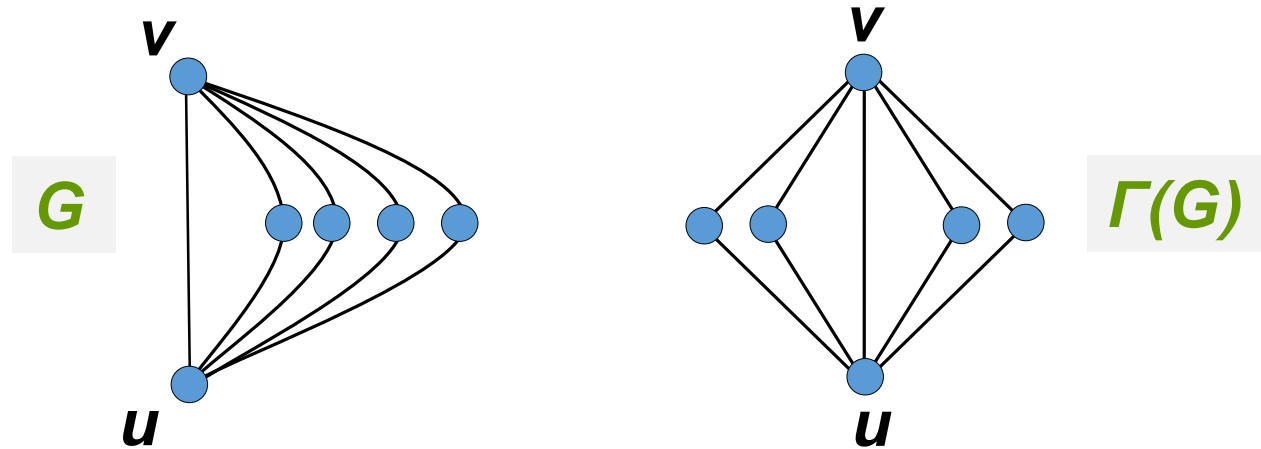
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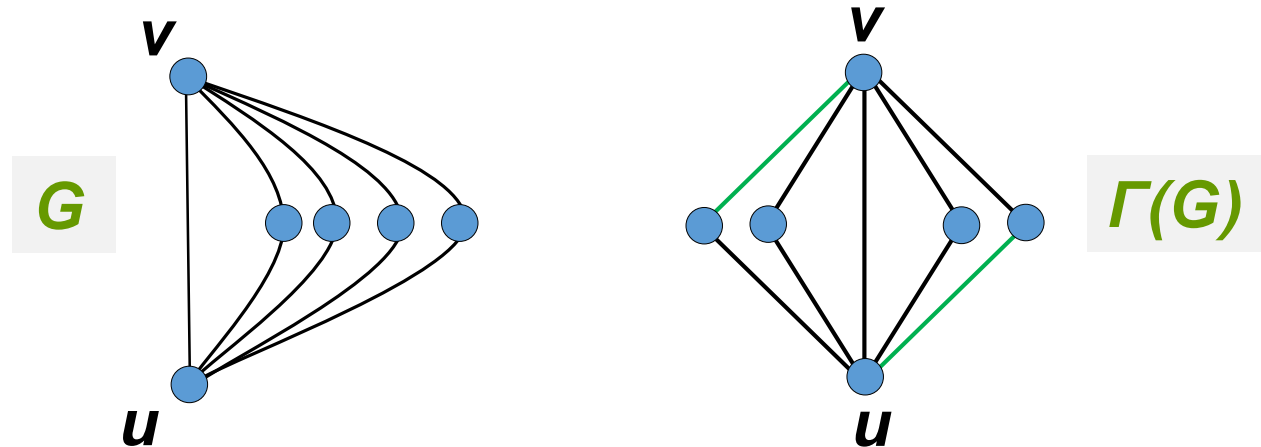
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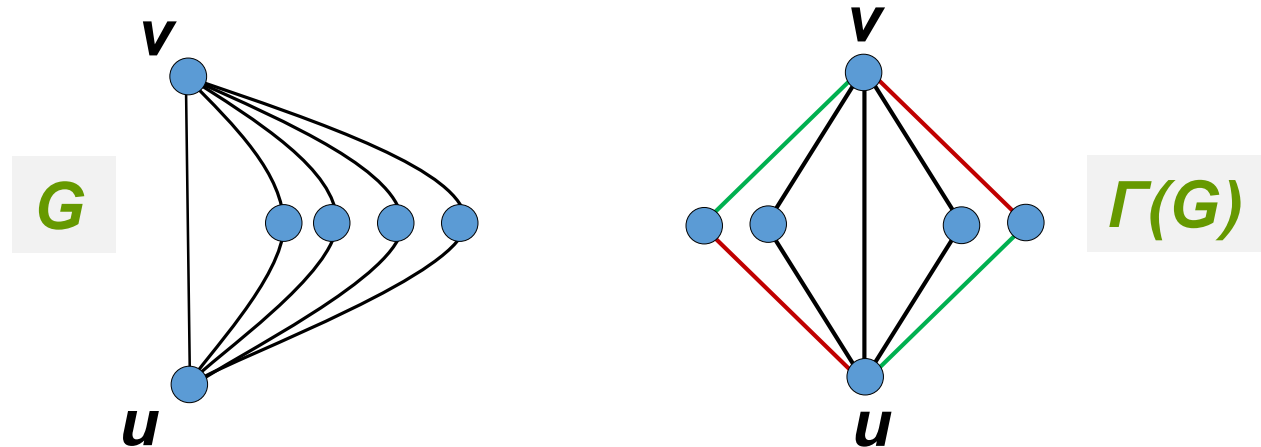
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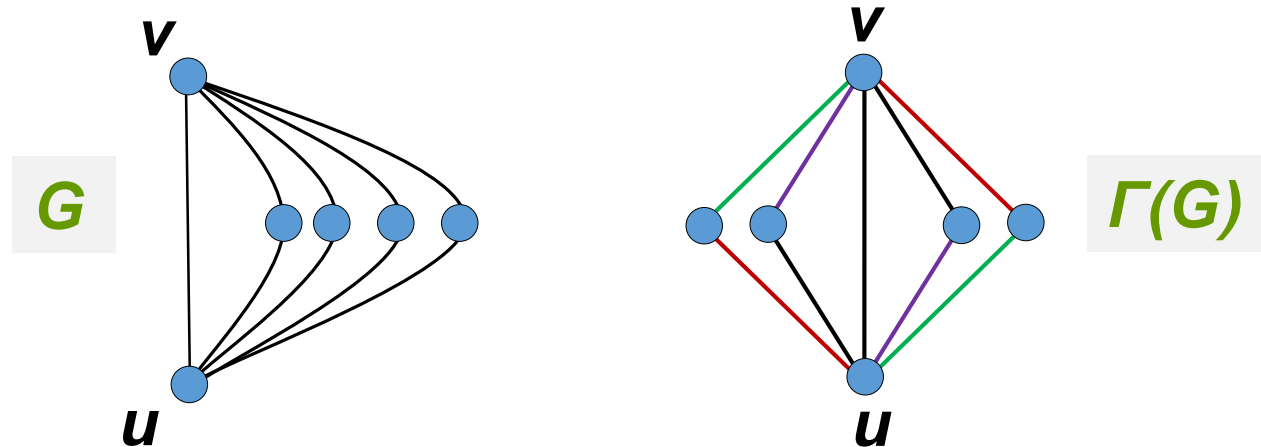
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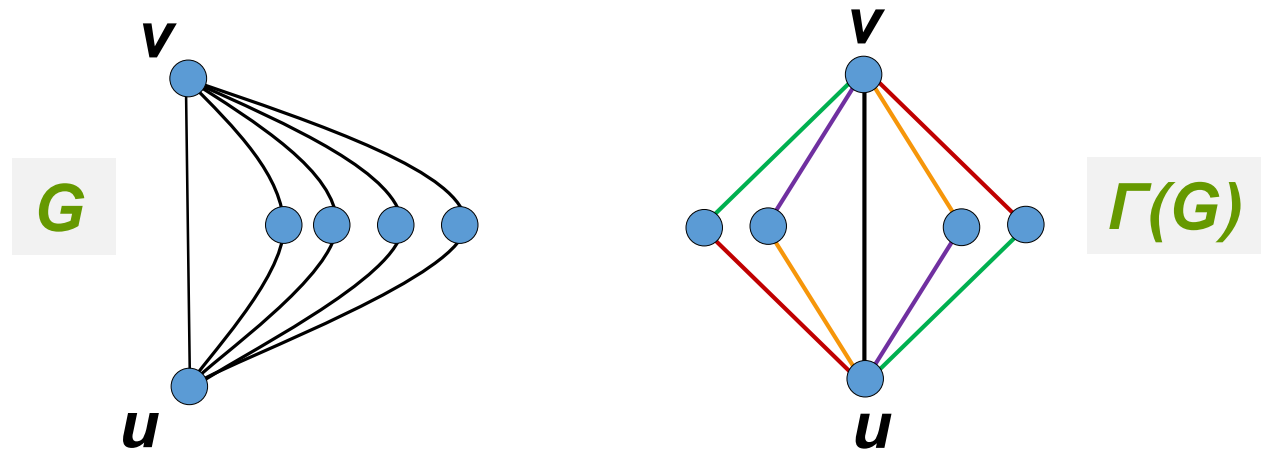
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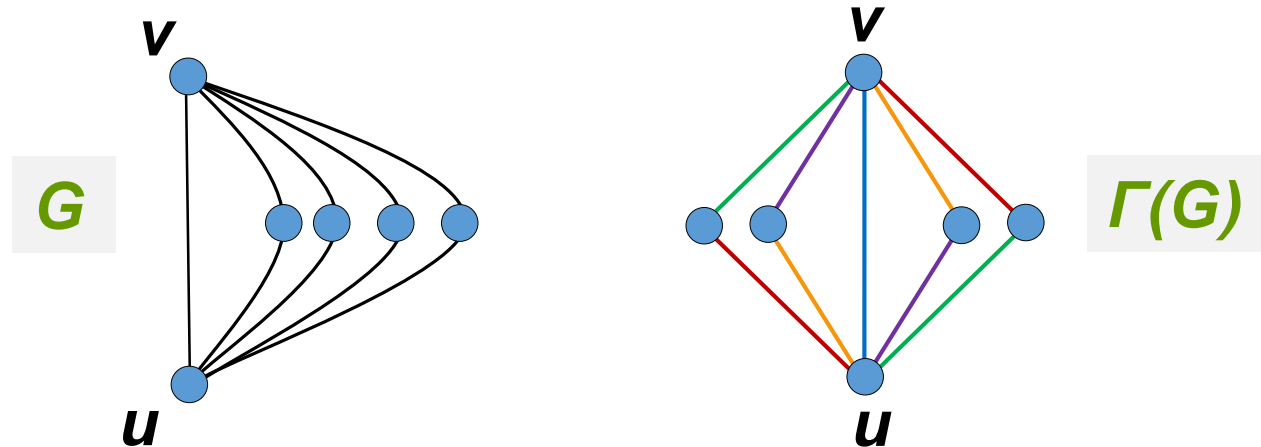
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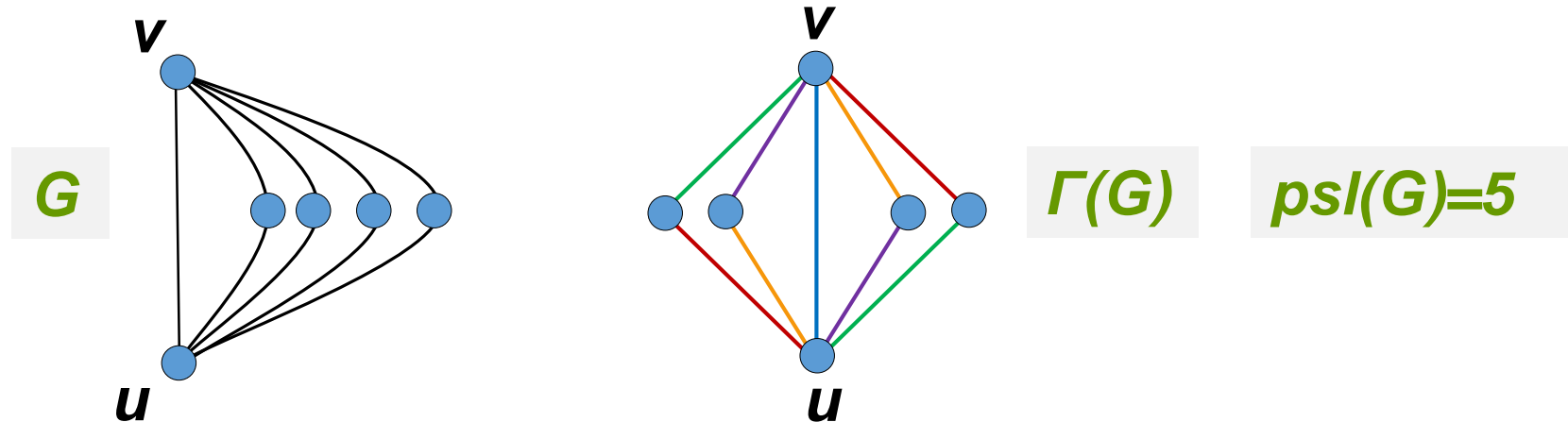
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Given a family of (planar) graphs, find upper and lower bounds on the (planar) slope number of any graph  $G$  in the family

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we focus on graphs of **bounded degree  $\Delta$**

$$\left\lceil \frac{\Delta}{2} \right\rceil \leq sl(G) \leq m \quad sl(G) \leq psl(G)$$

# Motivation and State of the Art

Problem introduced by Wade and Chu in 1994:

“some lines produce an undesired zig-zag effect...only lines with certain slopes appear smooth”...more in general **minimizing the number of slopes is a desirable aesthetics requirement**

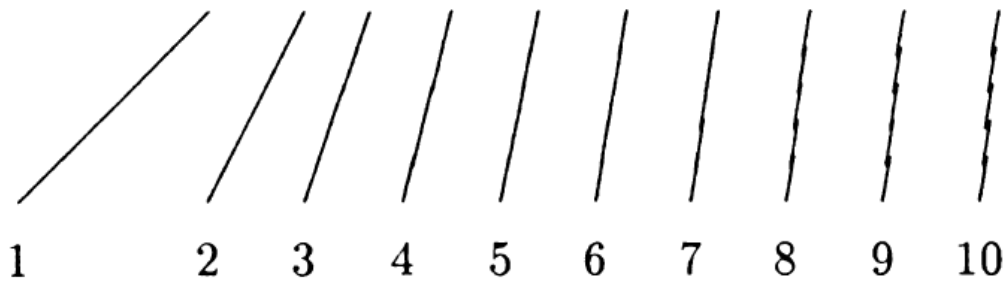


FIGURE 1. Lines with slopes 1 to 10 (LaTeX).

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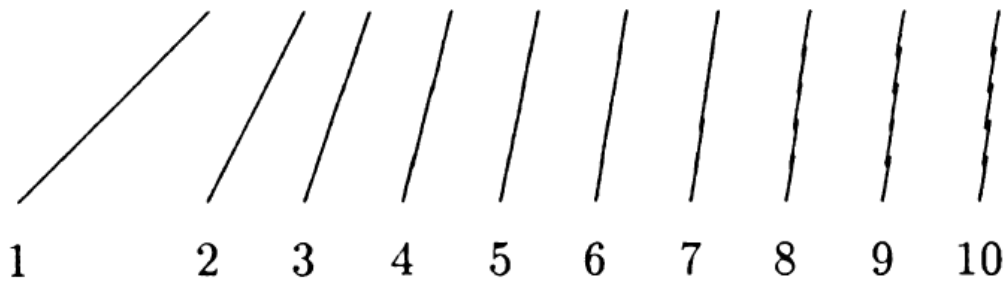
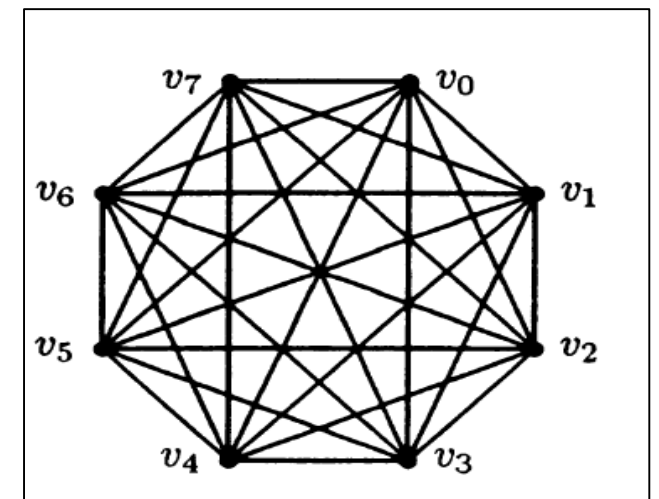


FIGURE 1. Lines with slopes 1 to 10 (LaTeX).

**Wade and Chu:** The slope number of  $K_n$  is  $n$   
[The Computer Journal, 1994]





# State of the Art: planar graphs

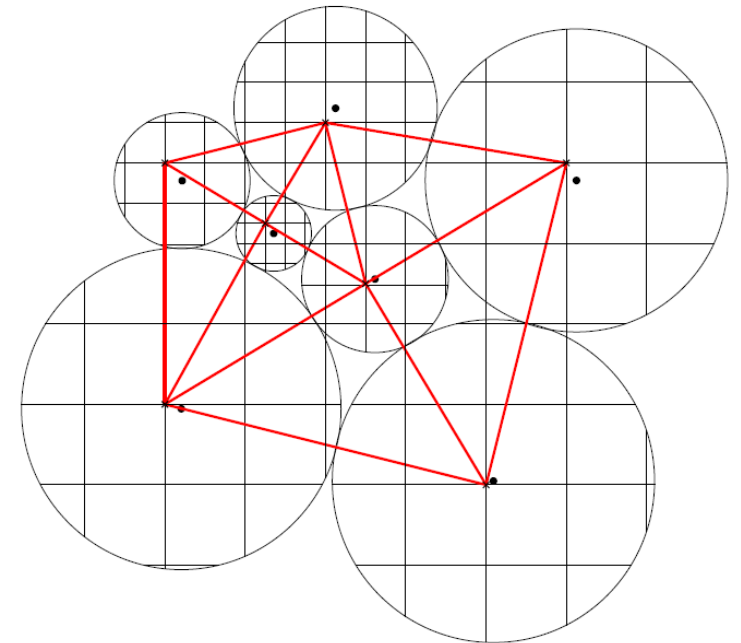
Keszegh, Pach and Pálvölgyi: **planar** bounded-degree graphs have **bounded planar slope number** -  $psl(G) \in O(2^{O(\Delta)})$  -  
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It relies on the following result.

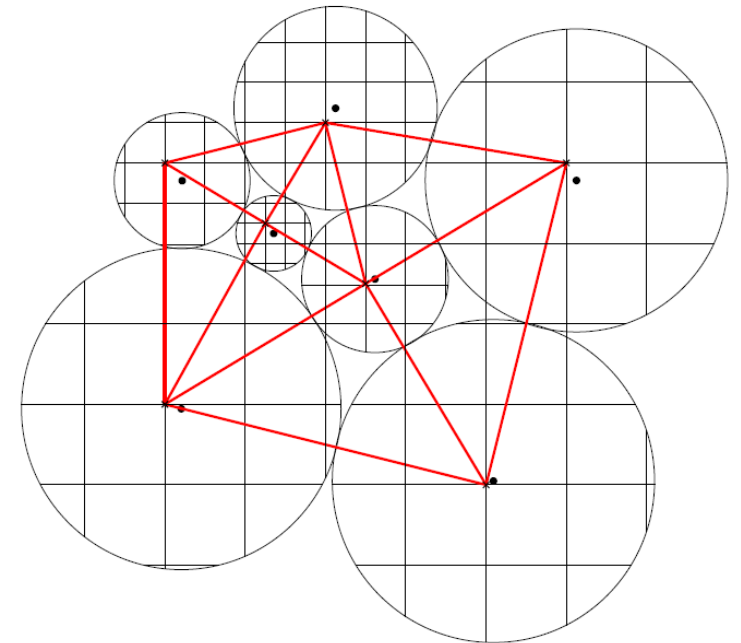
Malitz and Papakostas: every triangulated planar graph with maximum degree  $\Delta$  has a **disk contact representation** such that **the ratio of the radii of any two tangent disks is at least  $\alpha^{\Delta-2}$  ( $\alpha \approx 0,15$ )**  
[SIAM J. On Discrete Mathematics, 1994]



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- 1 – Triangulate the graph by adding vertices and edges so that the degree of each vertex increases only by a factor of at most three
- 2 - Compute a disk contact representation
- 3 – Place in each disk a grid whose side length is proportional to the radius of the disk
- 4 – Replace each disk center by the nearest grid point and connect the corresponding pairs of grid points

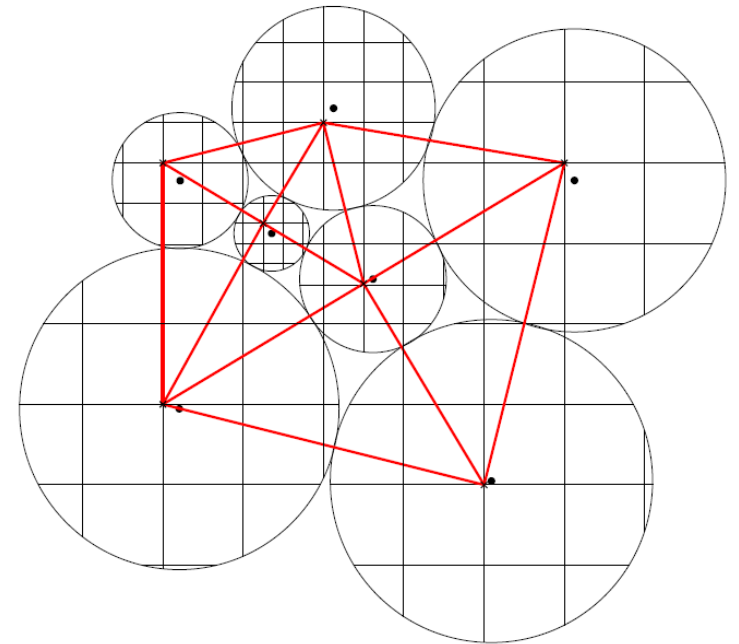


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Since the grids are of "comparable size", it can be proved that the number of slopes is bounded in  $\Delta$

However the proof is not constructive



# State of the Art: planar graphs

| Graphs                         | Upper bound        | Lower bound | References                      |
|--------------------------------|--------------------|-------------|---------------------------------|
| Planar                         | $O(2^{O(\Delta)})$ | $3\Delta-6$ | Keszegh <i>et al.</i> , 2013    |
| Planar partial 3-trees         | $O(\Delta^5)$      | $\Delta+1$  | Jelínek <i>et al.</i> , 2013    |
| Outer 1-planar                 | $O(\Delta^2)$      | $\Delta+1$  | Di Giacomo <i>et al.</i> , 2014 |
| Partial 2-trees                | $2\Delta$          | $\Delta+1$  | Lenhart <i>et al.</i> , 2013    |
| Outerplanar                    | $\Delta-1$         | $\Delta-1$  | Knauer <i>et al.</i> , 2014     |
| Subcubic ( $\Delta=3$ ) planar | 4                  | 4           | Di Giacomo <i>et al.</i> , 2014 |

**Reserch direction:** Can we improve the upper bound for planar graphs?

The planar slope number of  
subcubic ( $\Delta=3$ ) planar graphs

# Previous results

In 1992, Kant claimed that every **subcubic planar graph** (except for  $K_4$ ) has a straight-line drawing with at most **3 slopes** (except for one bent edge) and with every **vertex angle at least  $\pi/3$** .

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**Kant & (ind.) Dujmović *et al.***: If  $G$  is cubic and **3-connected**, then  **$psl(G) = 3$ , but 3 edges on the outerface that need bends**  
[WG 1992; Computational Geometry, 2007]

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**Jelínek *et al.***: If  $G$  is subcubic and **series-parallel**, then  **$psl(G) = 3$**   
[GD 2009]

# New result

Jelínek *et al.*: Can we find a tight bound for the planar slope number of subcubic planar graphs?

[GD 2009]

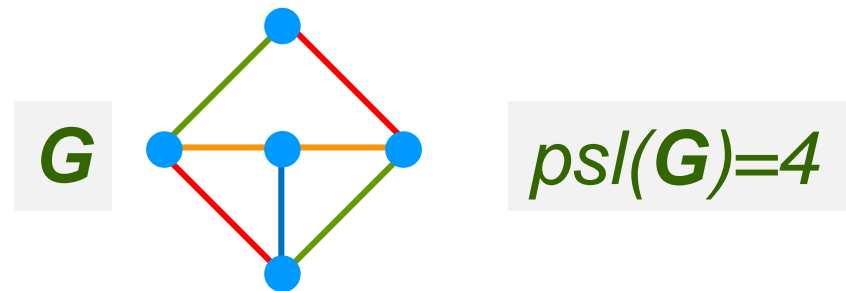
# New result

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Di Giacomo, Liotta, M.: If  $G$  is a subcubic planar graph and  $n \geq 5$ , then  $psl(G) \leq 4$  and this bound is **worst case optimal**

[LATIN 2014]



# An interesting byproduct

**Formann *et al.*:** Does every subcubic planar graph has a planar straight-line drawing such that the **smallest vertex angle is a constant?**

[FOCS 1990; SIAM J. on Computing, 1993]

**Kant:** Does every subcubic planar graph has a planar straight-line drawing such that the **smallest vertex angle is at least  $\frac{\pi}{4}$ ?**

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**Di Giacomo, Liotta, M.:** If  $G$  is a subcubic planar graph with  $n$  vertices, then  $G$  has a straight-line planar drawing with **smallest vertex angle at least  $\frac{\pi}{4}$**  on a grid of size  $O(n^2)$

[LATIN 2014]

# Overview of proof technique

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Prove  $psl(G) \leq 4$  for 2-connected subcubic graphs

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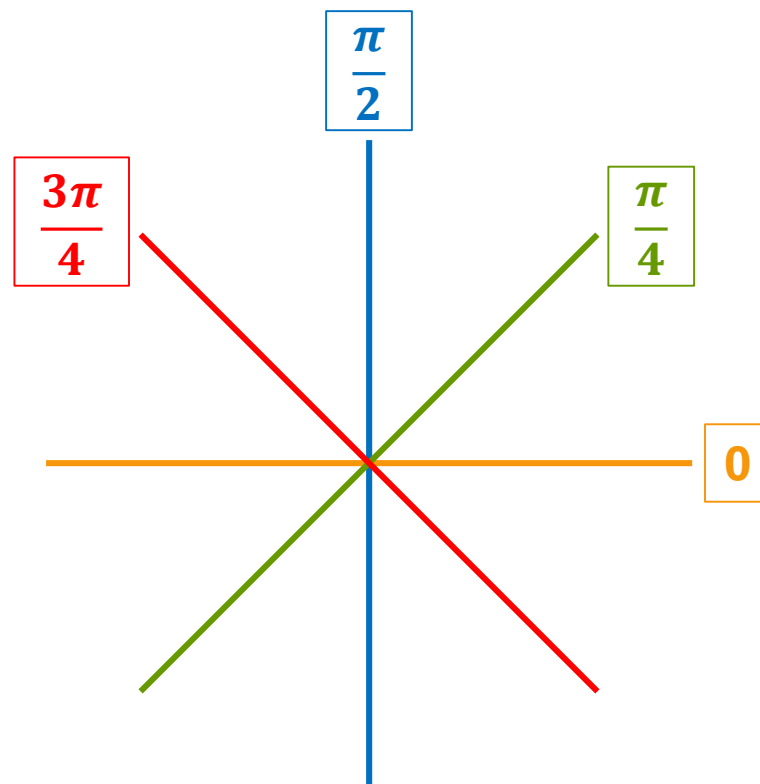
# Overview of proof technique

Prove  $psl(G) \leq 4$  for 2-connected subcubic graphs

Ingredients:

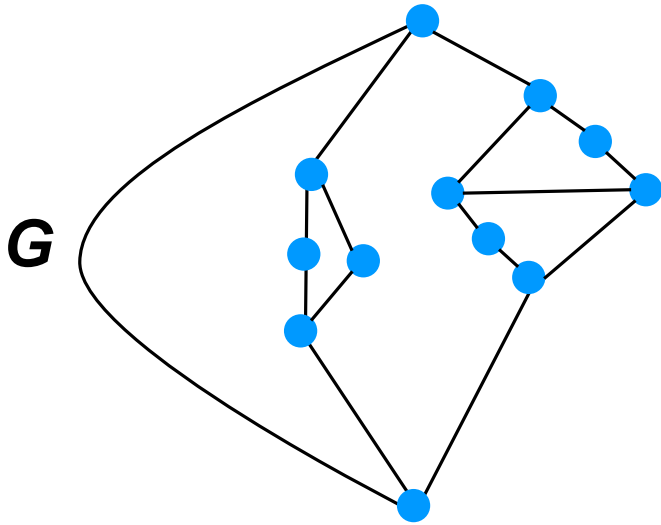
- The four canonical slopes  $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$
- SPQR-tree decomposition
- Canonical ordering

# The four canonical slopes



# SPQR-tree decomposition

**G** is a **2-connected** (subcubic planar) graph

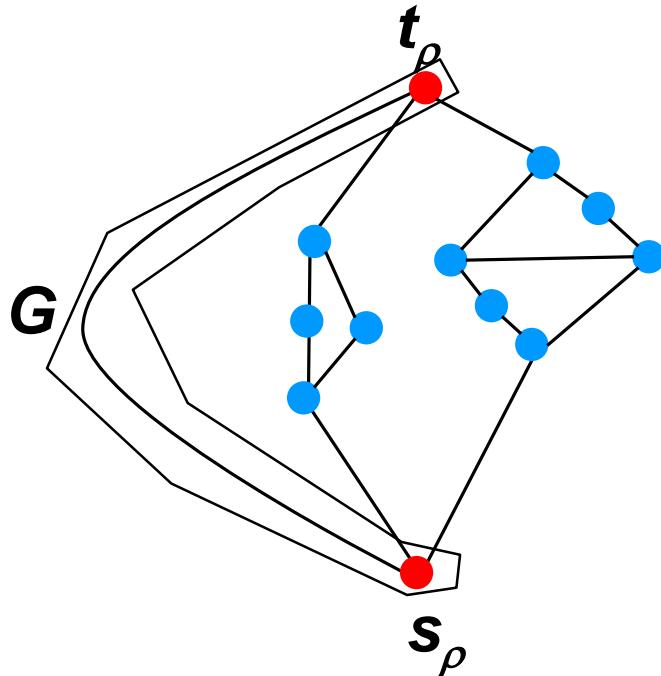


Recursive decomposition based on the **split pairs** (separation pairs or pairs of adjacent vertices)

Four types of nodes: S,P,Q,R

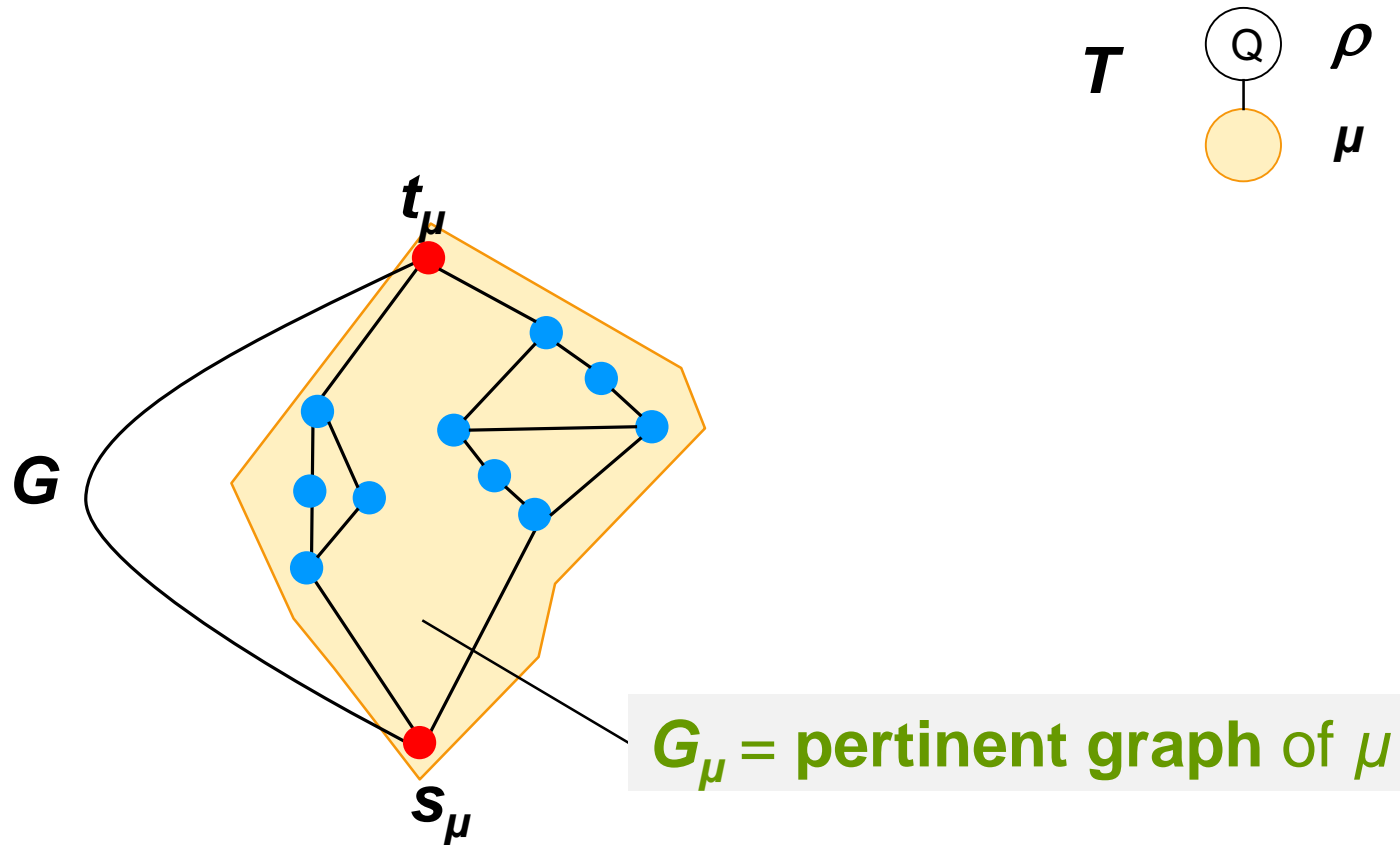
# SPQR-tree decomposition

$T$   $\textcircled{Q}$   $\rho$



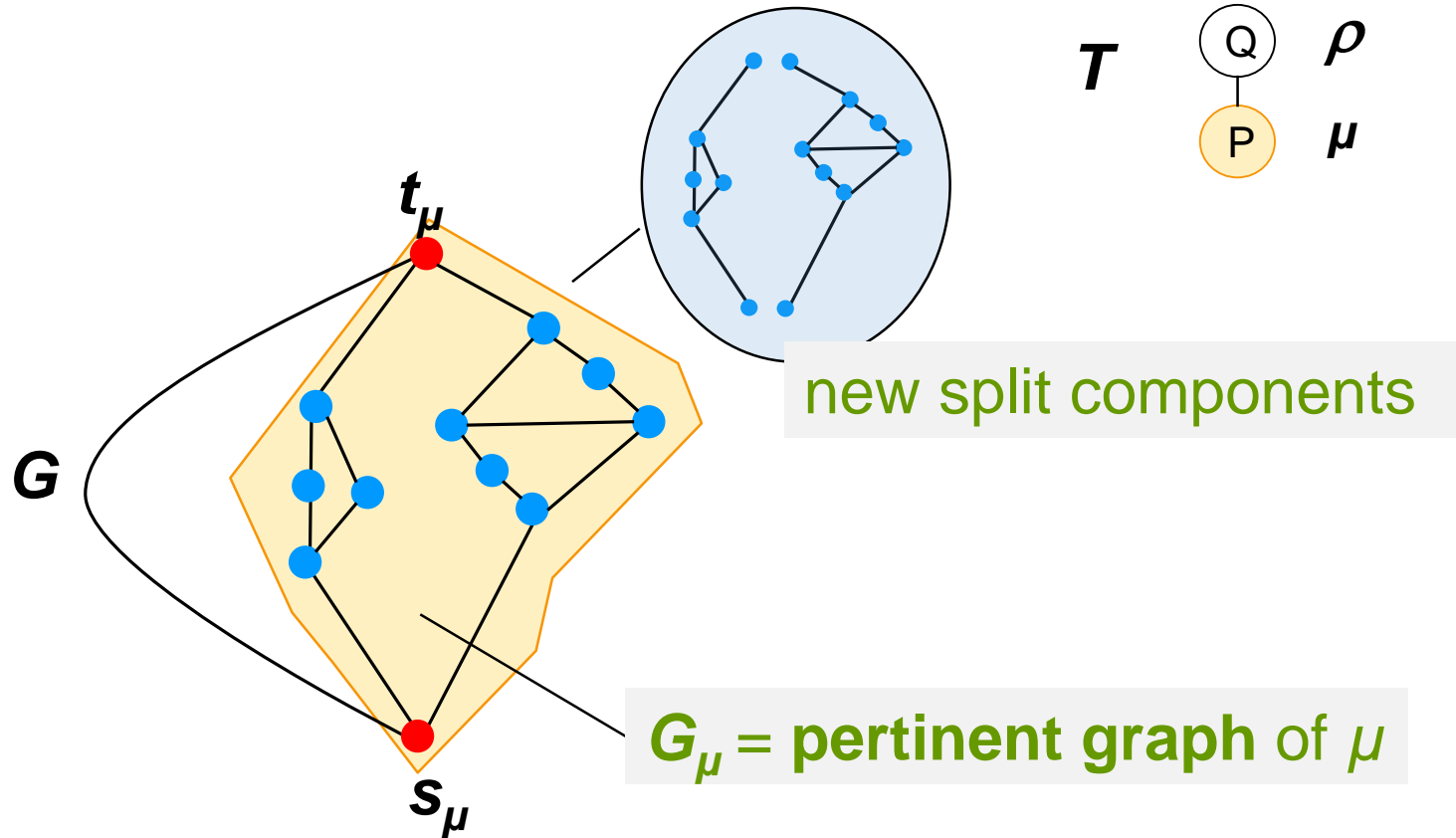
The root of  $T$  is **Q-node**, whose corresponding edge is the **reference edge**

# SPQR-tree decomposition



The **pertinent graph**  $G_\mu$  of  $\mu$  is the subgraph of  $G$  whose SPQR-tree is the subtree of  $T$  rooted at  $\mu$

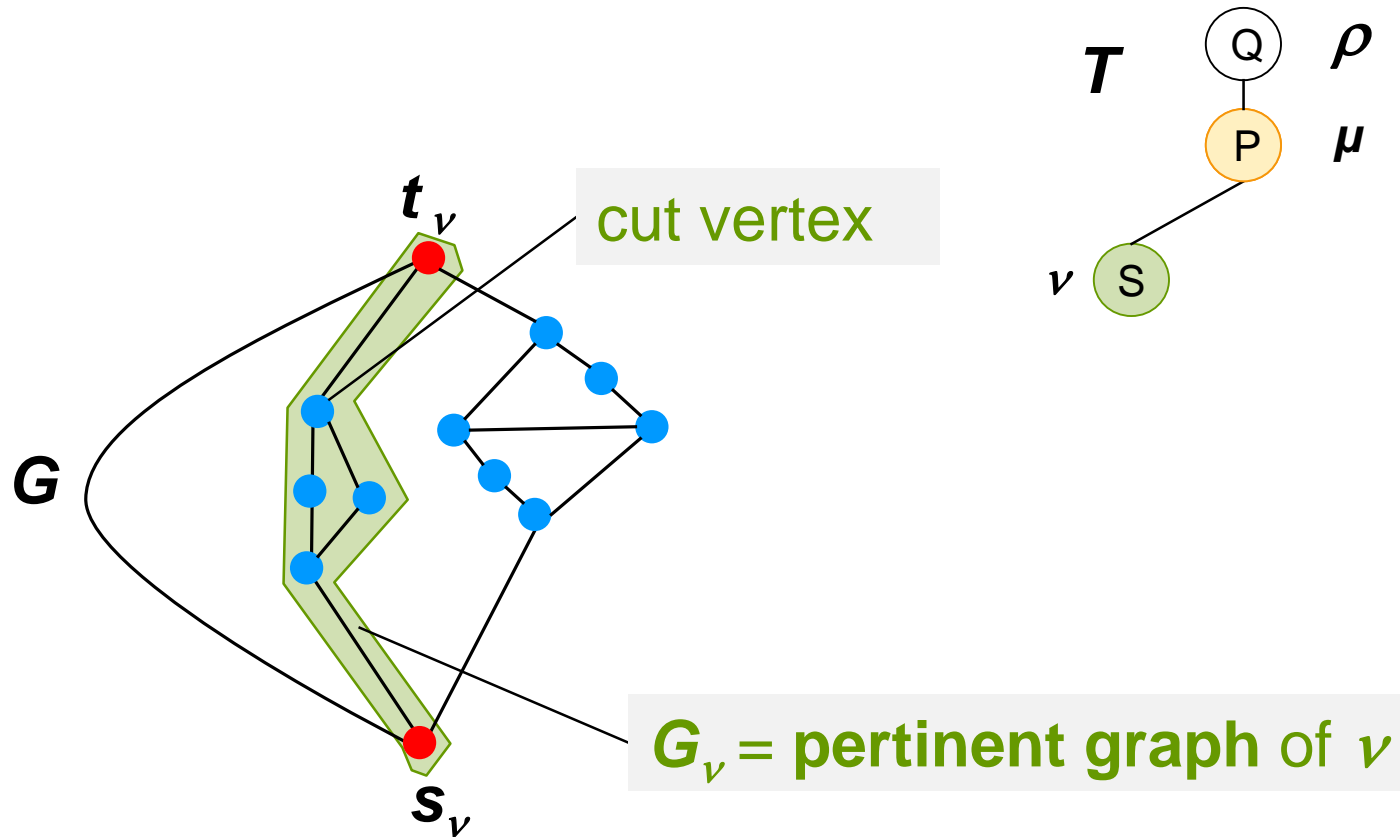
# SPQR-tree decomposition



**P-node** = the split pair  $(s_\mu, t_\mu)$  of  $G_\mu$  generates at least two split components

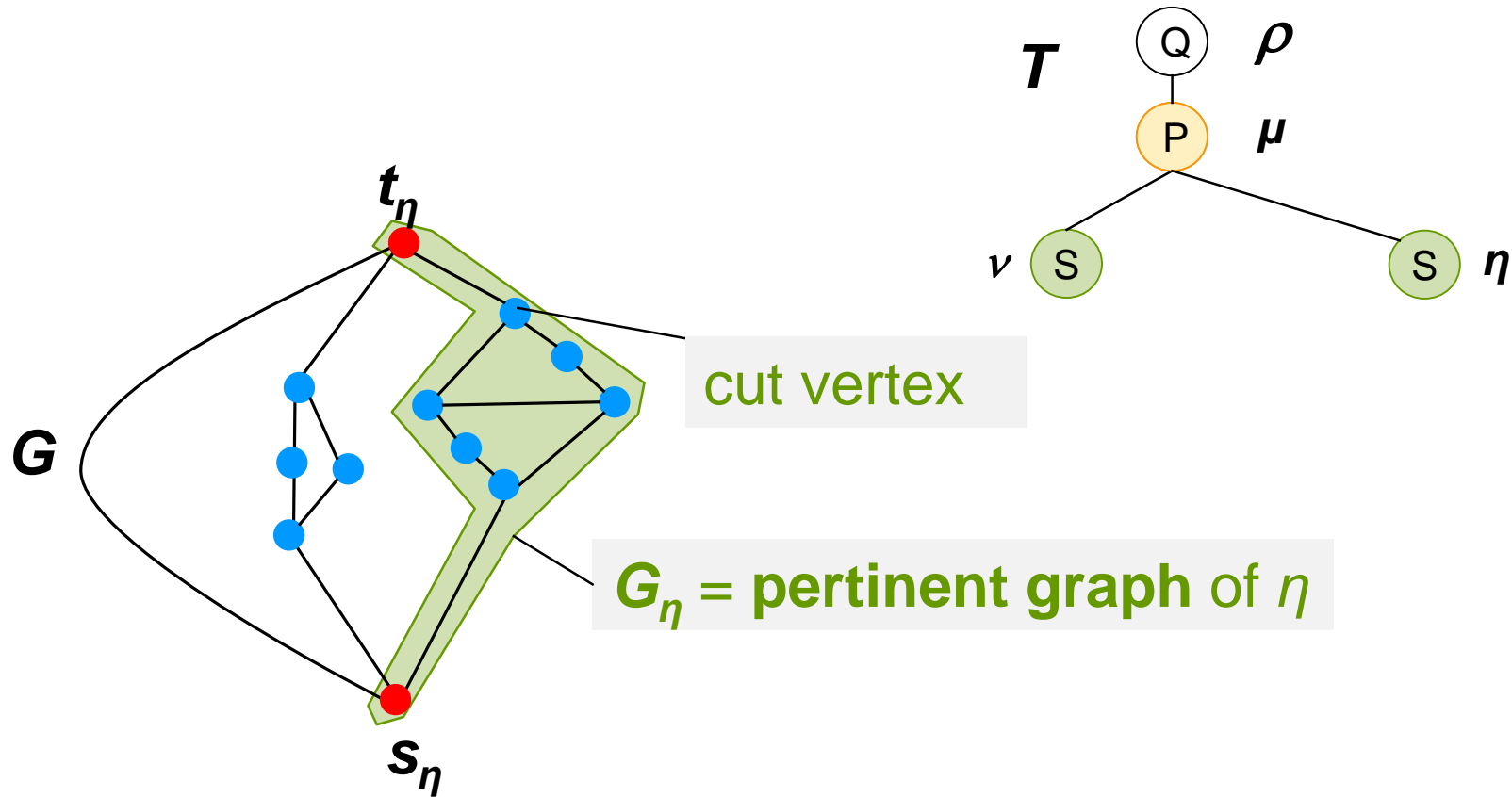


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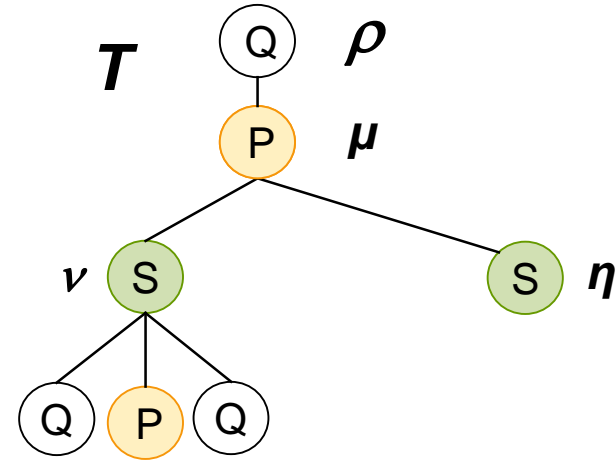
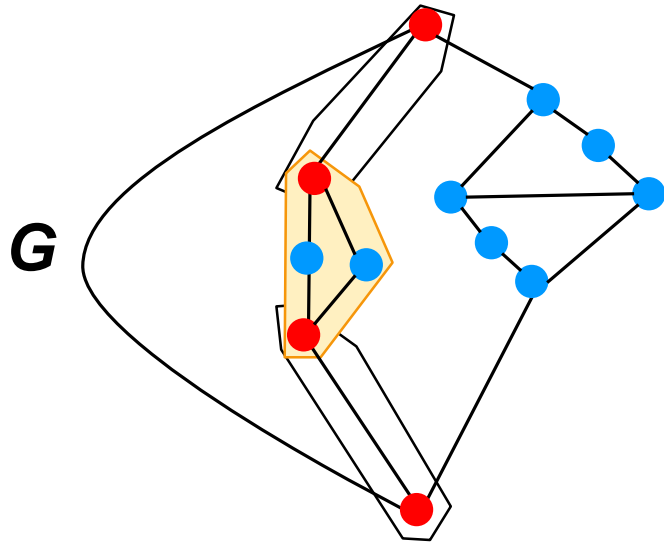


**S-node** =  $G_v$  is not 2-connected and contains at least a cut vertex

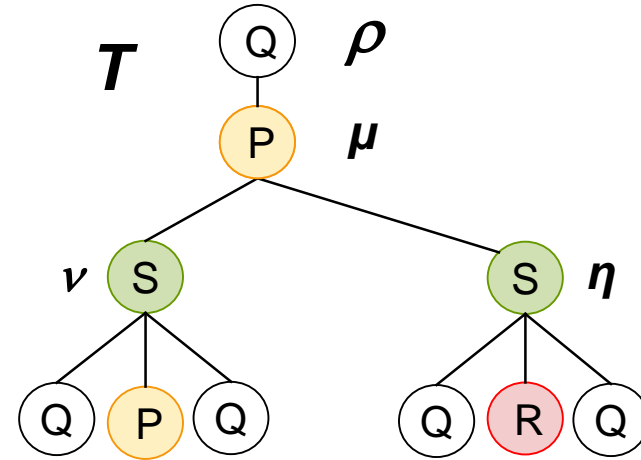
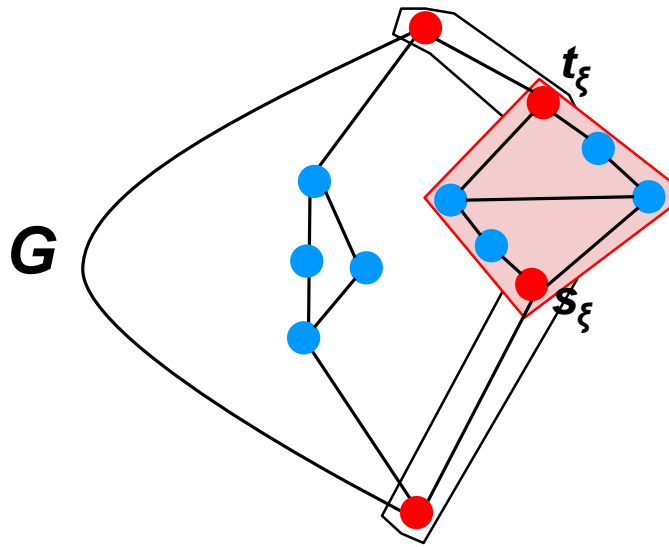
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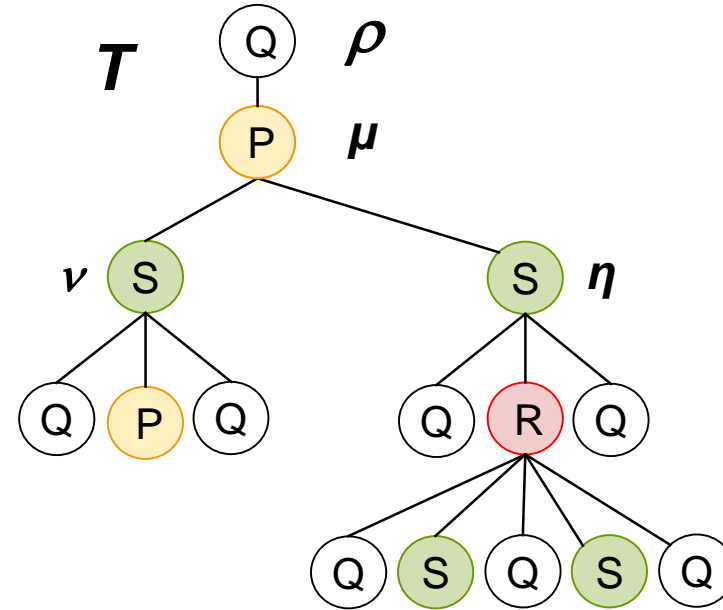
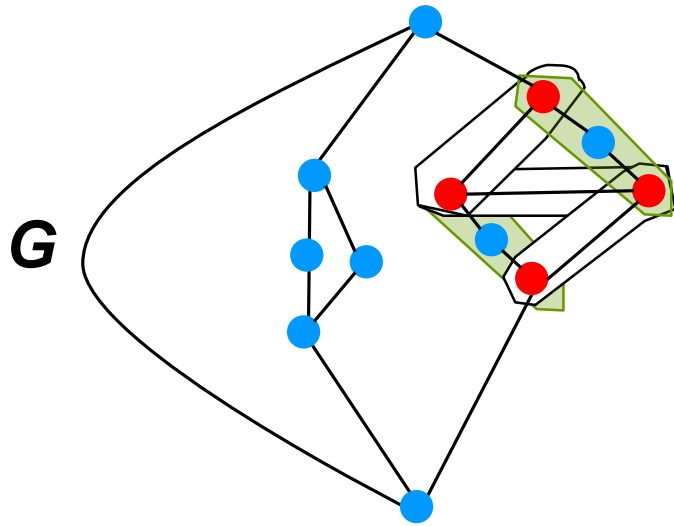


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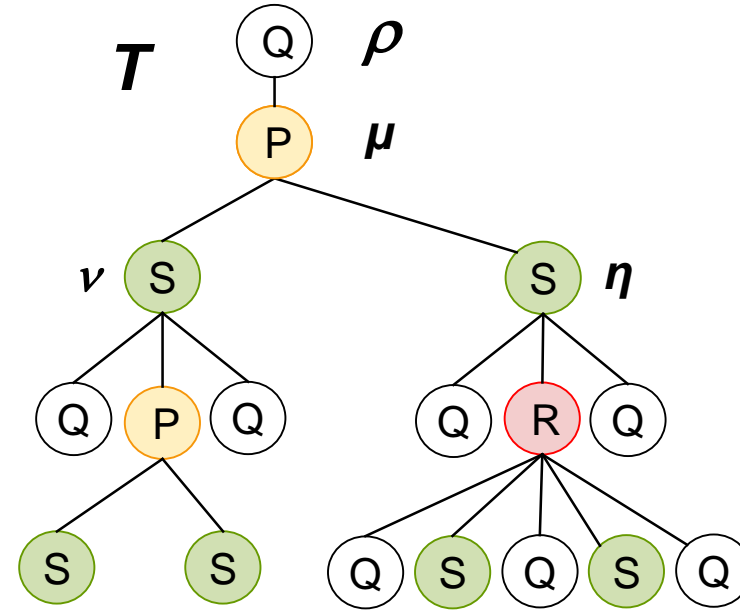
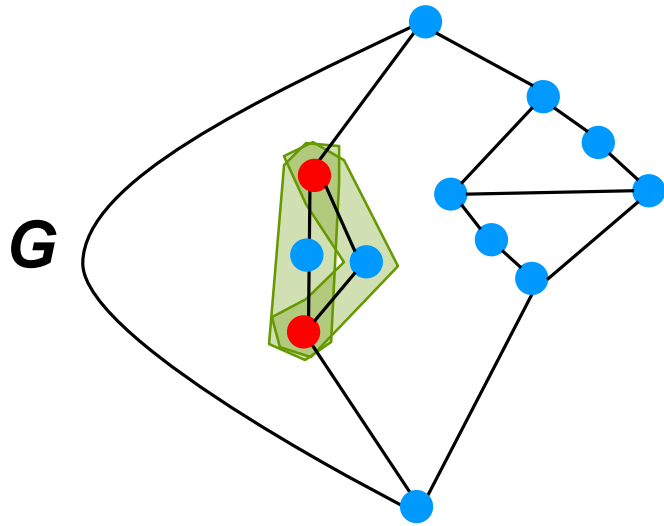


**R-node** = none of the other case is applicable. The decomposition recurs on each maximal split component.

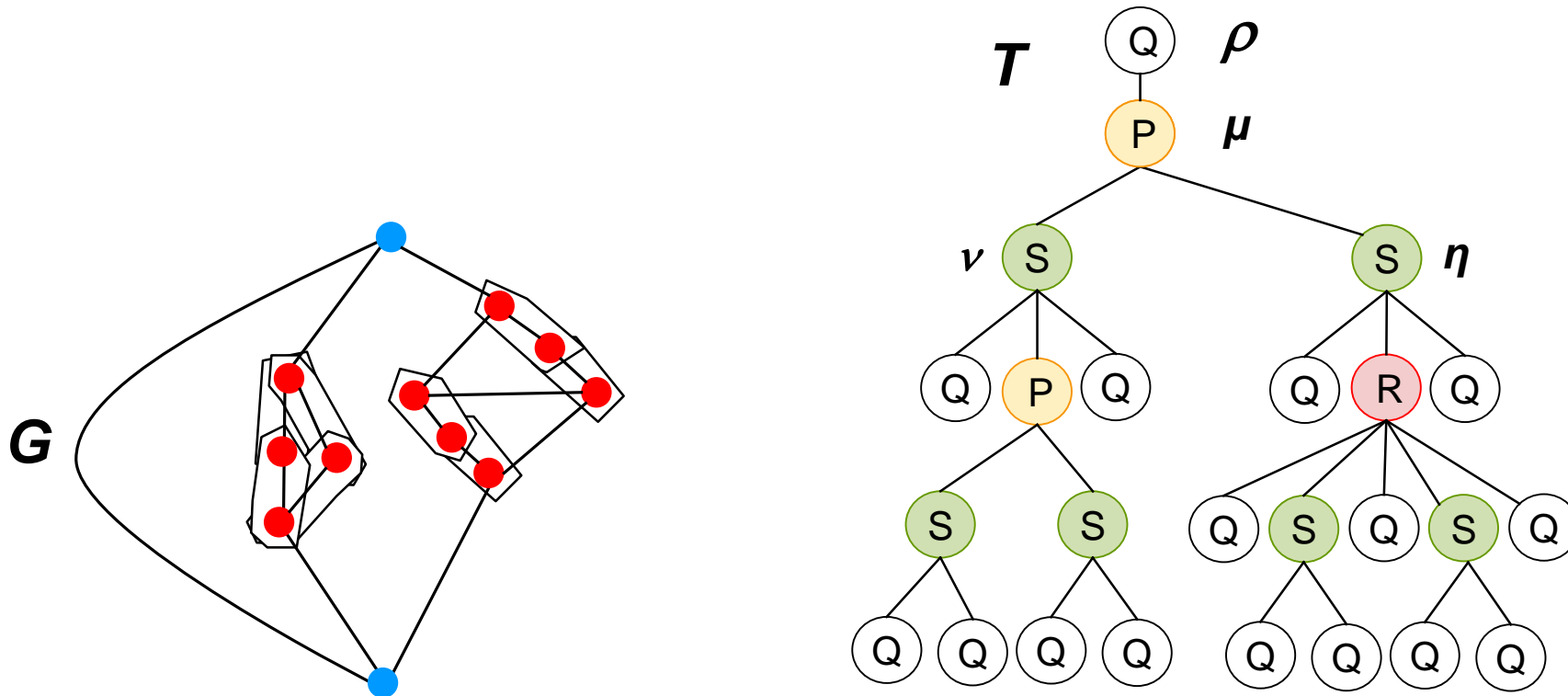
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The leaves of  $T$  are all Q-nodes and represents the edges in the graph  $G$ .

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Visit the nodes of  $T$  in a bottom-up order:

for each node  $\mu$  of  $T$

draw  $G_\mu$  by suitably combining the drawings of the children of  $\mu$

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## Invariants:

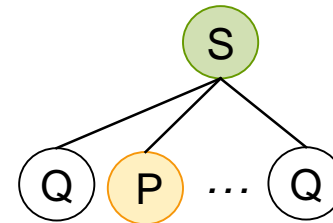
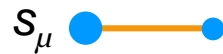
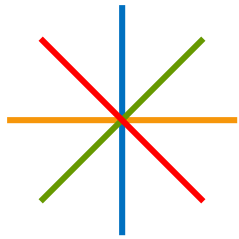
1.  $\Gamma_\mu$  is planar and uses the slopes  $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$
2.  $\Gamma_\mu$  is contained in an **isosceles right triangle**
3. The area of  $\Gamma_\mu$  is  $O(n_\mu) \times O(n_\mu)$



# How to draw S-nodes

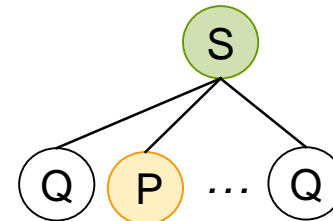
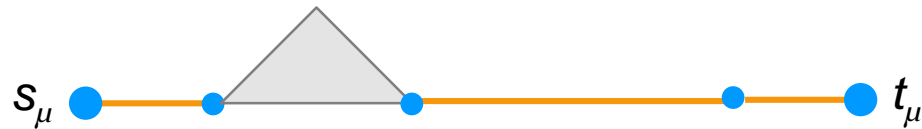
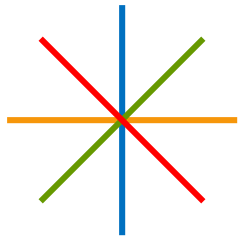
# How to draw S-nodes

**key property 1:** The leftmost and the rightmost children of an S-node (different from the child of the root) are two Q-nodes (their pertinent graph is simply an edge)



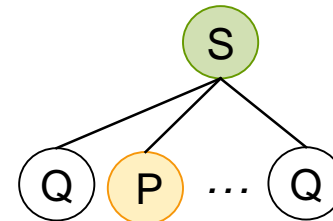
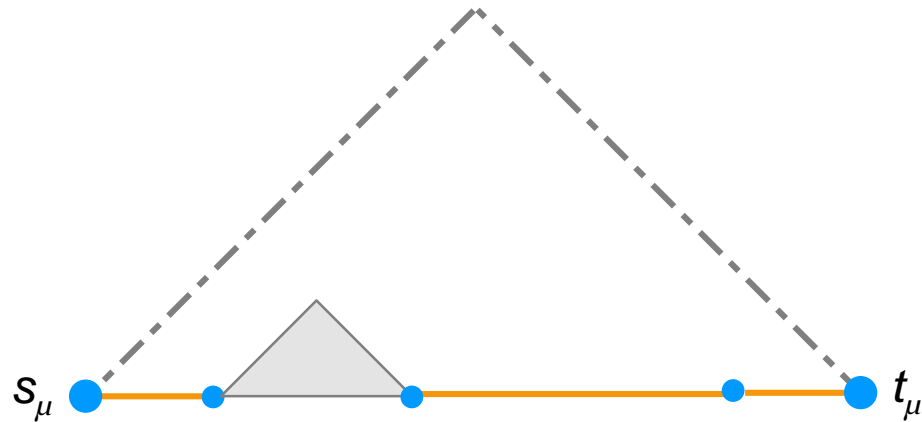
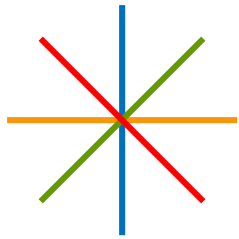
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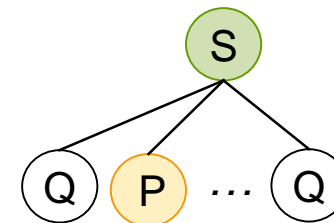
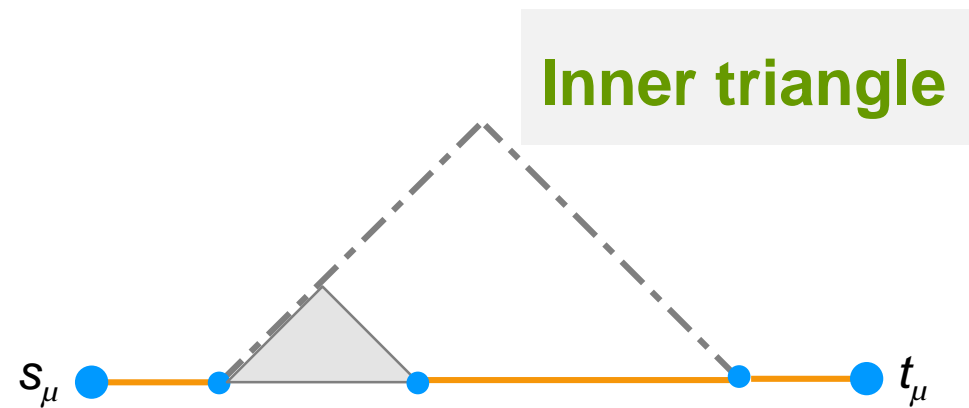
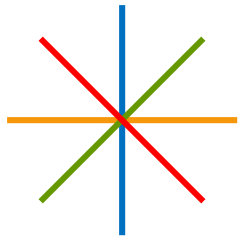
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**The three invariants hold**

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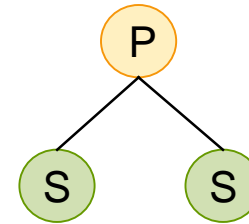
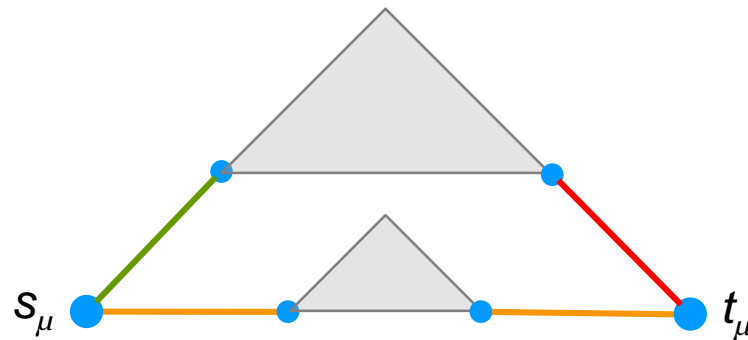
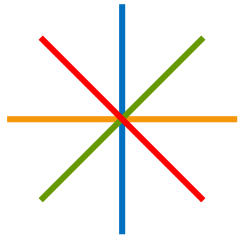


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# How to draw P-nodes

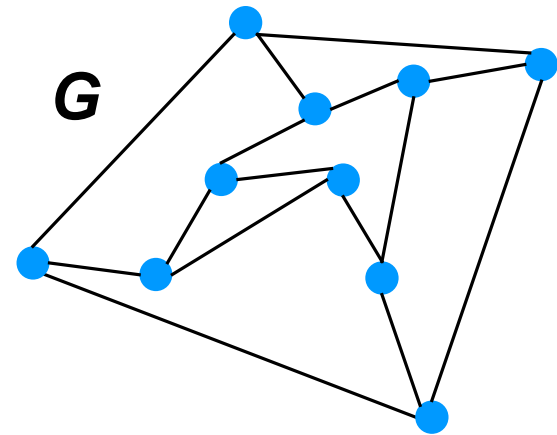
**key property 2:** The children of a P-node are either two S-nodes or an S-node and a Q-node



**The three invariants hold**

# canonical ordering

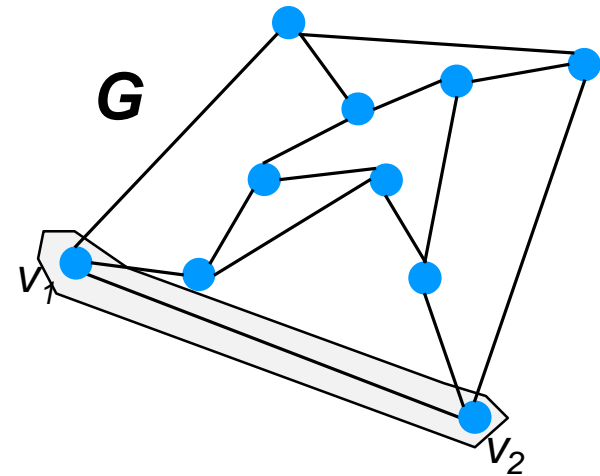
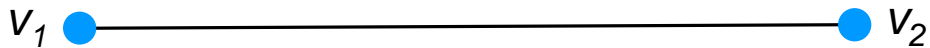
**G** is a **3-connected** cubic planar graph



# canonical ordering

$$\delta = \{V_1$$

$$V_1 = \{v_1, v_2\}$$



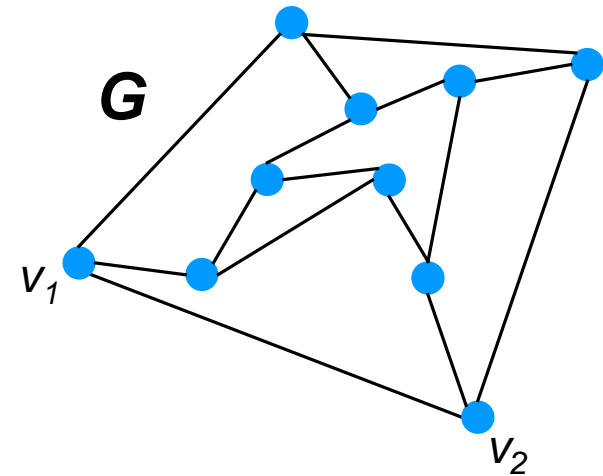
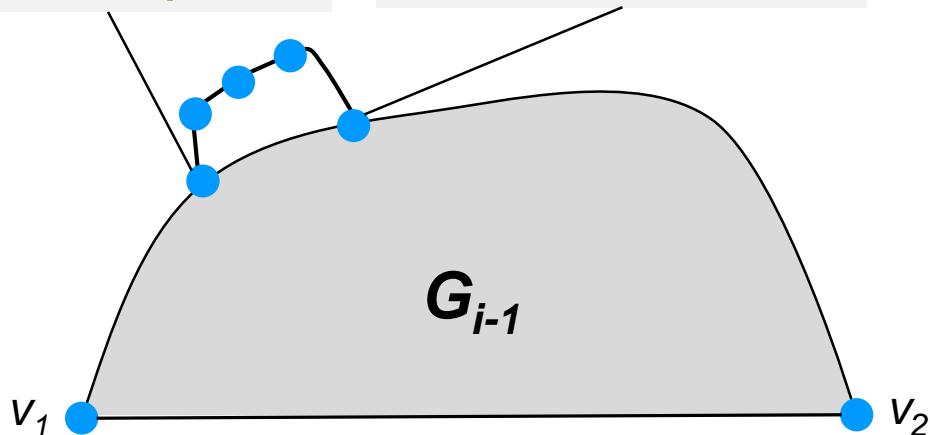
# canonical ordering

$$\delta = \{V_1, \dots, V_i\}$$

$V_i$  can be a CHAIN

leftmost pred.

rightmost pred.

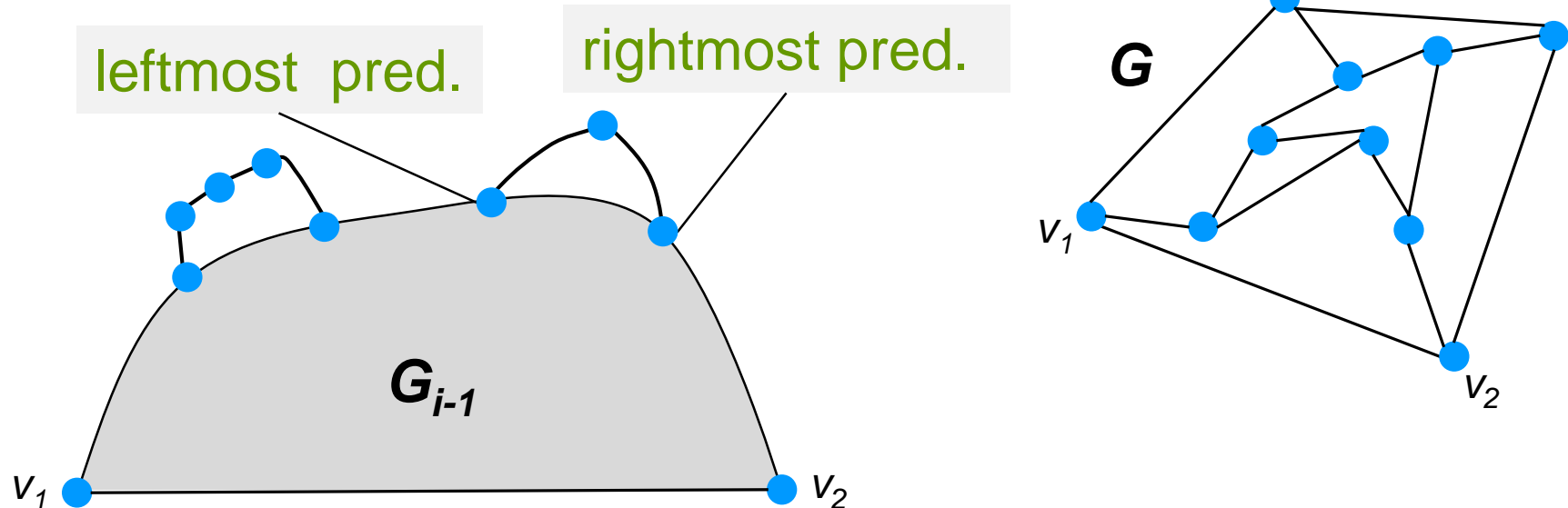


The leftmost and the rightmost predecessors appear consecutive walking clockwise along the outerface

# canonical ordering

$$\delta = \{V_1, \dots, V_i\}$$

$V_i$  can be a SINGLETON

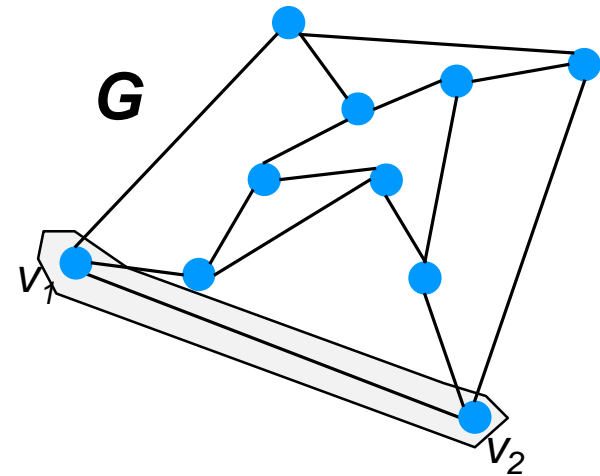
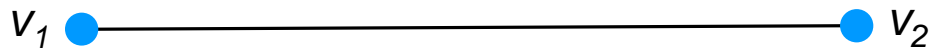


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# canonical ordering

$$\delta = \{V_1$$

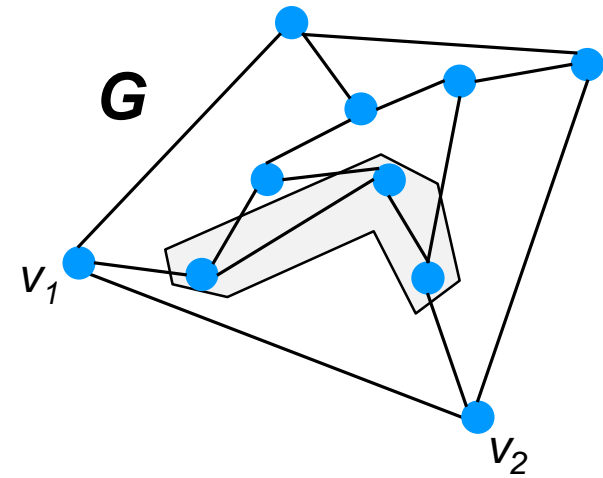
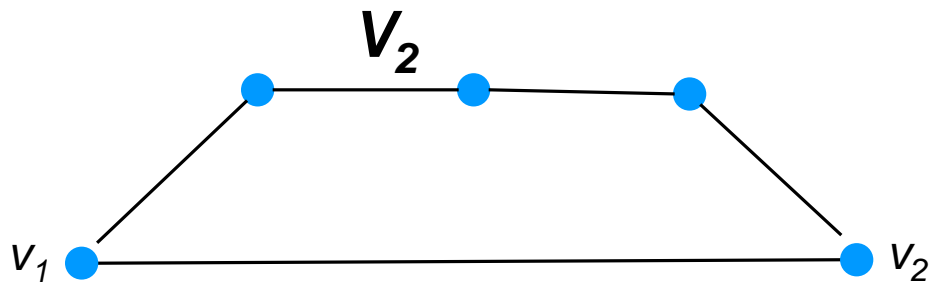
$$V_1 = \{v_1, v_2\}$$



# canonical ordering

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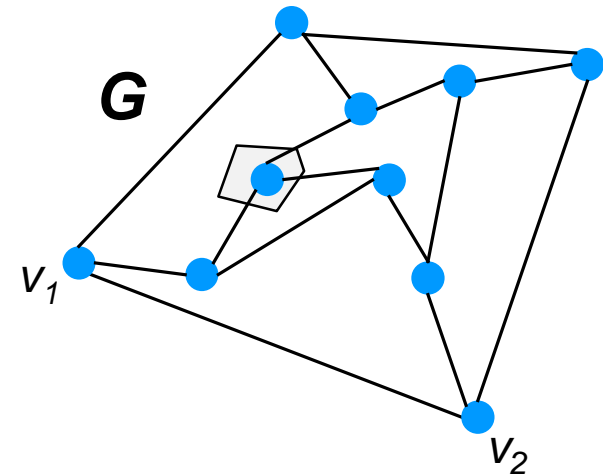
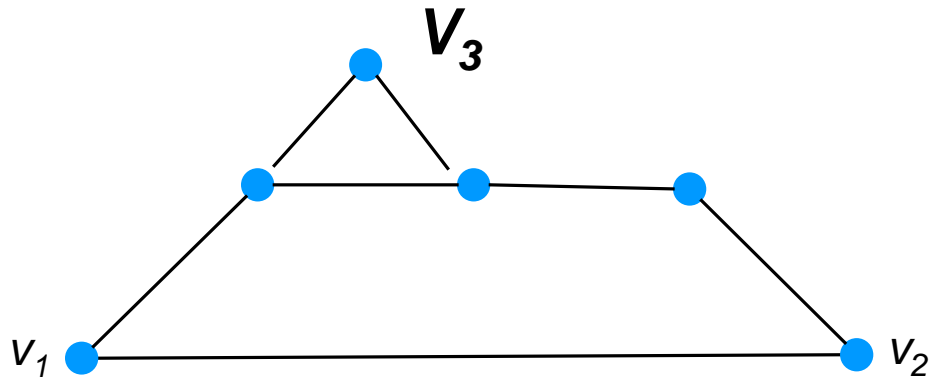
CHAIN



# canonical ordering

$$\delta = \{V_1, V_2, V_3\}$$

SINGLETON

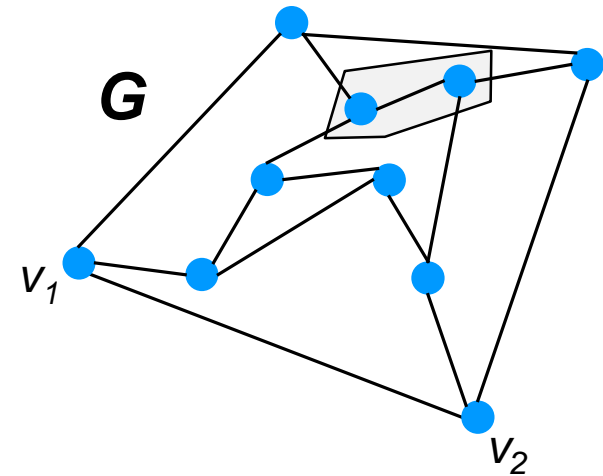
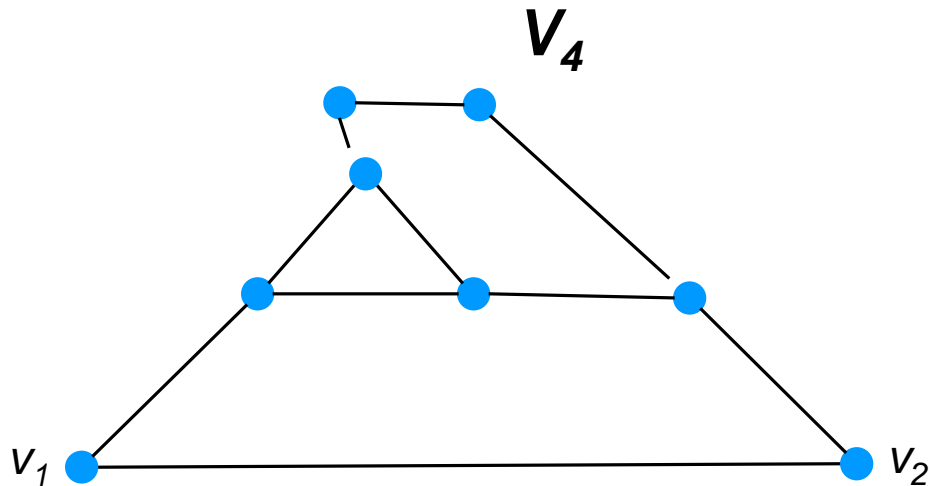




# canonical ordering

$$\delta = \{V_1, V_2, V_3, V_4\}$$

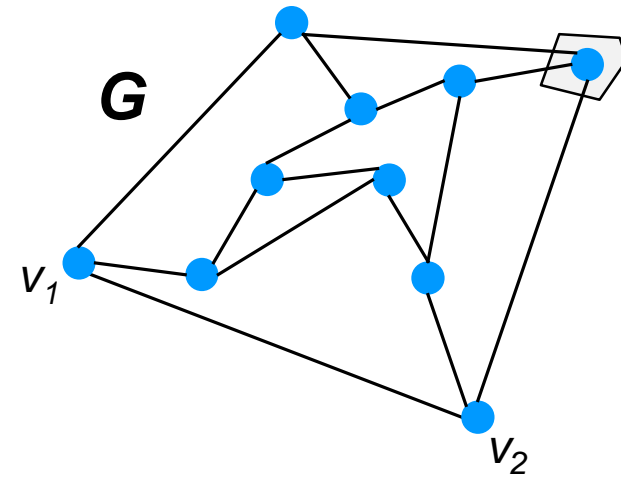
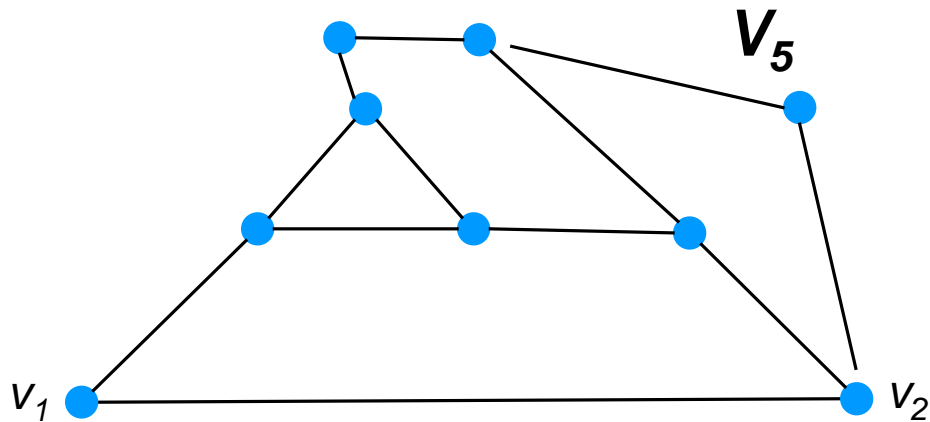
CHAIN



# canonical ordering

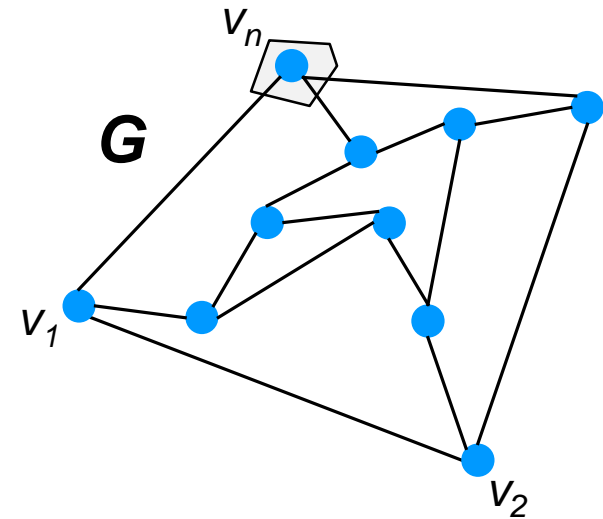
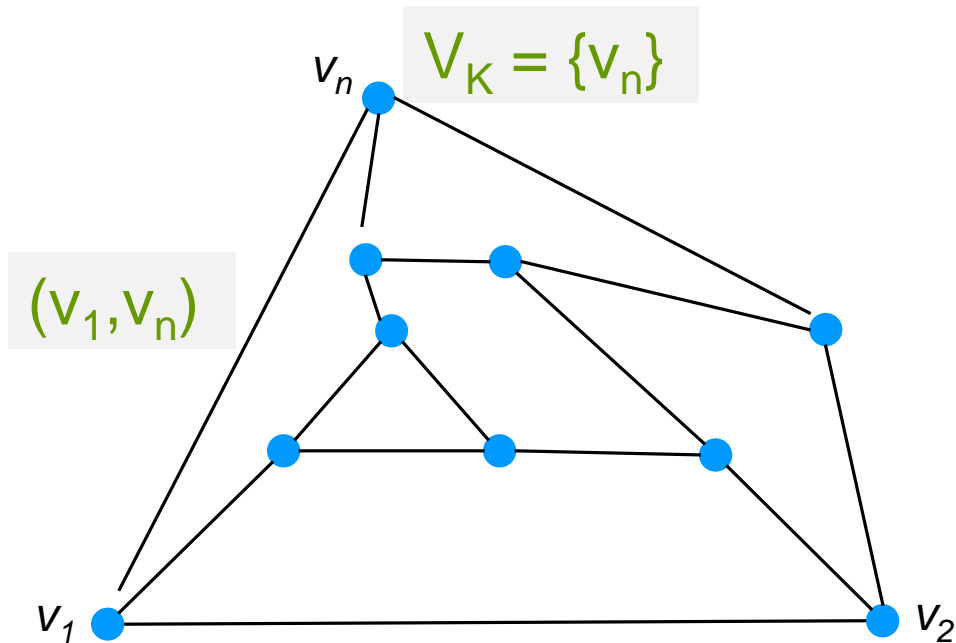
$$\delta = \{V_1, V_2, V_3, V_4, V_5\}$$

SINGLETON



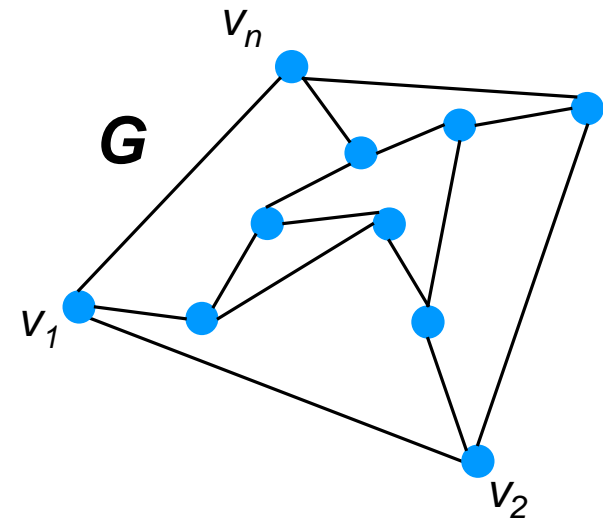
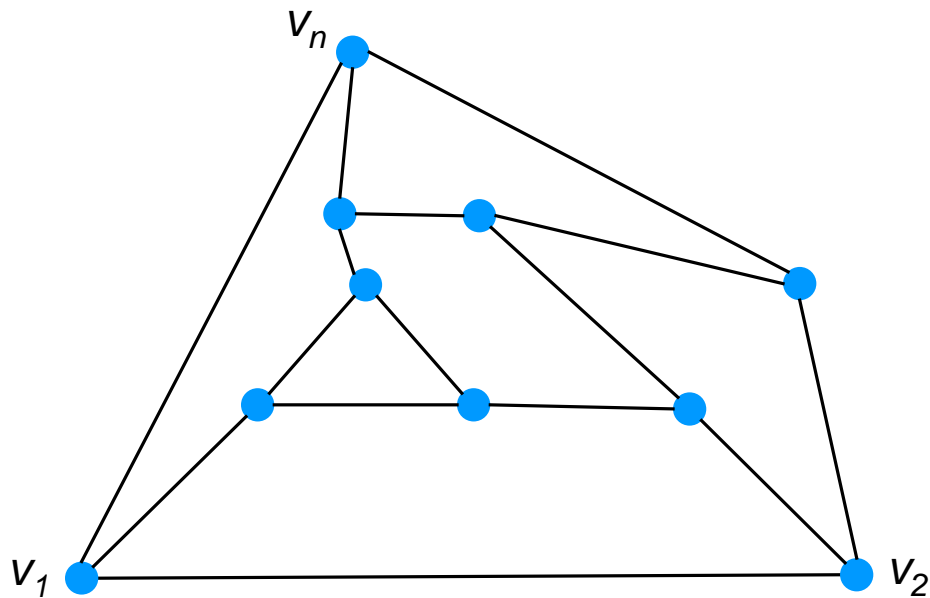
# canonical ordering

$$\delta = \{V_1, V_2, V_3, V_4, V_5, V_k\}$$



# canonical ordering

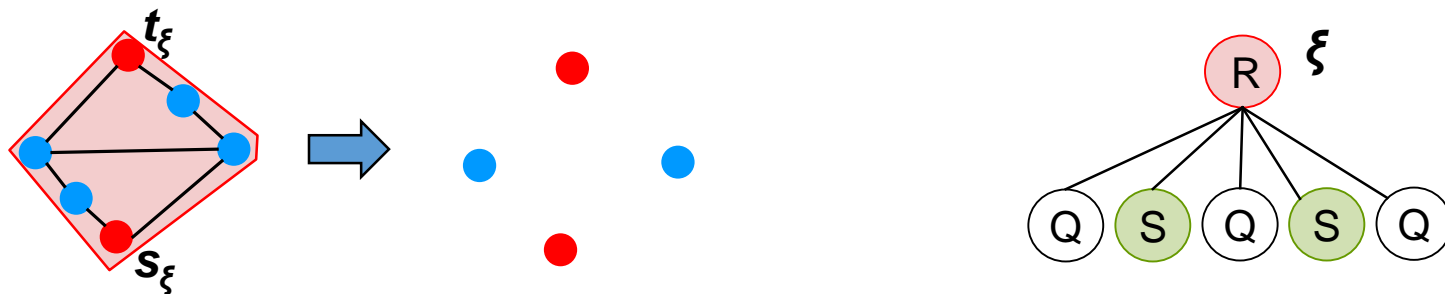
$$\delta = \{V_1, V_2, V_3, V_4, V_5, V_k\}$$



# How to draw R-nodes

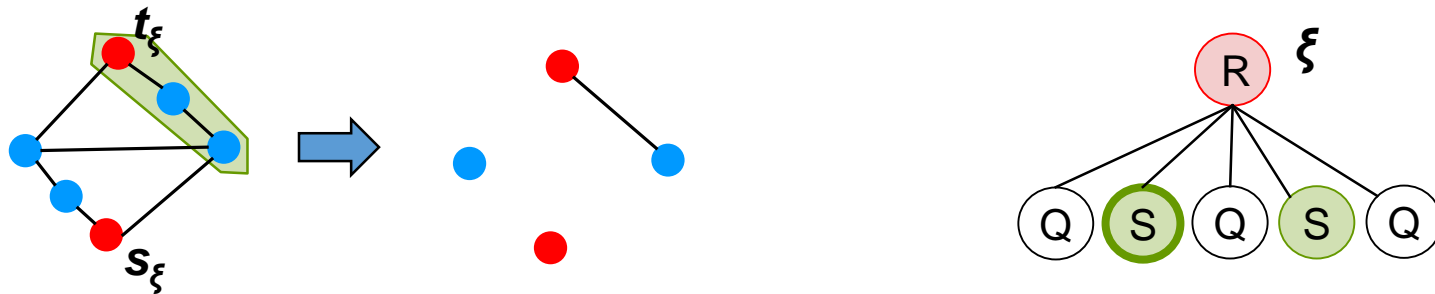
# How to draw R-nodes

**Frame graph:** substitute the pertinent graphs of the children of the node with a (virtual) edge: we obtain a 3-connected cubic graph (minus the edge between the poles)



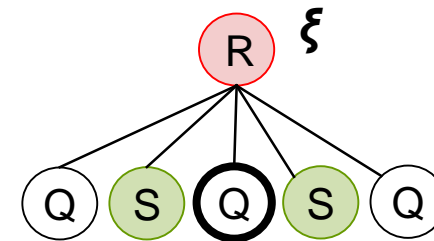
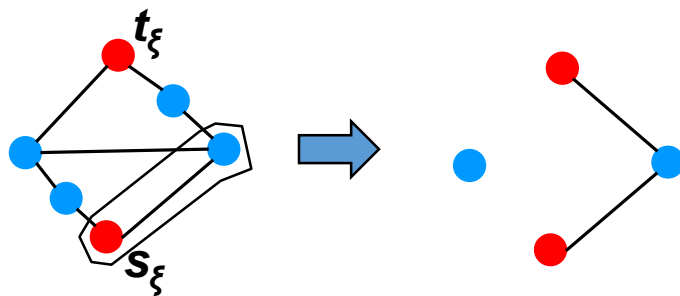
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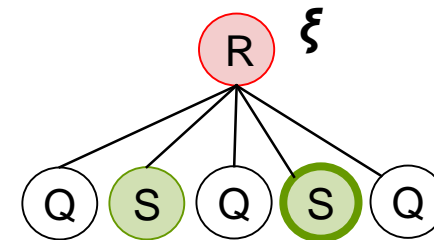
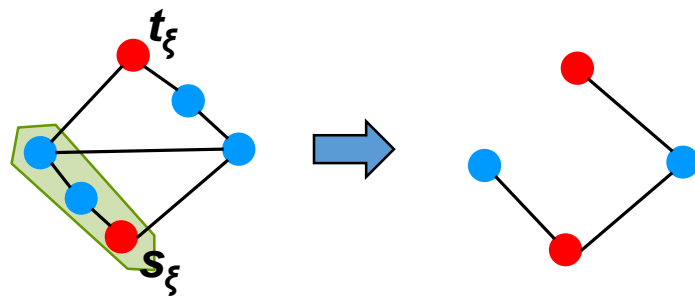
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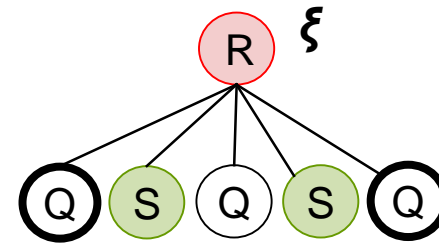
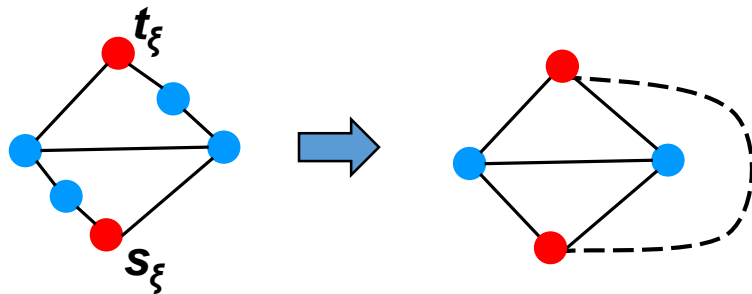
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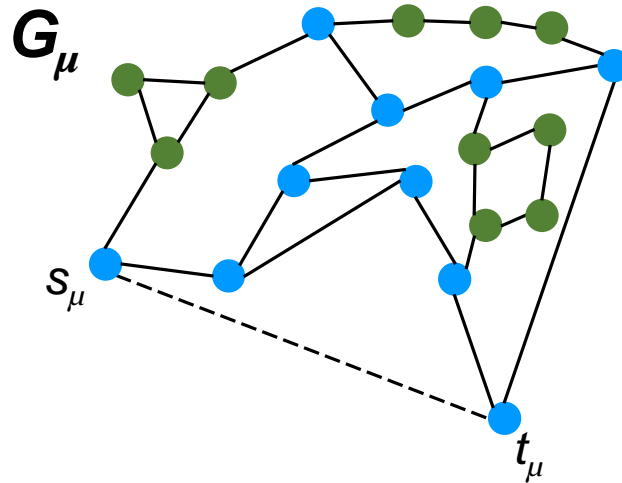


# How to draw R-nodes

1. Compute the frame  $F_\mu$  of  $G_\mu$
2. Draw the  $F_\mu$  exploiting the canonical ordering (add a dummy edge between the poles)
3. Replace the virtual edges of  $F_\mu$  with the drawings of the related pertinent graphs

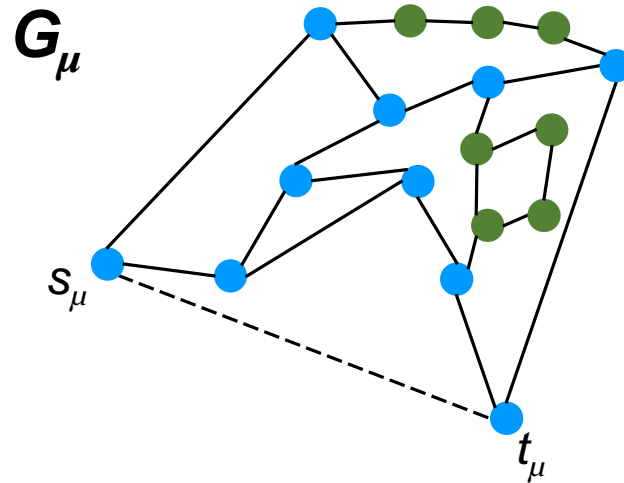
# How to draw R-nodes

1. Compute the frame  $F_\mu$  of  $G_\mu$  (the maximal split pairs are shown in blue)



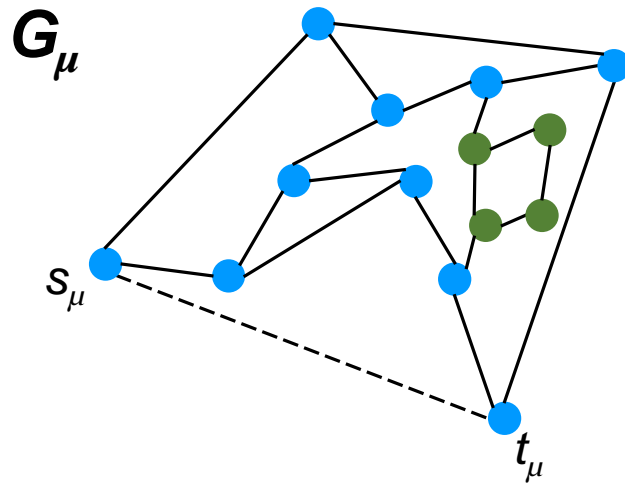
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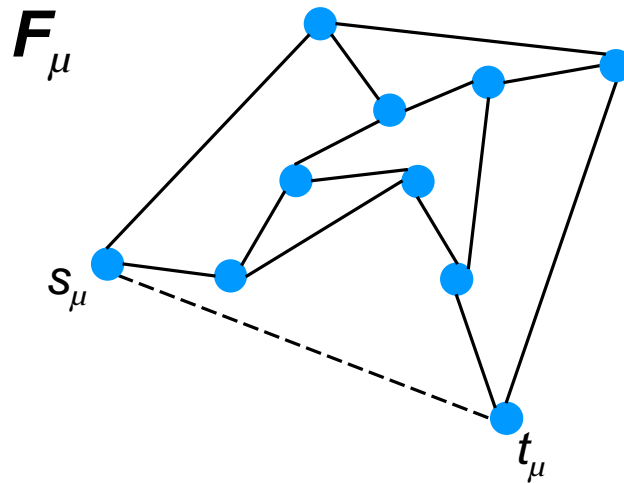
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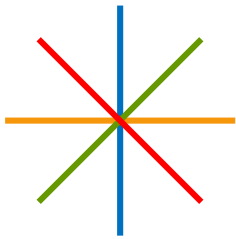
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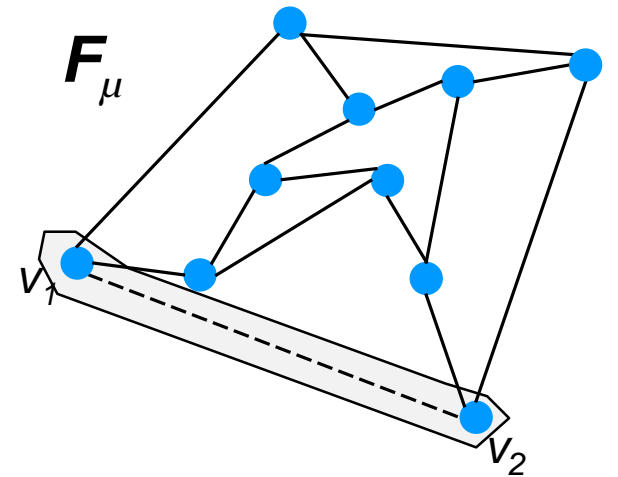
# How to draw R-nodes

## 2. Draw the frame



$$V_1 = \{v_1, v_2\}$$

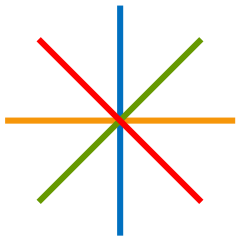
$$s_\mu = v_1 \text{---} t_\mu = v_2$$



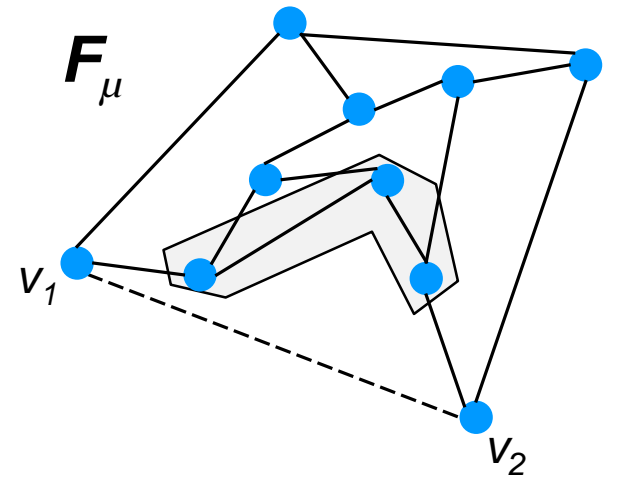


# How to draw R-nodes

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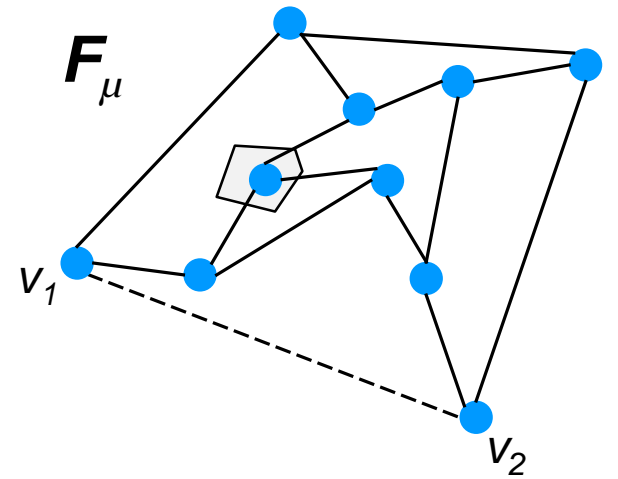
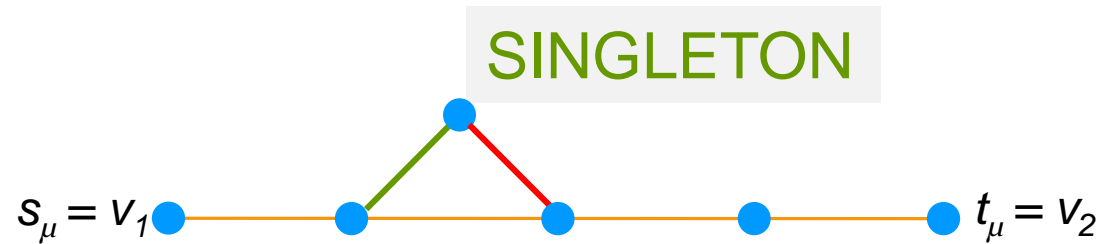
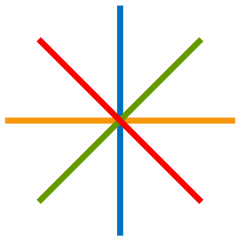


$V_2$  placed between the poles



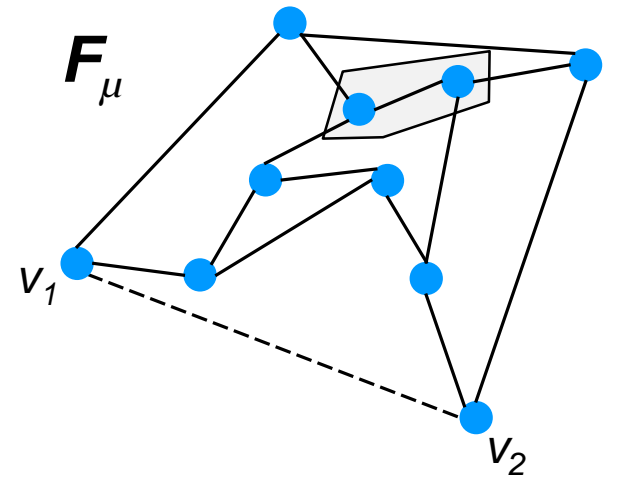
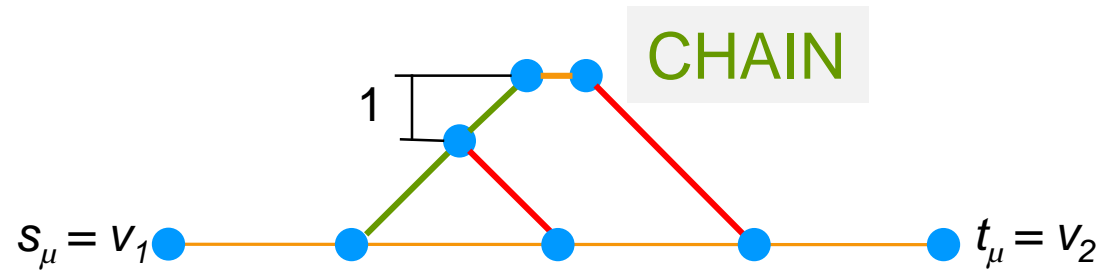
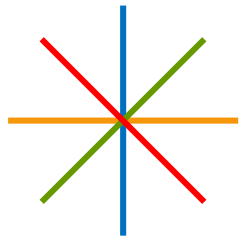
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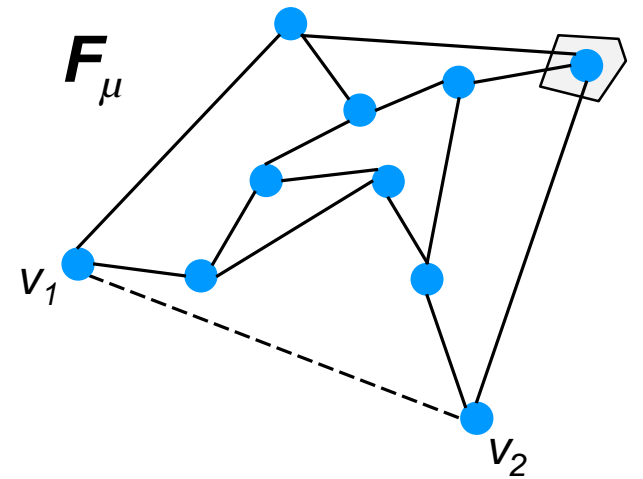
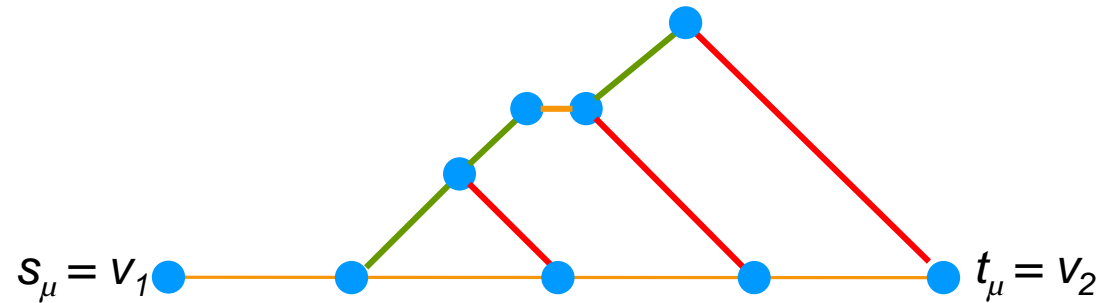
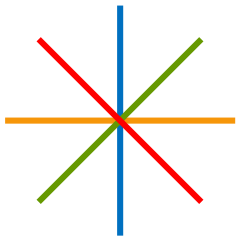
# How to draw R-nodes

## 2. Draw the frame



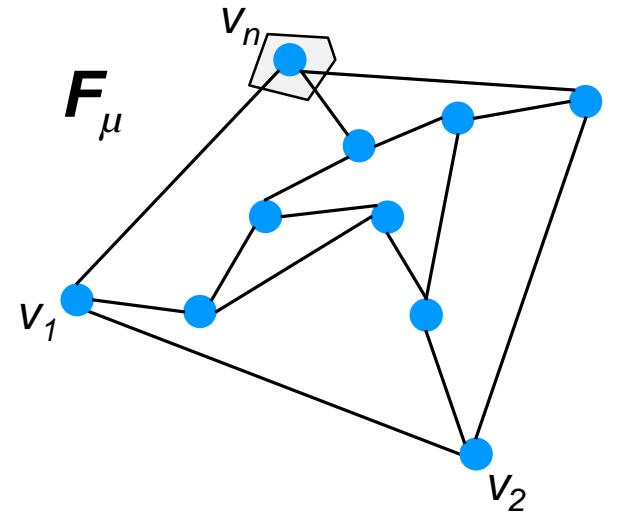
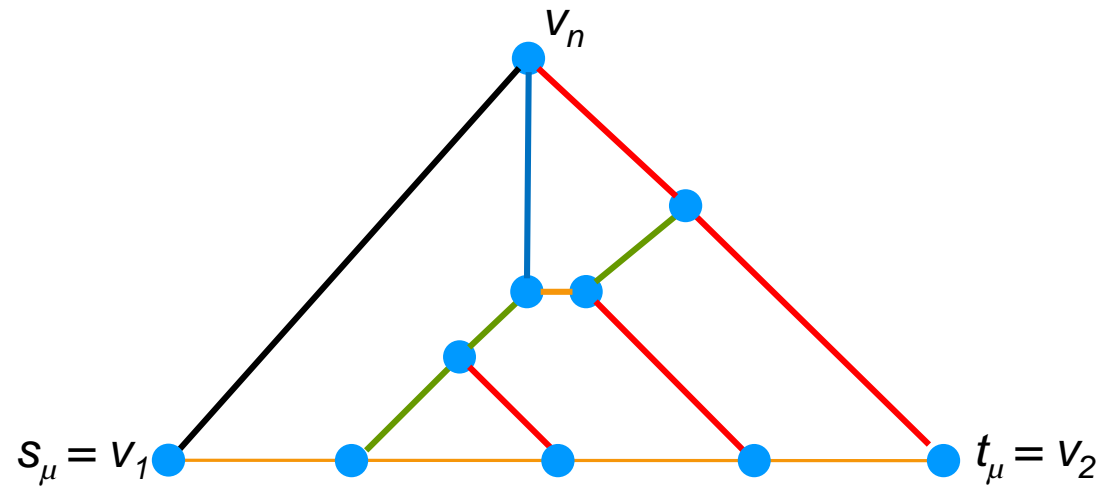
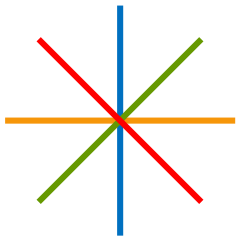
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## 2. Draw the frame



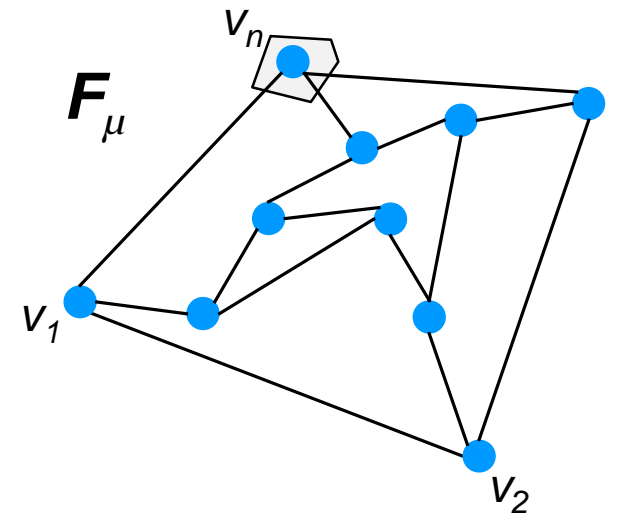
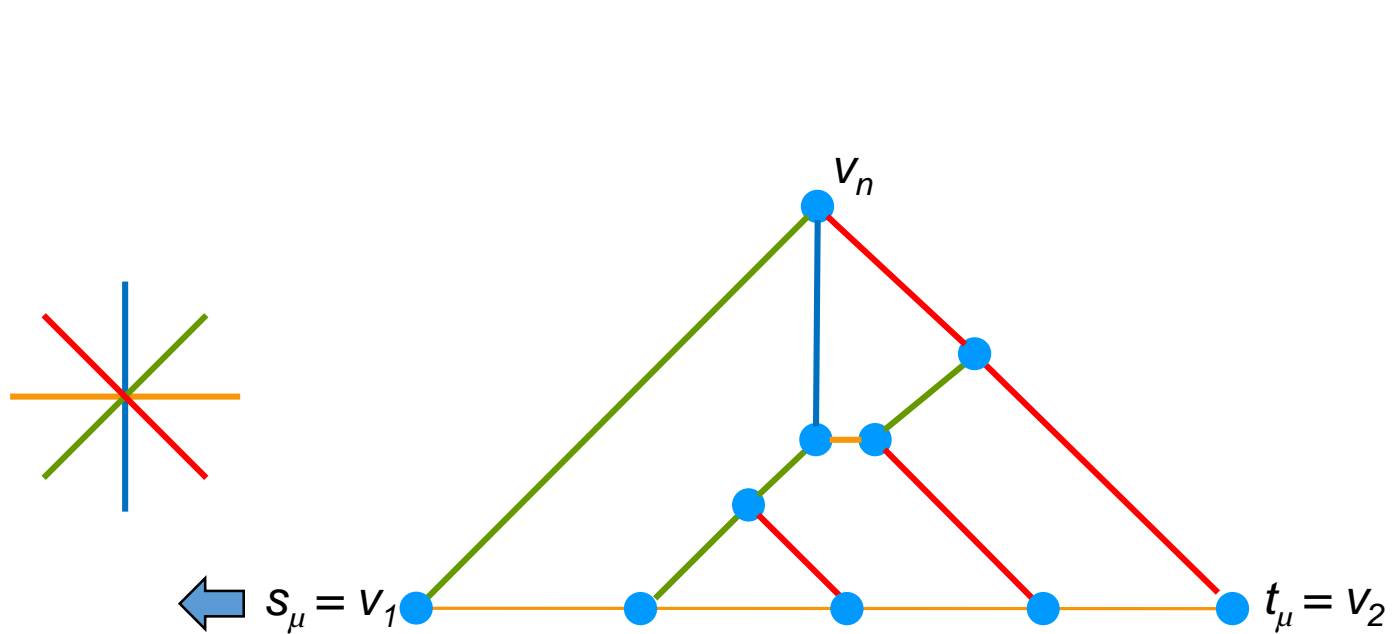
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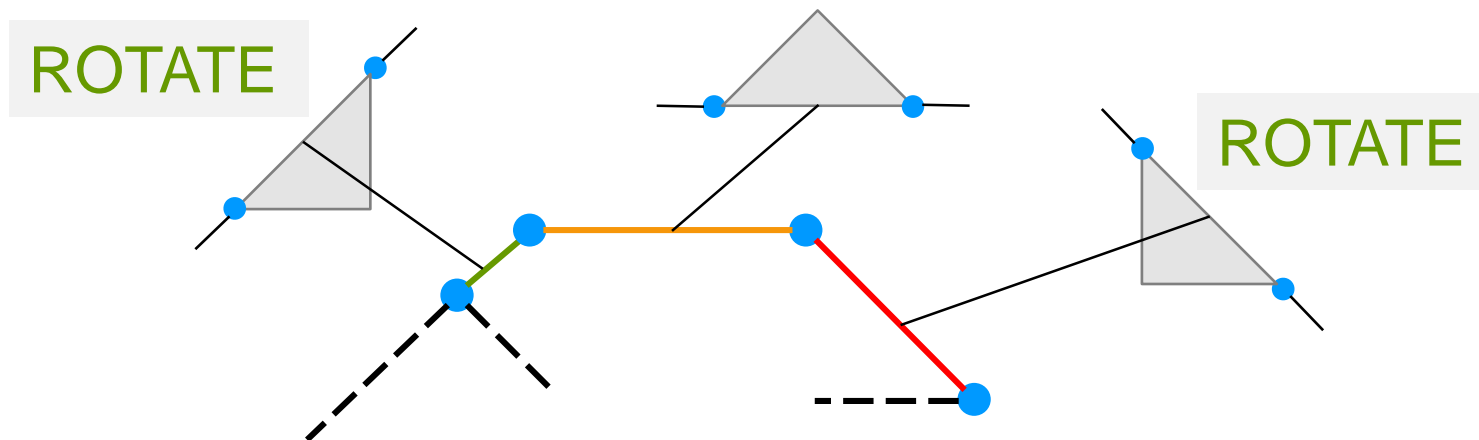
# How to draw R-nodes

## 2. Draw the frame



# How to draw R-nodes

3. Replace the virtual edges with their drawings

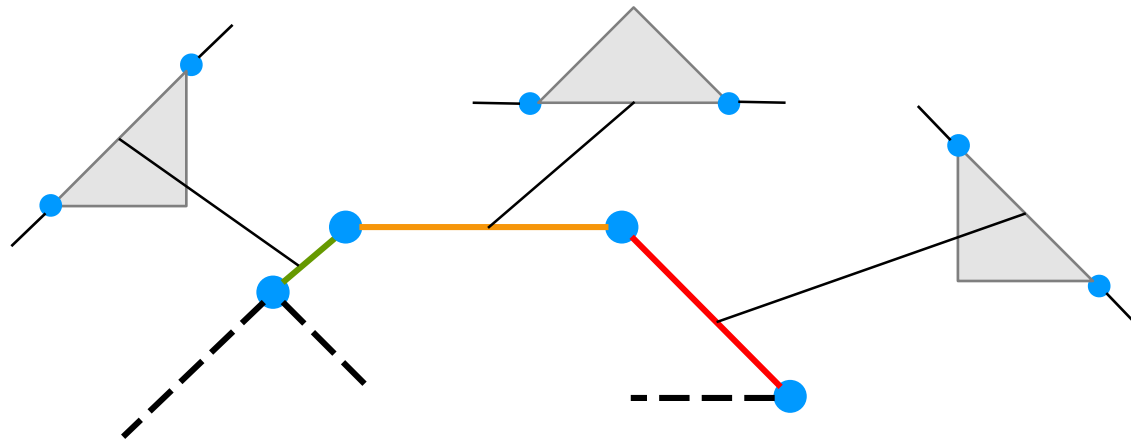


**key property 3:** The children of an R-node are either S-nodes or Q-nodes

# How to draw R-nodes

Replace the virtual edges with their drawings

Problem: How to guarantee quadratic area?

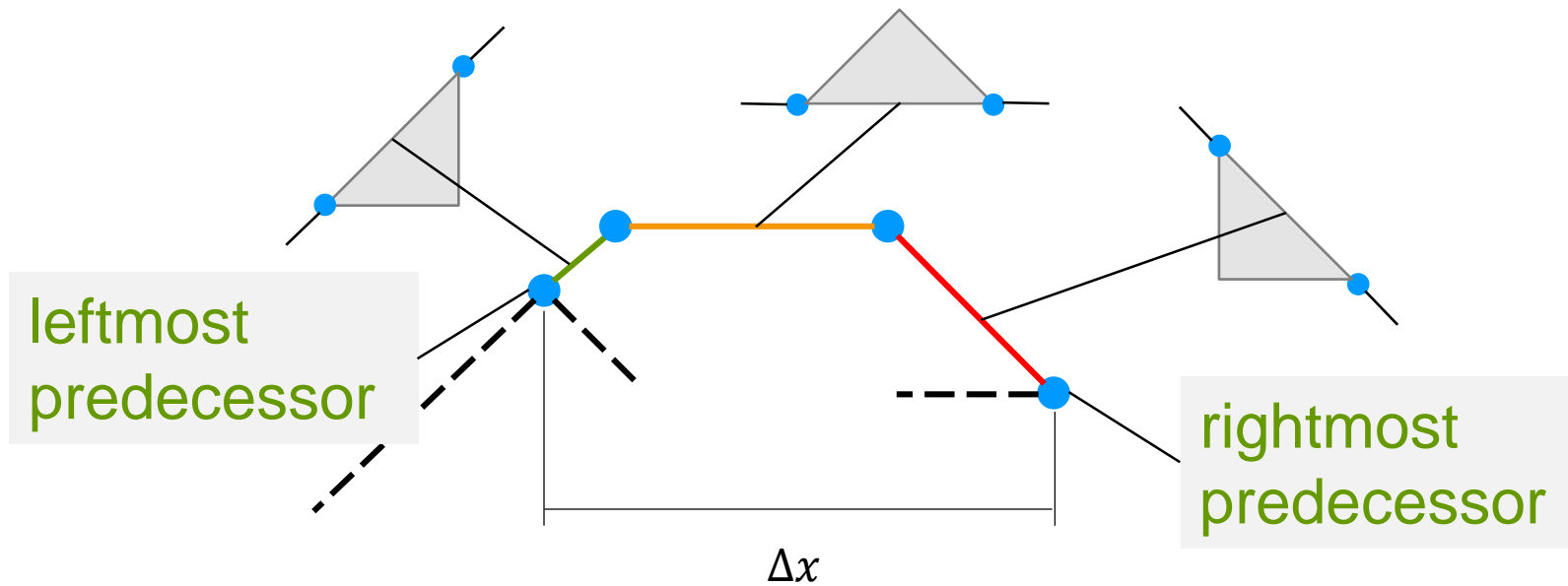


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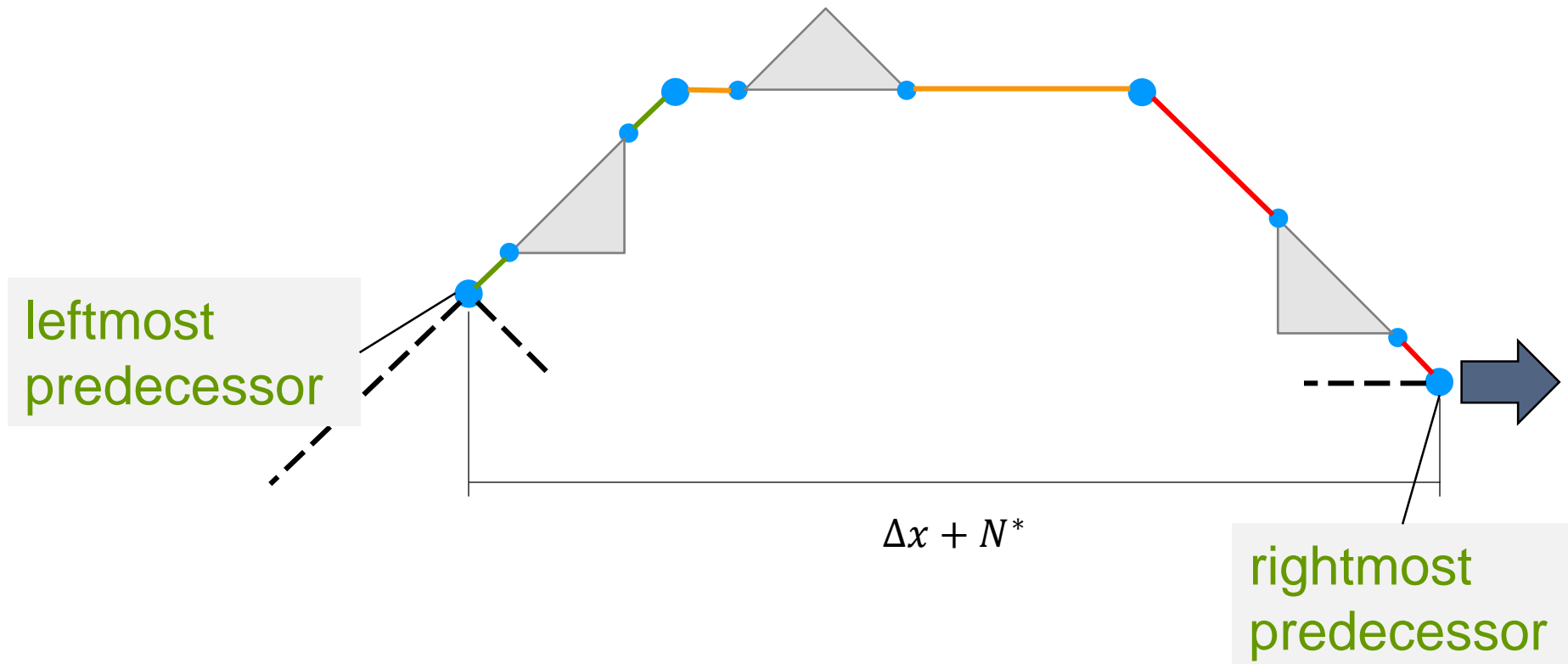
# How to draw R-nodes

Solution: expansion operation



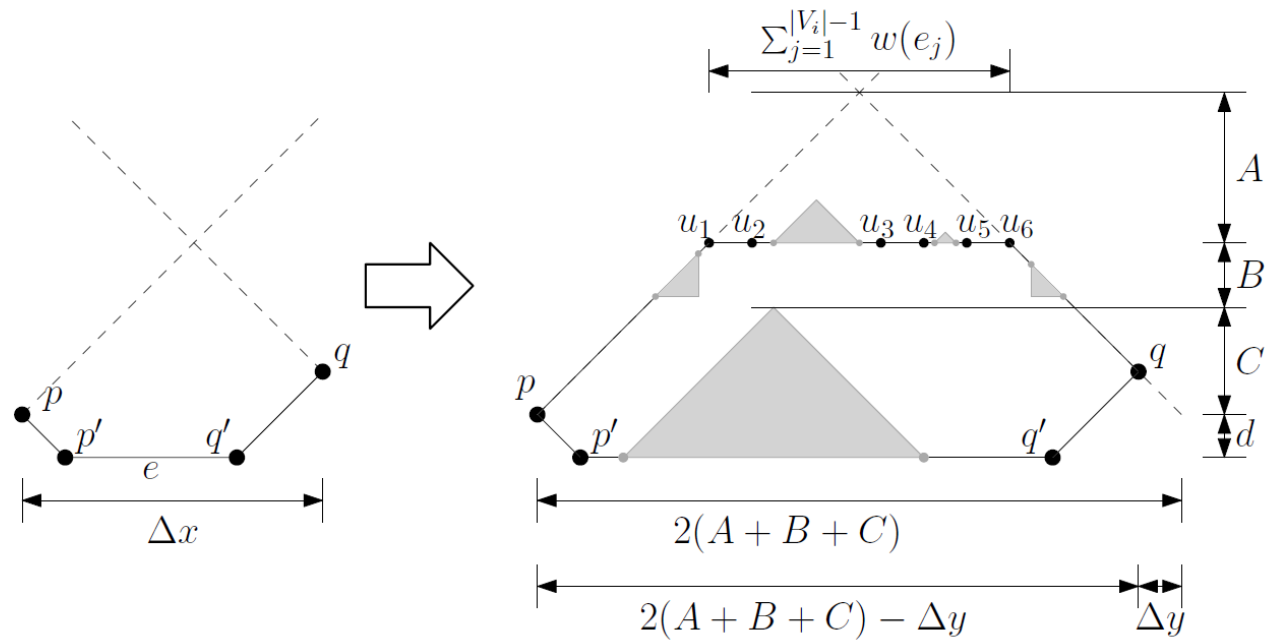
# How to draw R-nodes

Solution: expansion operation



# How to draw R-nodes

What's the value of  $N^*$ ? Some geometry may help...



$$A = \frac{\sum_{j=1}^{|V_i|-1} w(e_j)}{2} \quad B = \max\{w(e_0), w(e_{|V_i|})\}$$

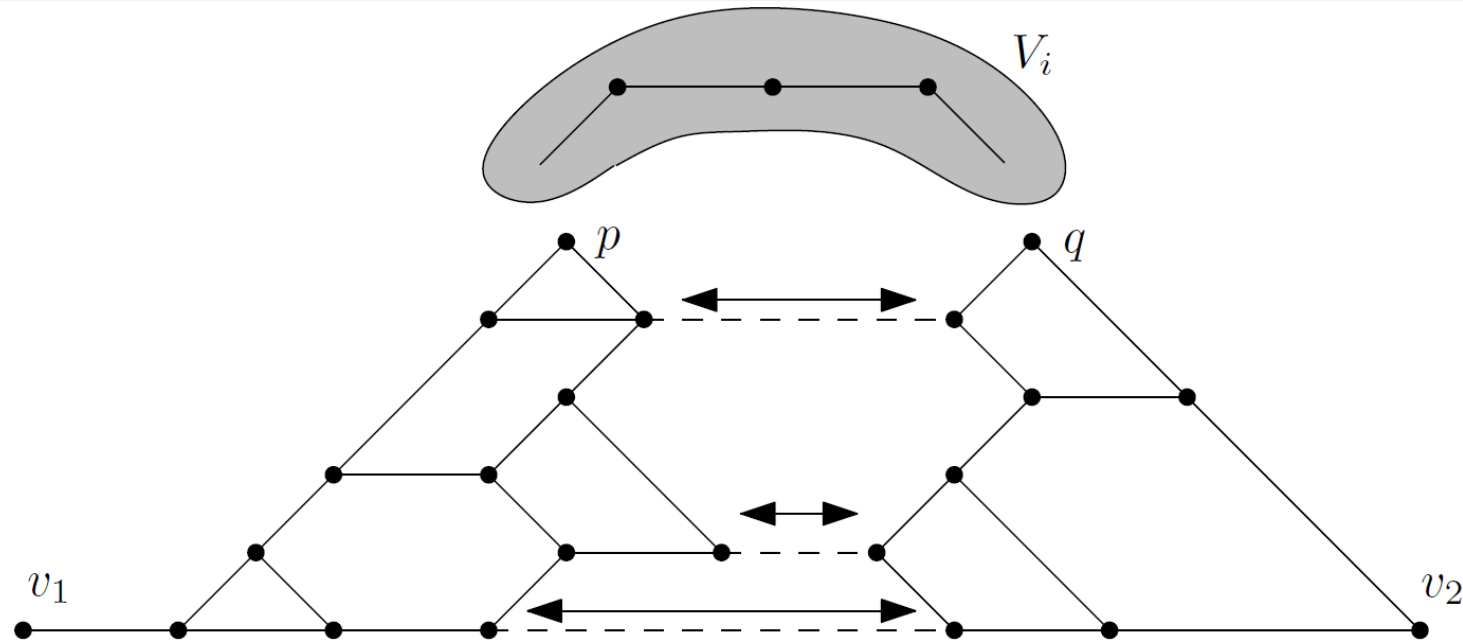
$$C = \max\{\Delta y + 1 - d, \frac{w(e)}{2} - 1\} \quad N^* = 2(A + B + C) - \Delta y - \Delta x$$

# How to draw R-nodes

Problem: how can I expand the drawing without breaking some slope?

# How to draw R-nodes

Problem: how can I expand the drawing without breaking some slope?



Solution: there exists a set of **horizontally drawn edges** that **separate the drawing** in two parts, one containing the leftmost pred. and one containing the rightmost pred.

Back to the state of the art...

# State of the Art: nonplanar graphs

Barát *et al.* & (ind.) Pach, Pálvölgyi: Bounded-degree ( $\Delta \geq 5$ ) **non-planar** graphs can have **arbitrarily large slope number** (unb. in  $\Delta$ )

[The Electronic J. of Combinatorics, 2006]

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Mukkamala and Pálvölgyi: **non-planar** graphs with  $\Delta = 3$  can be drawn with at most **4 slopes**.  
[GD 2011]



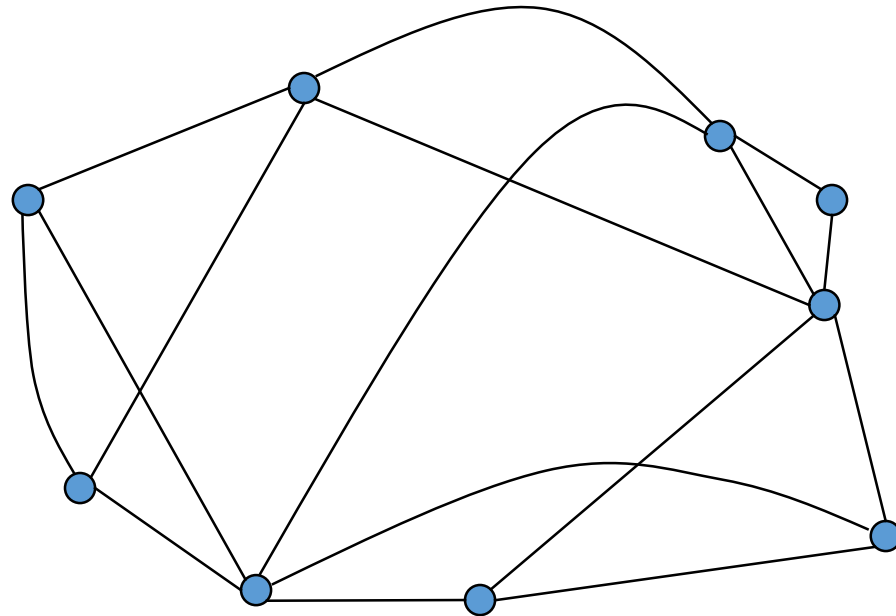
# State of the Art: nonplanar graphs

| Graphs          | Upper bound      | Lower bound       | References   |
|-----------------|------------------|-------------------|--|
| $\Delta = 3$    | 4                | 2                 | Barát <i>et al.</i> & (ind.) Pach, Pálvölgyi, 2006 |
| $\Delta = 4$    | <i>bounded?</i>  | <i>unbounded?</i> | -  |
| $\Delta \geq 5$ | <i>unbounded</i> | <i>unbounded</i>  | Mukkamala and Pálvölgyi, 2013                      |
| Outer 1-planar  | $6\Delta+12$     | $\Delta - 1$      | Di Giacomo <i>et al.</i> , 2014                    |

**Reserch direction:** Is the slope number of “nearly-planar” graphs bounded in  $\Delta$ ?

# Our research

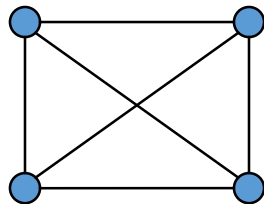
**outer 1-planar graphs:** graphs that admit drawings where each edge is crossed at most once and each vertex is on the boundary of the outer face



# Our research

- there exists a linear-time algorithm for testing outer 1-planarity (returns an o1p embedding)

[Auer et al., GD 2013] [Hong et al., GD 2013]



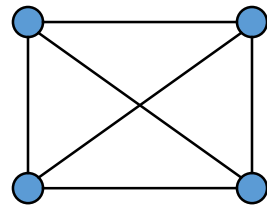
$K_4$

o1p drawing

# Our research

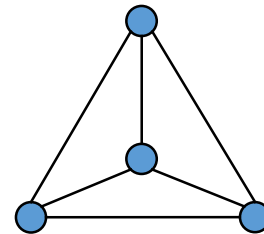
- there exists a linear-time algorithm for testing outer 1-planarity (returns an o1p embedding)
- o1p graphs are planar partial 3-trees

[Auer et al., GD 2013] [Hong et al., GD 2013]



o1p drawing

$K_4$



planar drawing

# New results

Di Giacomo, Liotta, M.: The **o1p slope number**,  $\text{o1p-sl}(G)$ , of o1p graphs with maximum degree  $\Delta$  is at most  **$6\Delta + 12$**

[GD 2014]

- This result also generalizes the result on the planar s. n. of outerplanar graphs which is  **$\Delta - 1$**

[Knauer et al., COCOON 2012 - CGTA, 2014]

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# New results

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[GD 2014]

# Overview of proof technique



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- Prove the result for 2-connected o1p graphs
- Extend the result to 1-connected o1p graphs

# Overview of proof technique

- Prove the result for 2-connected o1p graphs
  - Define a universal set of slopes
  - Decompose the graph by SQPR-tree
  - Draw each pertinent graph (bottom-up) using only the universal slopes
- Extend the result to 1-connected o1p graphs

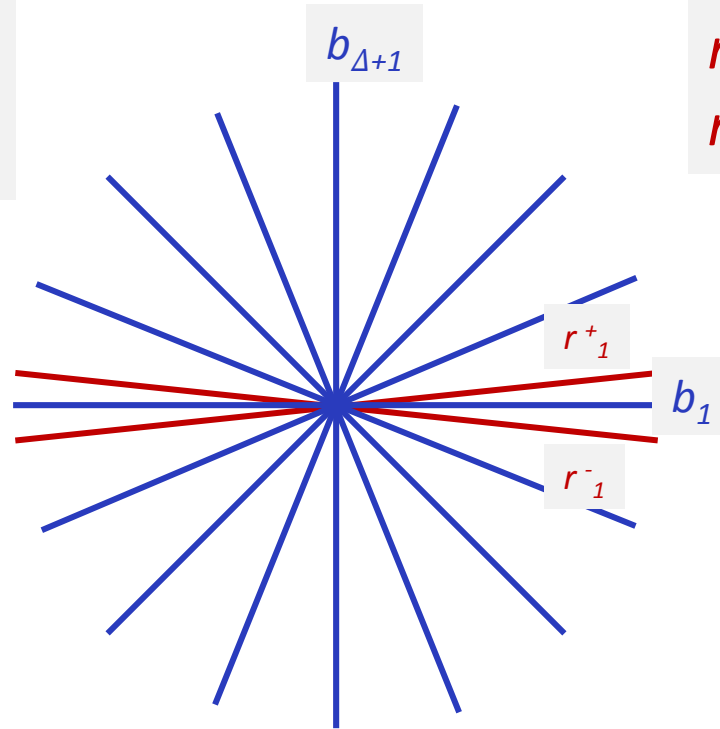
A universal set of slopes

# A universal set of slopes

**$2\Delta$  blue slopes**

$$b_i = (i-1)\alpha$$

$$\alpha = \frac{\pi}{2\Delta}$$



**$4\Delta$  red slopes**

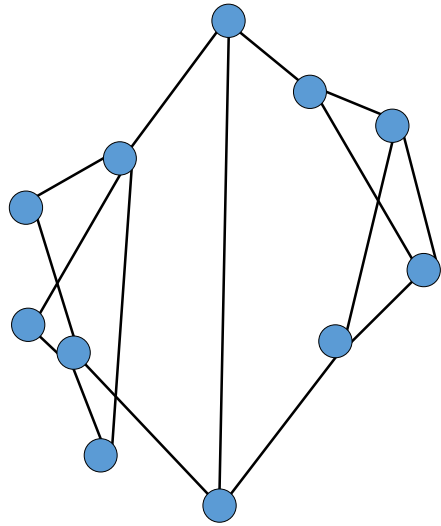
$$r_i^- = b_i - \varepsilon$$

$$r_i^+ = b_i + \varepsilon$$

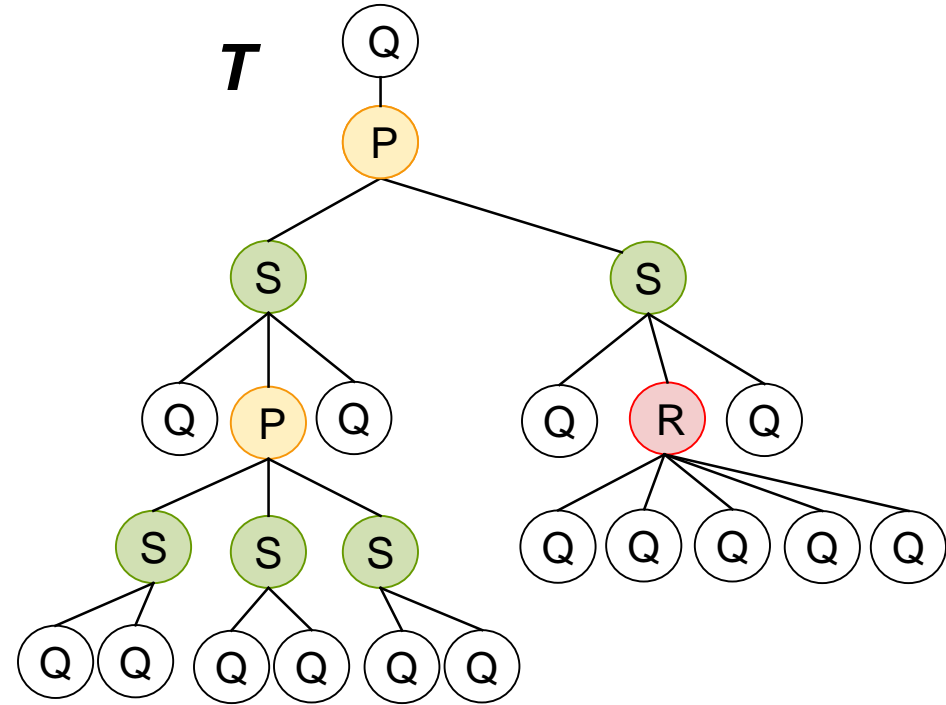
In the following: edges using **blue** (**red**) slopes will **never** (**always**) receive crossings

# SPQR-tree decomposition

**G**

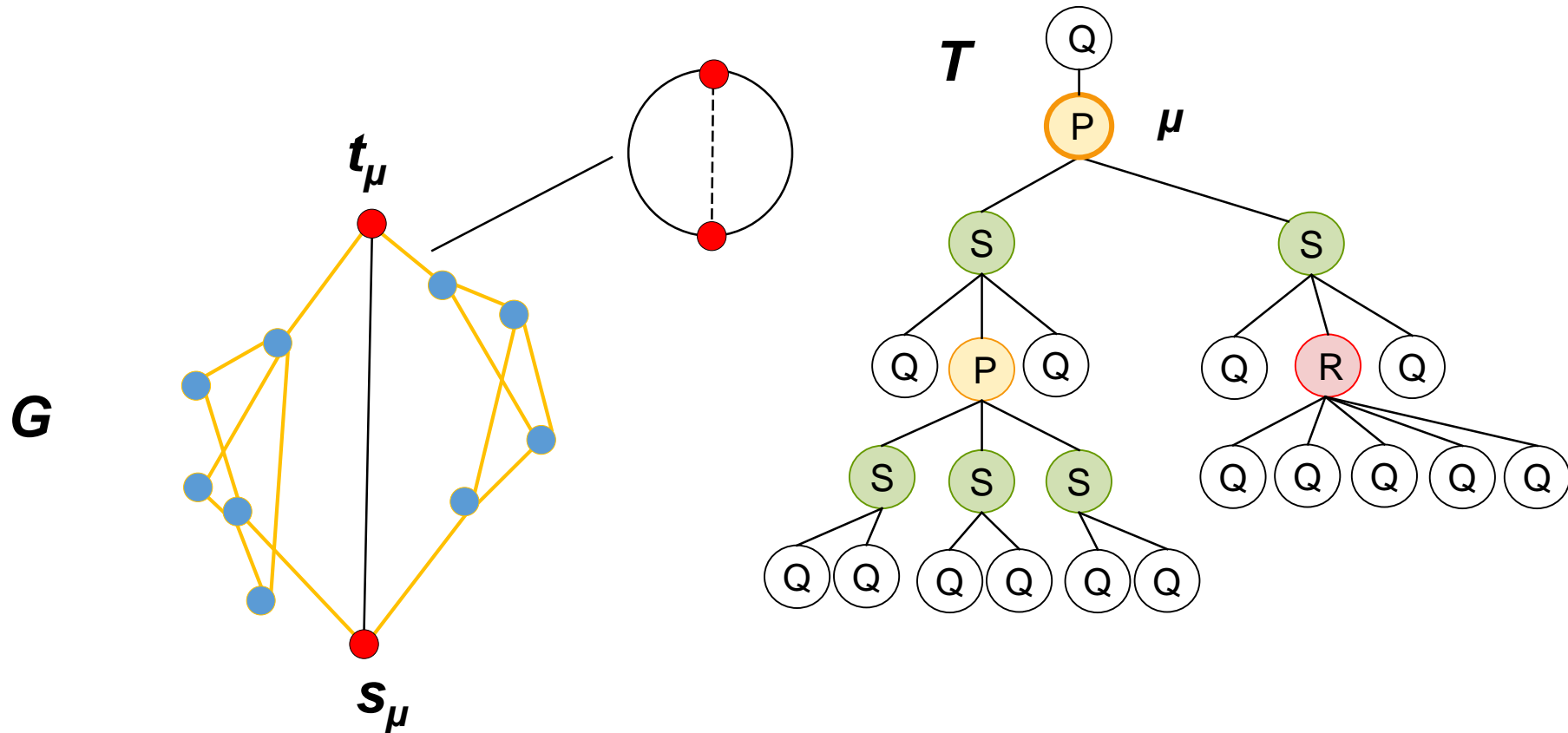


**T**



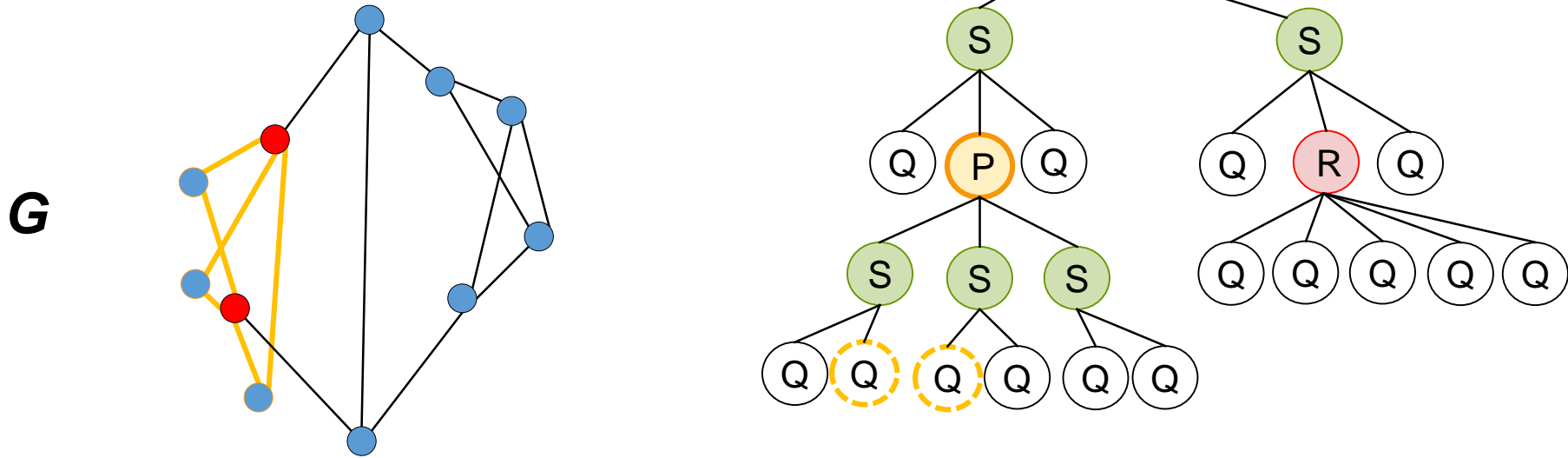
**G** is a **2-connected** (outer 1-plane) graph

# SPQR-tree decomposition



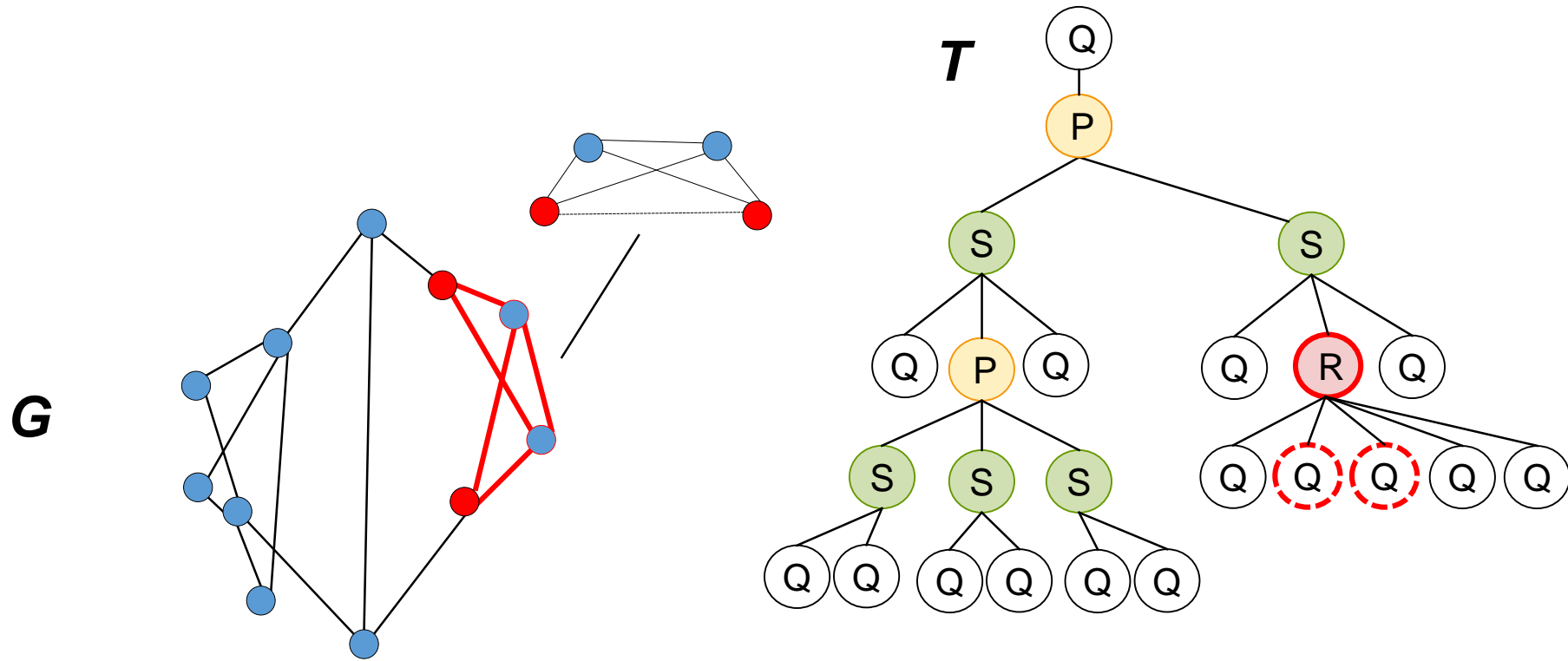
The **skeleton** of  $\mu$  is the biconnected (multi)graph obtained from  $G_\mu$  by collapsing the pertinent graphs of the children of  $\mu$  with a virtual edge, plus the ref. edge

# SPQR-tree decomposition



**key property 1:** internal P-nodes have **at most three children** and, in this case, **two are S-nodes that cross** and the other one is either a Q-node or an S-node

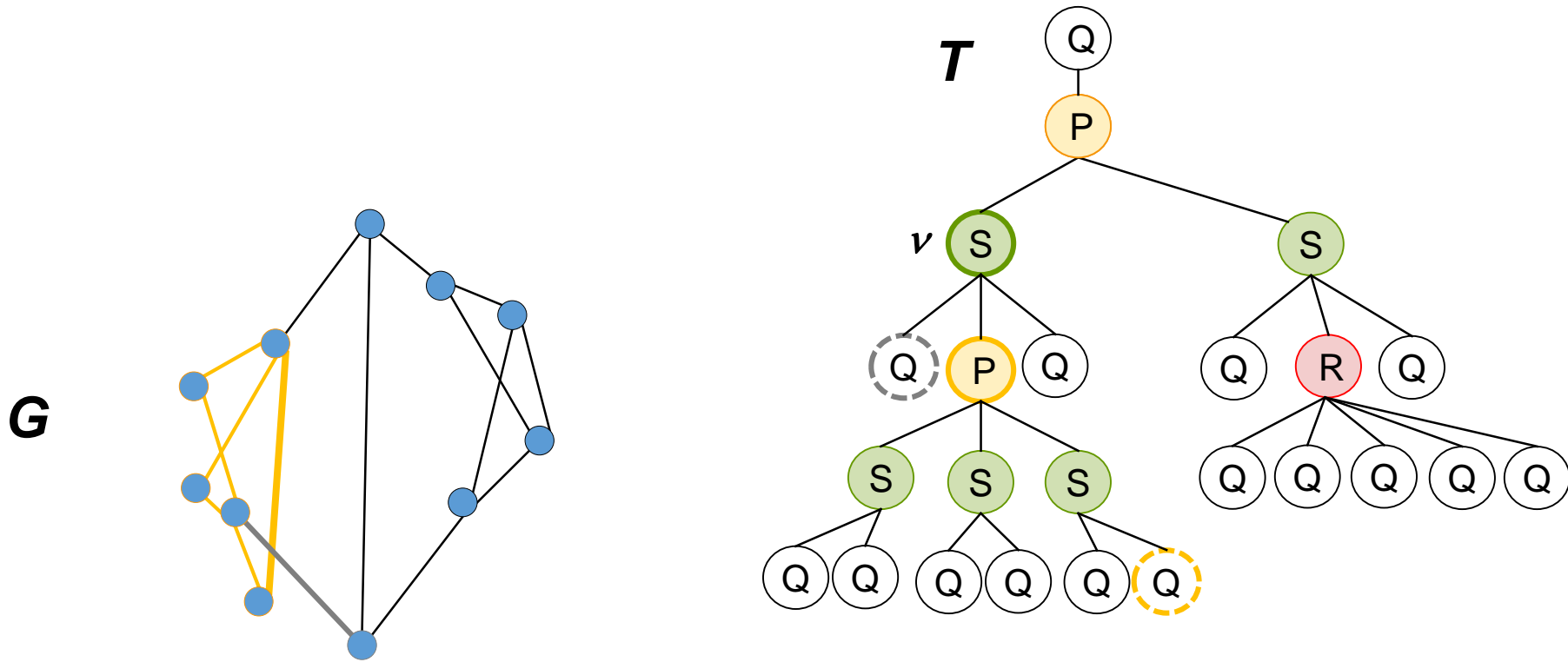
# SPQR-tree decomposition



**key property 2:** the skeleton of an R-node is isomorphic to  $K_4$  and embedded with one crossing

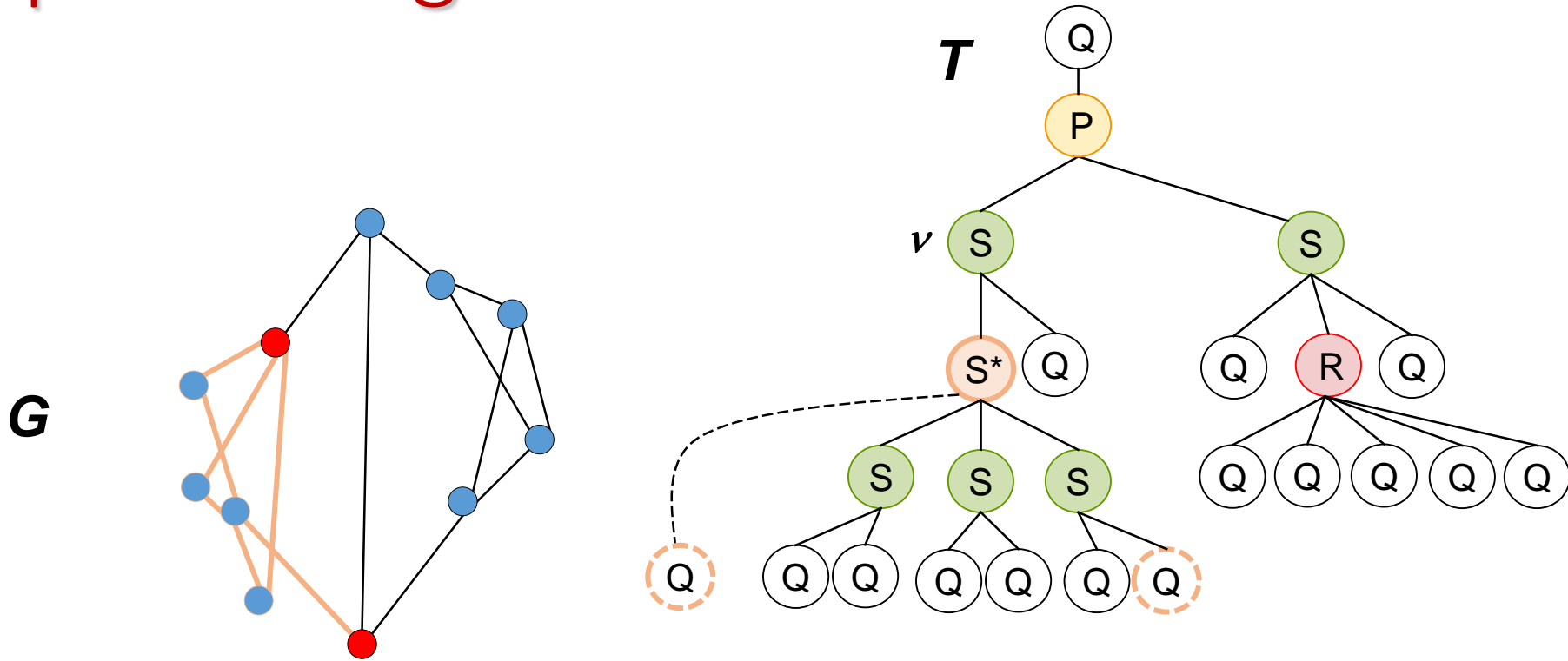


# SPQR-tree decomposition



**key property 3:** two  $Q$ -nodes at different levels of  $T$  cross only if one of them is the child of an  $S$ -node  $v$  and the other one the grandchild of a  $P$ -node that is also a child of  $v$

# SPQR-tree decomposition: Postprocessing



**S\*-node** = merge of a P-node + a Q-node that are children of the same S-node and that 'cross' each other

# 2-connected o1p graphs: Proof scheme

Let  $T$  be an SPQR-tree of an embedded o1p graph  $G$

Visit the nodes of  $T$  in a bottom-up order:

for each node  $\mu$  of  $T$

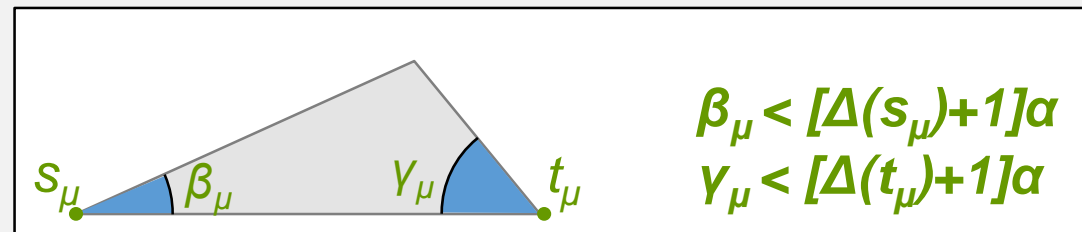
draw  $G_\mu$  by combining the drawings of the children of  $\mu$

**Invariants:**

**I1.**  $\Gamma_\mu$  is o1p

**I2.**  $\Gamma_\mu$  uses only slopes of the universal slope set

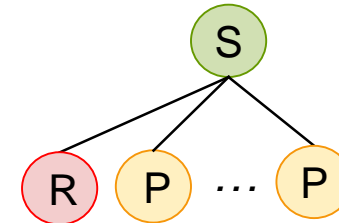
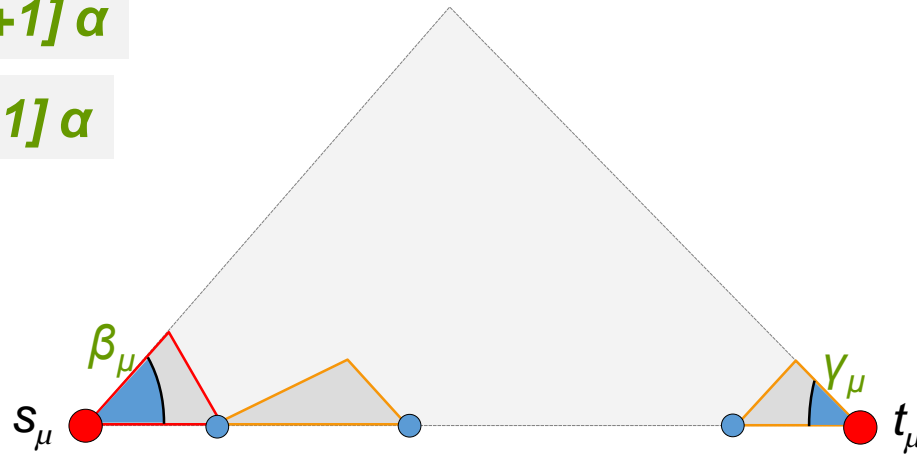
**I3.**



# How to draw S-nodes

$$\beta_\mu < [\Delta(s_\mu) + 1] \alpha$$

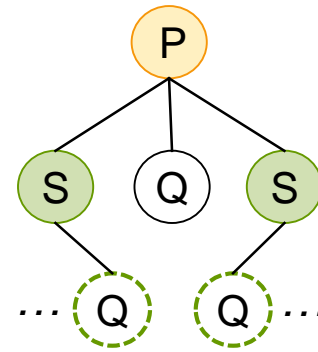
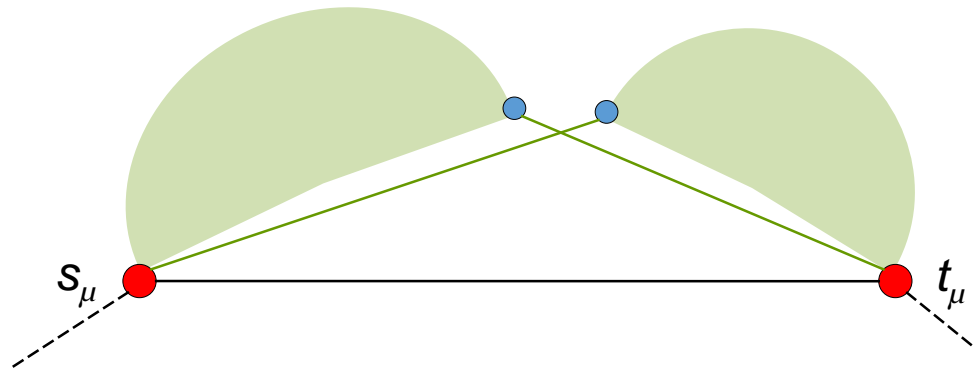
$$\gamma_\mu < [\Delta(t_\mu) + 1] \alpha$$



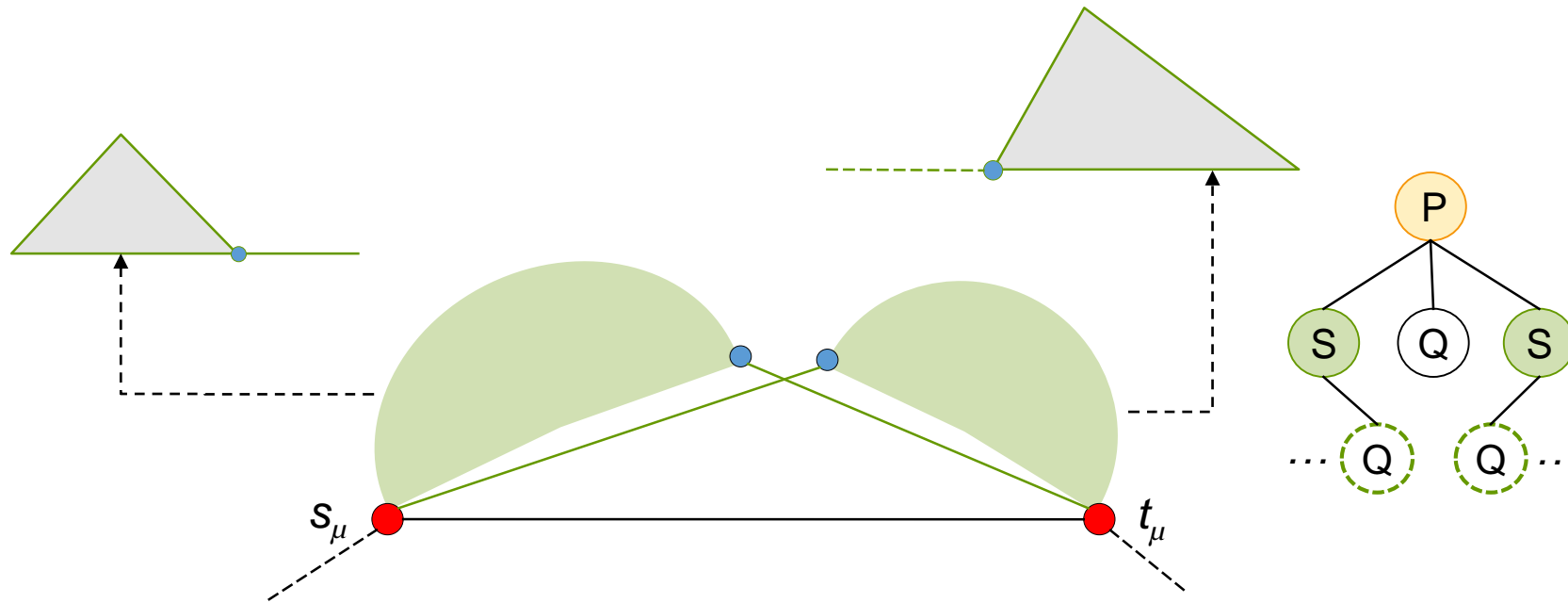
All the invariants are maintained

# How to draw P-nodes

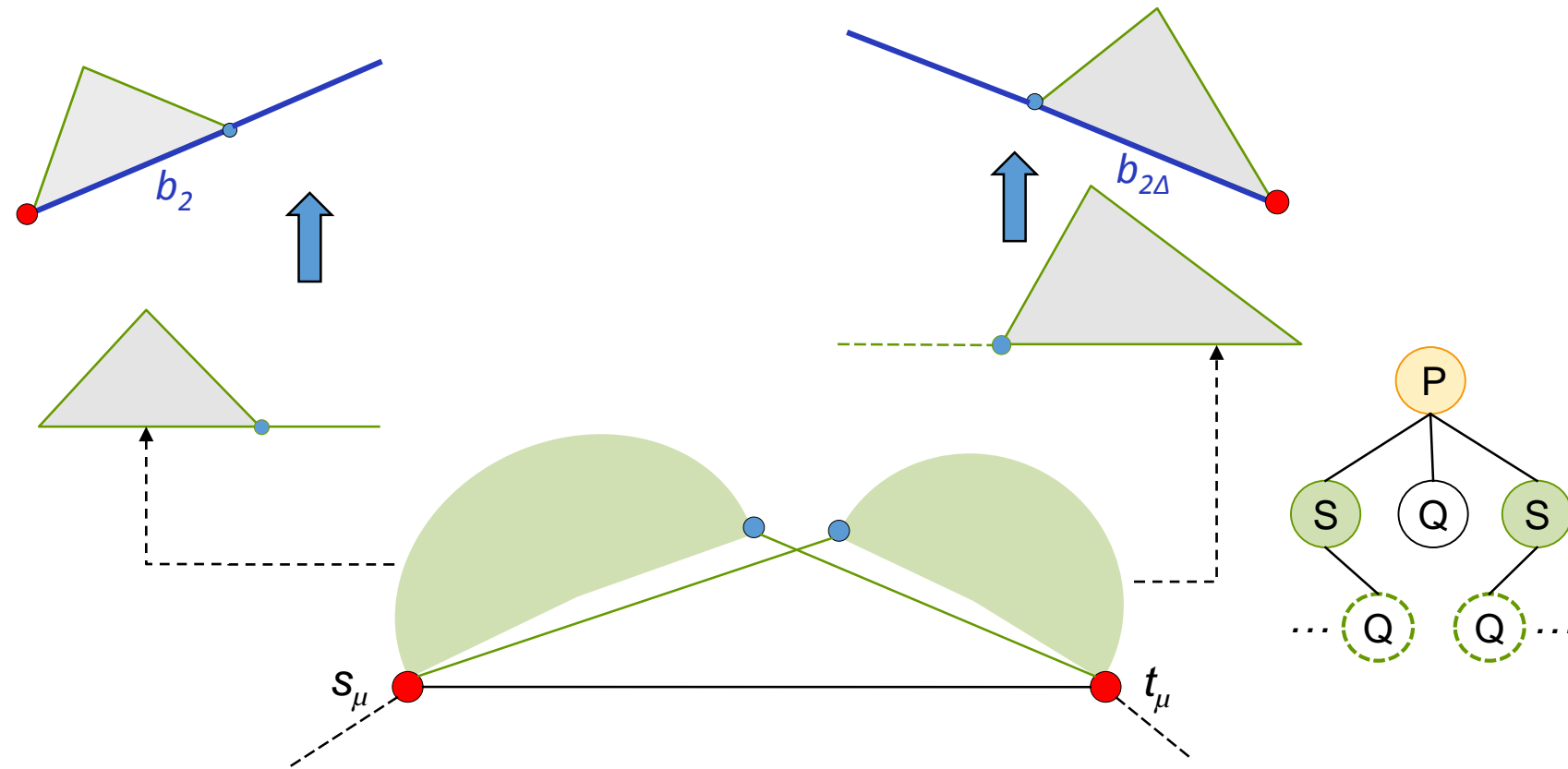
# How to draw P-nodes



# How to draw P-nodes

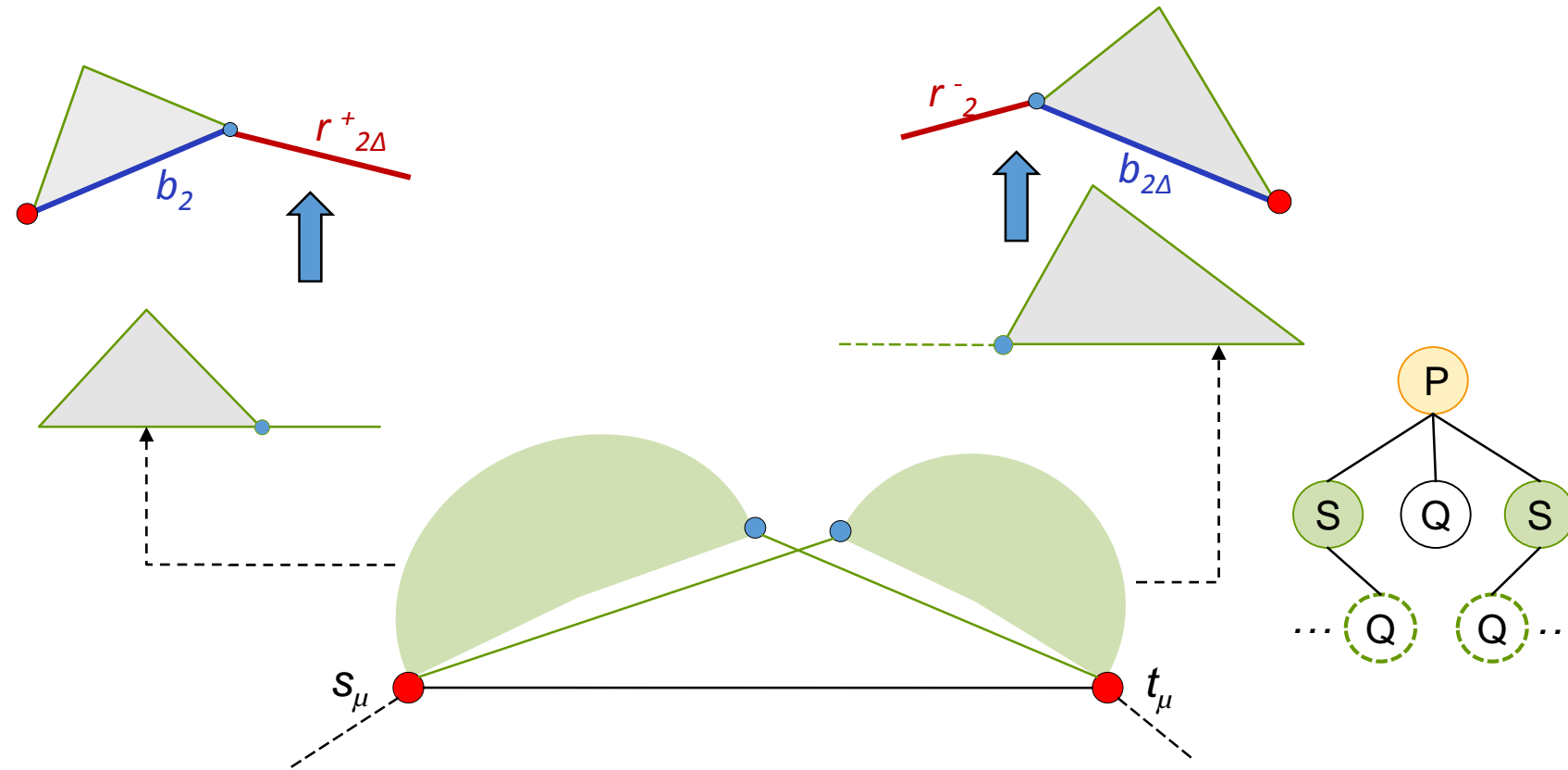


# How to draw P-nodes

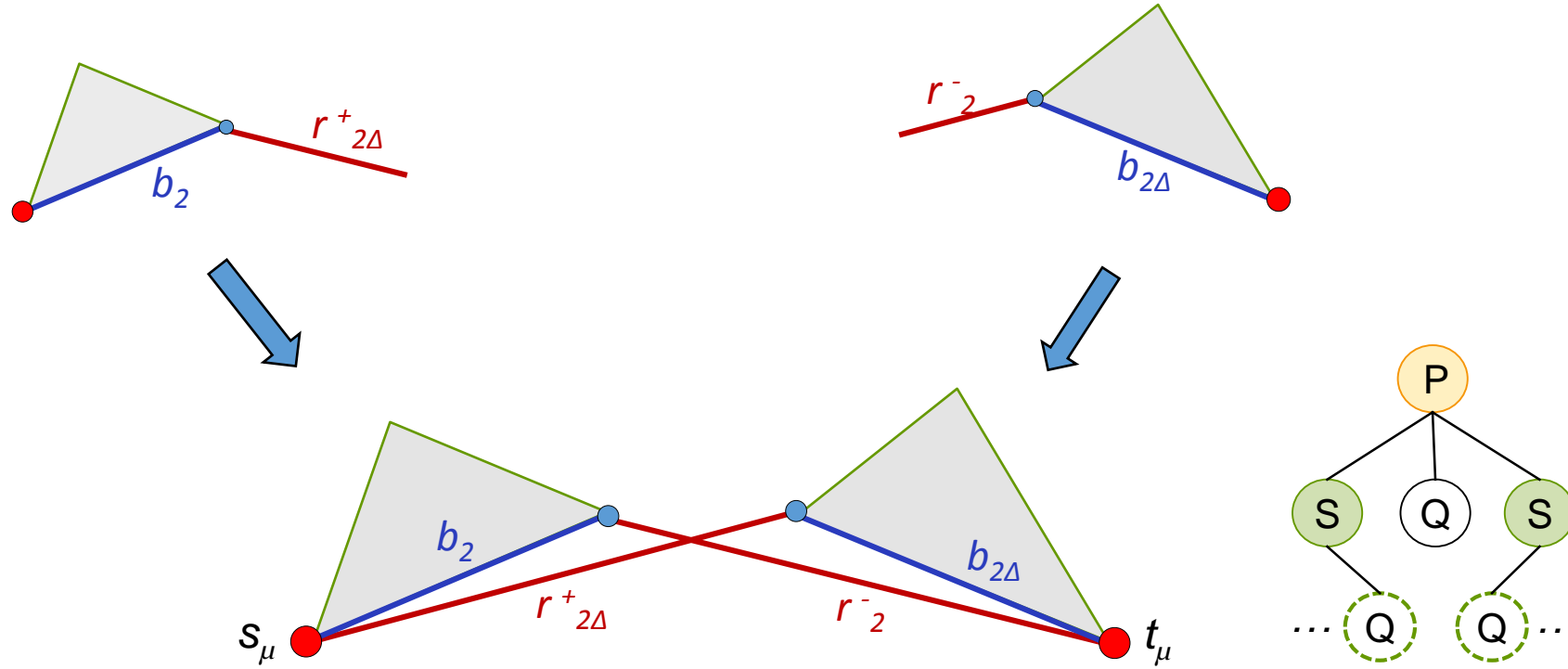




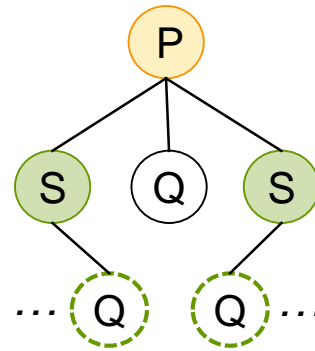
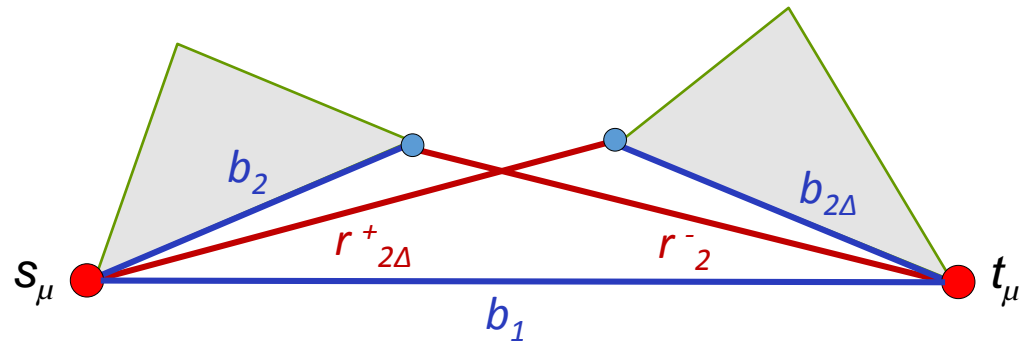
# How to draw P-nodes



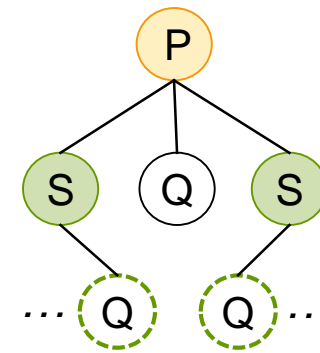
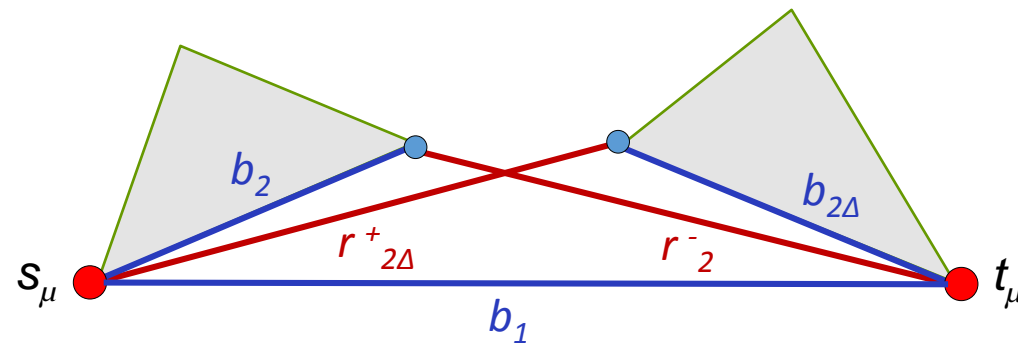
# How to draw P-nodes



# How to draw P-nodes



# How to draw P-nodes

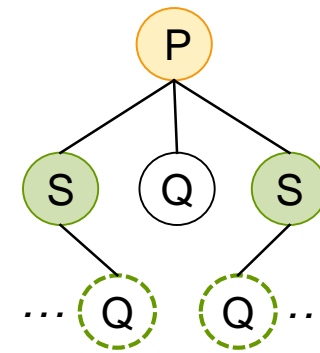
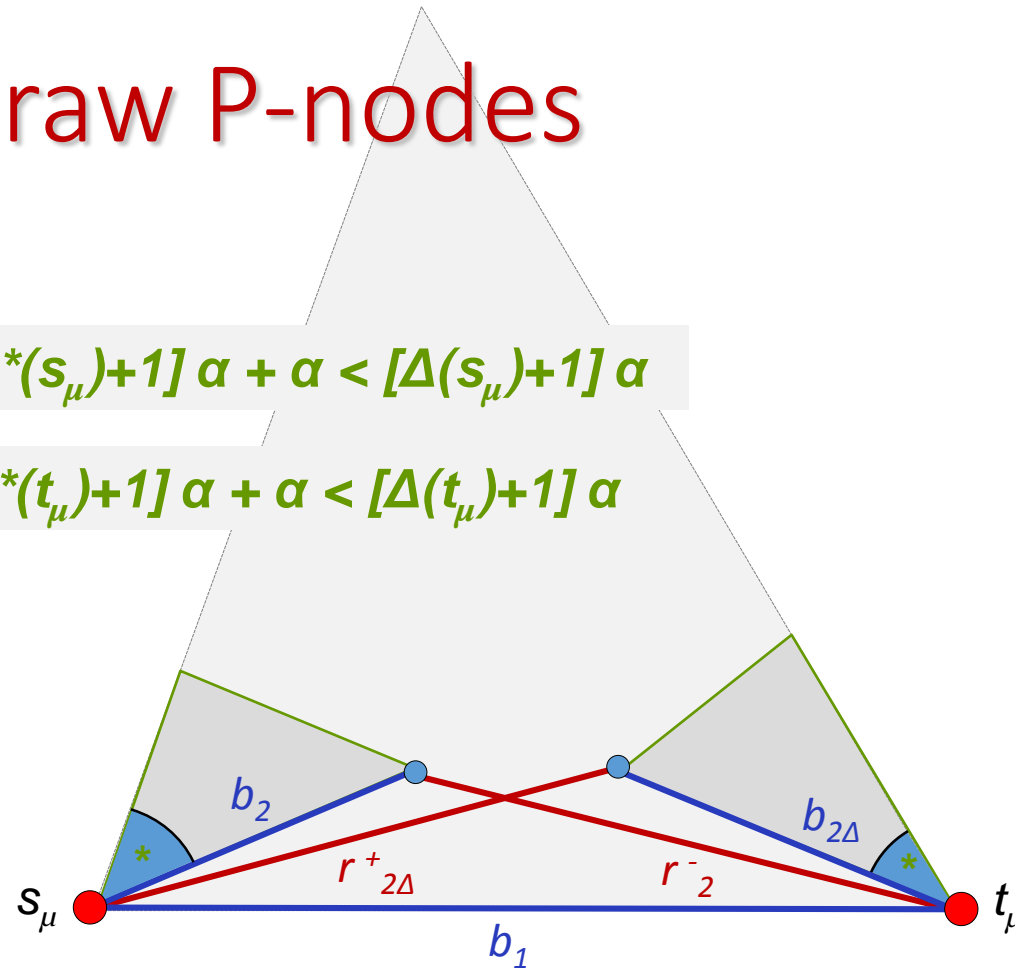


The drawing is o1p (11.) and uses only slopes in the univ. slope set (12.)

# How to draw P-nodes

$$\beta_\mu < [\Delta^*(s_\mu)+1] \alpha + \alpha < [\Delta(s_\mu)+1] \alpha$$

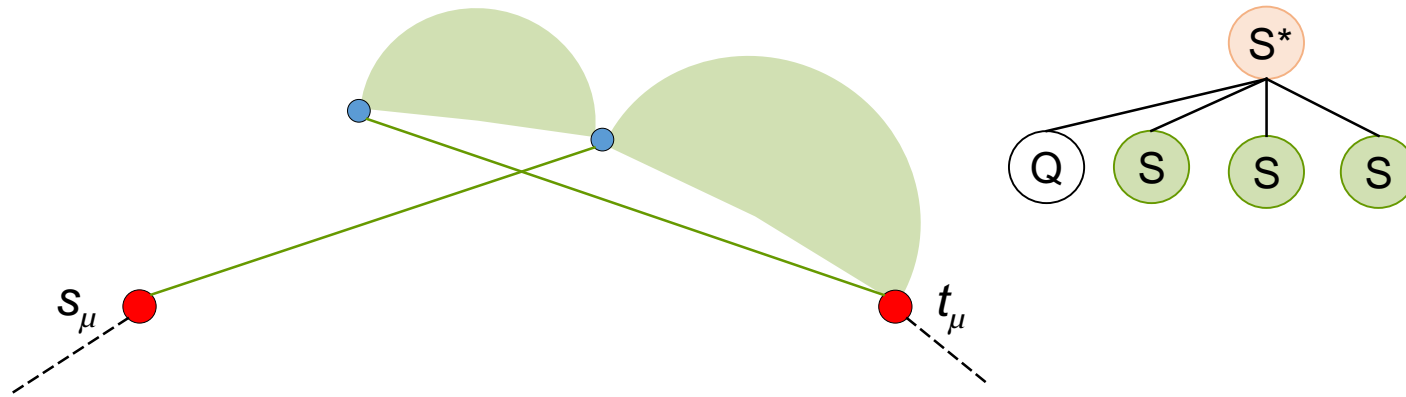
$$\gamma_\mu < [\Delta^*(t_\mu)+1] \alpha + \alpha < [\Delta(t_\mu)+1] \alpha$$



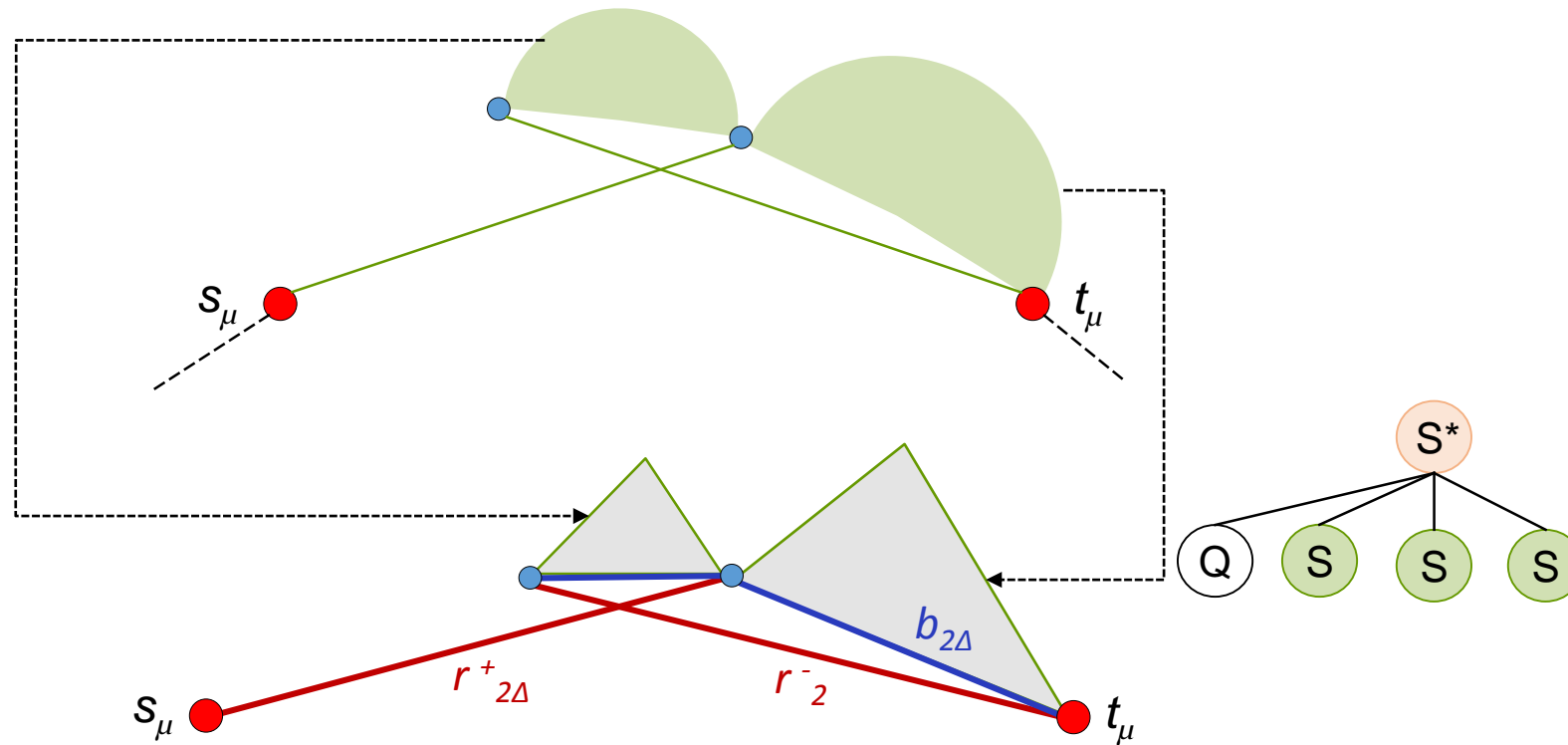
The drawing is contained in a triangle that respects the third invariant (**I3.**)

How to draw  $S^*$ -nodes

# How to draw $S^*$ -nodes



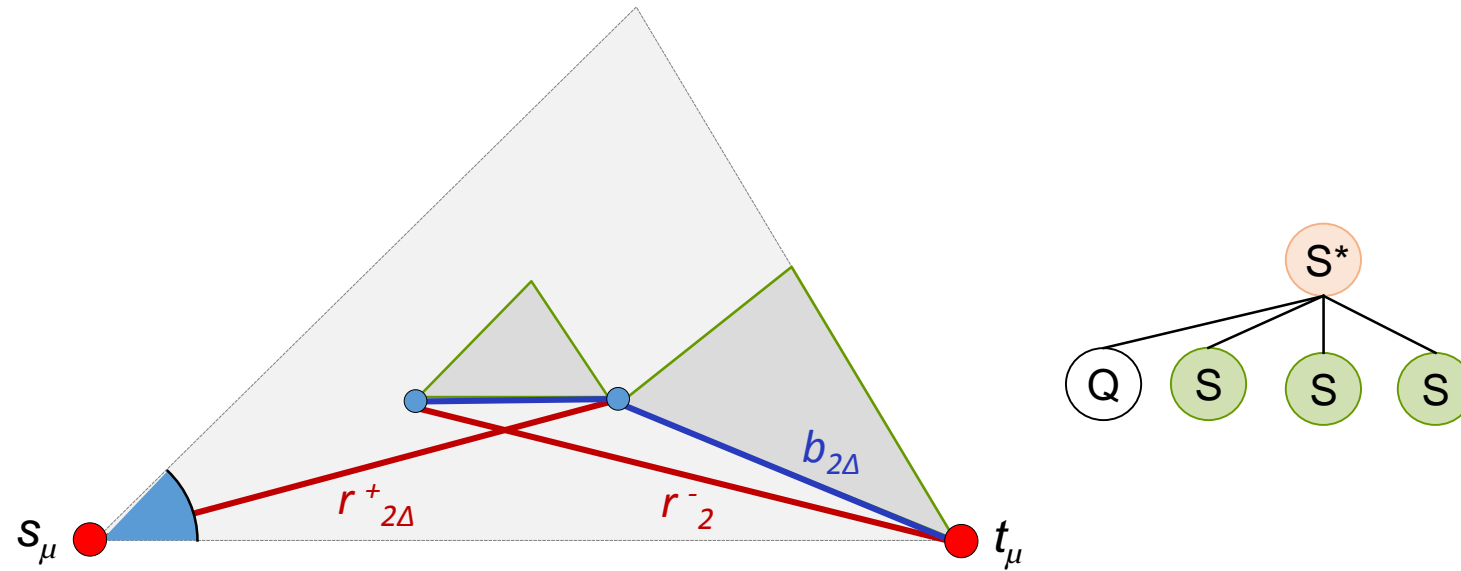
# How to draw $S^*$ -nodes



The drawing is o1p (**I1.**) and uses only slopes in the univ. slope set (**I2.**)

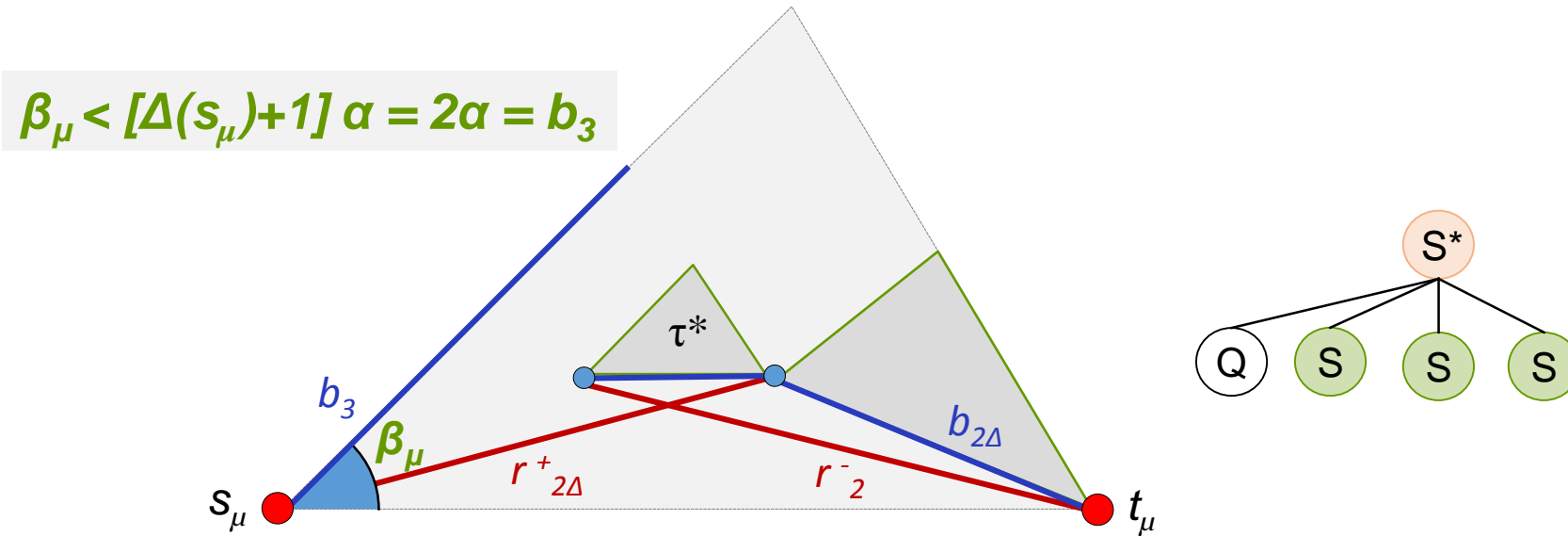


# How to draw $S^*$ -nodes



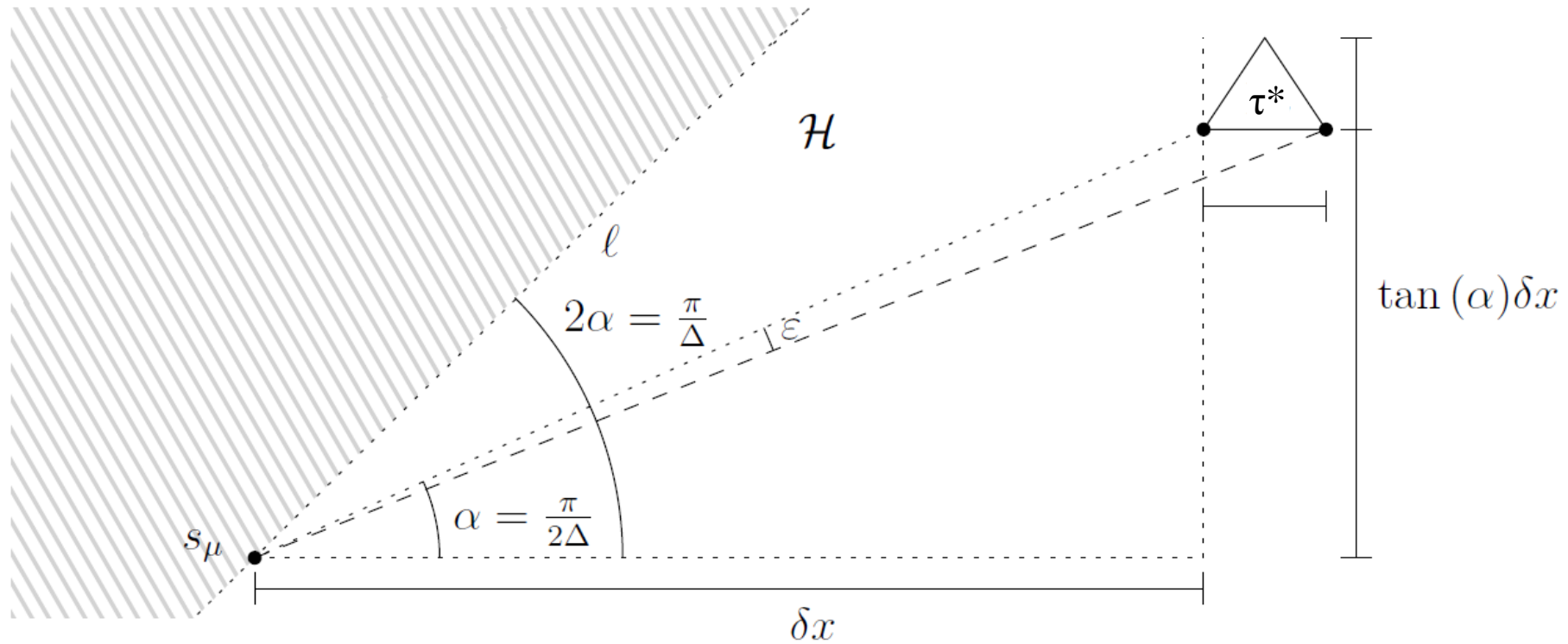
Is the drawing contained in a triangle that respects the third invariant (**I3.**)?

# How to draw $S^*$ -nodes



We need the line with slope  $b_3$  passing through  $s_\mu$  not to intersect  $\tau^*$ : **the height of  $\tau^*$  depends on  $\epsilon$**

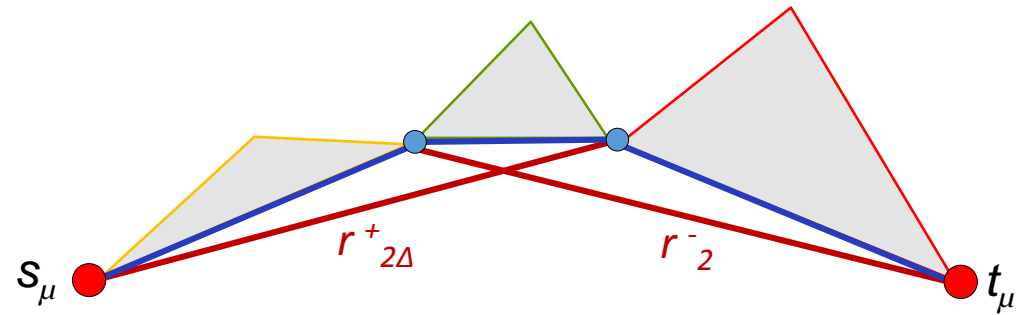
# How to draw $S^*$ -nodes



$$\varepsilon \leq \alpha - \arctan\left(\frac{\tan(\alpha)}{2 \tan(2\alpha) \tan(\alpha) - 2 \tan(\alpha) \tan(\alpha) + 1}\right)$$

$$0 < \varepsilon < \alpha$$

# How to draw R-nodes



Similar construction as for  $S^*$ -nodes

Open problems

# Open problems: (non-)planar graphs

## Planar graphs:

- Improve the upper bound  $O(2^{O(\Delta)})$  (or prove a better lower bound than  $3\Delta-6$ ) for the planar slope number of planar graphs
- Fill the gap between upper -  $O(\Delta^5)$  - and lower -  $2\Delta$  - bound for planar partial 3-trees

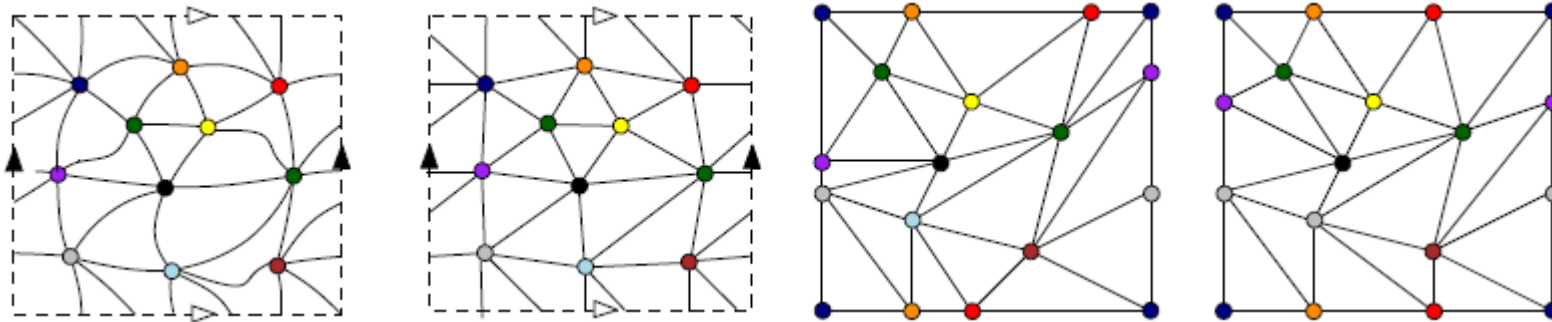
## Non-planar graphs:

- The case for  $\Delta = 4$  is still open
- Investigate new nearly-planar models (e.g., the 1-planar slope number)

# Open problems: toroidal graphs

Study the toroidal slope number of toroidal graphs

(...since toroidal graphs can be drawn with straight-line edges in a rectangle with periodic boundary conditions...)



Castelli Aleardi, Fusy, Kostygin,  
LATIN 2014