Partition identities and $q$-difference equations

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JCB 2016
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   - Schur’s theorem for overpartitions
   - Andrews’ generalisations of Schur’s theorem
   - Generalisation of Andrews’ theorems to overpartitions
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Introduction

Integer partitions

Definition

A *partition* of a positive integer $n$ is a finite non-increasing sequence of positive integers $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = n$. The integers $\lambda_1, \ldots, \lambda_m$ are called the *parts* of the partition.

Example

There are 5 partitions of 4:

$$4, 3 + 1, 2 + 2, 2 + 1 + 1 \text{ and } 1 + 1 + 1 + 1.$$  

Let $p(n)$ denote the number of partitions of $n$. 

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Generating functions

Question of Naudé (1740): How many partitions of 50 into 7 distinct parts?

Let $n, k$ be positive integers and let $Q(n, k)$ denote the number of partitions of $n$ into $k$ distinct parts. Then

$$1 + \sum_{n \geq 1} \sum_{k \geq 1} Q(n, k) z^k q^n = (1 + zq)(1 + zq^2)(1 + zq^3)(1 + zq^4) \cdots = \prod_{n \geq 1} (1 + zq^n).$$

Recurrence relation:

$$Q(n, k) = Q(n-k, k) + Q(n-k, k-1).$$

$\Rightarrow$ There are 522 partitions of 50 into 7 distinct parts.
Generating functions

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Solution of Euler: generating functions
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= \prod_{n \geq 1} (1 + zq^n).
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Recurrence relation: \( Q(n, k) = Q(n - k, k) + Q(n - k, k - 1) \).
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Recurrence relation: \( Q(n, k) = Q(n - k, k) + Q(n - k, k - 1) \).

\[\Rightarrow\] There are 522 partitions of 50 into 7 distinct parts.
Let \( p(n, k) \) denote the number of partitions of \( n \) into \( k \) parts. Then

\[
1 + \sum_{n \geq 1} \sum_{k \geq 1} p(n, k)z^k q^n = (1 + zq + z^2 q^2 + \cdots)(1 + zq^2 + z^2 q^4 + \cdots) \cdots
\]

\[
= \prod_{n \geq 1} \left(1 + zq^n + z^2 q^{2n} + \cdots\right)
\]

\[
= \prod_{n \geq 1} \frac{1}{(1 - zq^n)}.
\]
Theorem (Euler 1748)

For every integer $n$, the number of partitions of $n$ into distinct parts equals the number of partitions of $n$ into odd parts.
Introduction

Partition identities

Theorem (Euler 1748)

For every integer $n$, the number of partitions of $n$ into distinct parts equals the number of partitions of $n$ into odd parts.

Proof.

\[
\prod_{n \geq 1} (1 + q^n) = \prod_{n \geq 1} \frac{(1 + q^n)(1 - q^n)}{1 - q^n}
\]

\[
= \prod_{n \geq 1} \frac{1 - q^{2n}}{1 - q^n}
\]

\[
= \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}}.
\]
The Rogers-Ramanujan identities

The first Rogers-Ramanujan identity is the following $q$-series identity:

**Theorem (Rogers 1894, Rogers-Ramanujan 1919)**

$$
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 - q)(1 - q^2) \cdots (1 - q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1 - q^{5k+1})(1 - q^{5k+4})},
$$
The Rogers-Ramanujan identities

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**Theorem (Rogers 1894, Rogers-Ramanujan 1919)**

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}.$$ 

**Theorem (Partition version)**

For every positive integer $n$, the number of partitions of $n$ such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of $n$ into parts congruent to 1 or 4 modulo 5.
The Rogers-Ramanujan identities

The first Rogers-Ramanujan identity is the following $q$-series identity:

**Theorem (Rogers 1894, Rogers-Ramanujan 1919)**

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 - q)(1 - q^2) \cdots (1 - q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1 - q^{5k+1})(1 - q^{5k+4})},
\]

**Theorem (Partition version)**

*For every positive integer $n$, the number of partitions of $n$ such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of $n$ into parts congruent to 1 or 4 modulo 5.*

Rogers-Ramanujan type identity: “for all $n$, the number of partitions of $n$ satisfying some difference conditions is equal to the number of partitions of $n$ satisfying some congruence conditions.”
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   - Siladić’s theorem
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Theorem (Schur 1926)

For any positive integer $n$, let $A(n)$ denote the number of partitions of $n$ into distinct parts congruent to 1 or 2 modulo 3 and $B(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_m$ of $n$ such that

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3 & \text{if } \lambda_{i+1} \equiv 1, 2 \mod 3, \\ 4 & \text{if } \lambda_{i+1} \equiv 0 \mod 3. \end{cases}$$

Then $A(n) = B(n)$.

Example

The partitions counted by $A(10)$ are 10, 8 + 2, 7 + 2 + 1 and 5 + 4 + 1. The partitions counted by $B(10)$ are 10, 9 + 1, 8 + 2 and 7 + 3. There are 4 partitions in both cases.

Several proofs: Schur, Andrews, Bressoud, Bessenrodt, Alladi-Gordon, ...
Schur’s theorem and its generalisations

Idea of Andrews’ first proof

- $b_i(m, n)$: number of partitions $\lambda_1 + \cdots + \lambda_m$ of $n$ counted by $B(n)$, having $m$ parts, such that $\lambda_m \geq i$.

  $$f_i(x) = f_i(x, q) := 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_i(m, n)x^m q^n.$$  

- $f_1$ satisfies the following $q$-difference equation

  $$f_1(x) = (1 + xq + xq^2)f_1(xq^3) + xq^3(1 - xq^3)f_1(xq^6).$$

- We solve it and obtain:

  $$\sum_{n \geq 0} B(n)q^n = f_1(1) = \prod_{j=0}^{\infty} (1 + q^{3j+1})(1 + q^{3j+2}) = \sum_{n \geq 0} A(n)q^n.$$
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Overpartitions

Definition
Let $n$ be a positive integer. An overpartition of $n$ is a partition of $n$ in which the first occurrence of a number may be overlined.

Example
There are 8 overpartitions of 3:

$$3, \overline{3}, 2 + 1, \overline{2} + 1, 2 + \overline{1}, \overline{2} + \overline{1}, 1 + 1 + 1, \text{ and } \overline{1} + 1 + 1.$$
Overpartitions

Definition
Let $n$ be a positive integer. An overpartition of $n$ is a partition of $n$ in which the first occurrence of a number may be overlined.

Example
There are 8 overpartitions of 3:

$3$, $\bar{3}$, $2 + 1$, $\bar{2} + 1$, $2 + \bar{1}$, $\bar{2} + \bar{1}$, $1 + 1 + 1$, and $\bar{1} + 1 + 1$.

Let $\overline{p}(n, k)$ denotes the number of overpartitions of $n$ with $k$ non-overlined parts. Then

$$1 + \sum_{n \geq 1} \sum_{k \geq 0} \overline{p}(n, k) d^k q^n = \prod_{n \geq 1} \frac{1 + q^n}{1 - dq^n}.$$
Theorem (Lovejoy 2005)

Let $A(k, n)$ denote the number of overpartitions of $n$ into parts congruent to 1 or 2 modulo 3, with $k$ non-overlined parts. Let $B(k, n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of $n$, having $k$ non-overlined parts and satisfying the difference conditions

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 
0 + 3\chi(\lambda_{i+1}) & \text{if } \lambda_{i+1} \equiv 1, 2 \mod 3, \\
1 + 3\chi(\lambda_{i+1}) & \text{if } \lambda_{i+1} \equiv 0 \mod 3,
\end{cases}$$

where $\chi(\lambda_{i+1}) = 1$ if $\lambda_{i+1}$ is overlined and 0 otherwise.

Then for all $k, n$, $A(k, n) = B(k, n)$.

The case $k = 0$ corresponds to Schur’s theorem.
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Generalisations of Schur’s theorem

Notation:

\[(a; q)_\infty = \prod_{k \geq 0} (1 - aq^k).\]

**Schur**

\[(-q; q^3)_\infty (-q^2; q^3)_\infty\]

**Andrews**

\[(-q; q^N)_\infty \cdots (-q^{2r-1}; q^N)_\infty\]

**Lovejoy**

\[(-q; q^3)_\infty (-q^2; q^3)_\infty\]

\[\frac{(dq; q^3)_\infty (dq^2; q^3)_\infty}{(dq^N)_\infty \cdots (dq^{2r-1}; q^N)_\infty}\]

**D.**

\[\frac{(-q; q^N)_\infty \cdots (-q^{2r-1}; q^N)_\infty}{(dq; q^N)_\infty \cdots (dq^{2r-1}; q^N)_\infty}\]
Notation

- In the following, $r$ is a positive integer and $N \geq 2^r - 1$.
- $\beta_N(m) :=$ the least positive residue of $m \mod N$.
- For $\alpha \in \{1, 2, \ldots, 2^r - 1\}$,
  $w(\alpha) :=$ the number of powers of 2 appearing in the binary expansion of $\alpha$,
  $v(\alpha) :=$ the smallest power of 2 appearing in this expansion.
Andrews’ first theorem

**Theorem (Andrews)**

Let $D(r, N; n)$ denote the number of partitions of $n$ into distinct parts congruent to $2^0, 2^1, \ldots, 2^{r-1}$ modulo $N$.

Let $E(r, N; n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of $n$ into parts congruent to $1, 2, \ldots, 2^r - 1$ modulo $N$ such that

$$\lambda_i - \lambda_{i+1} \geq Nw(\beta_N(\lambda_{i+1})) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Then for all $n$, $D(r, N; n) = E(r, N; n)$.

Schur’s theorem corresponds to the case $r = 2, N = 3$. 
Andrews’ second theorem

Theorem (Andrews)

Let $F(r, N; n)$ denote the number of partitions of $n$ into distinct parts congruent to $-2^0, -2^1, \ldots, -2^{r-1}$ modulo $N$.

Let $G(r, N; n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of $n$ into parts congruent to $-1, -2, \ldots, -2^r + 1$ modulo $N$ such that

$$\lambda_i - \lambda_{i+1} \geq N w(\beta_N(-\lambda_i)) + v(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

and $\lambda_s \geq N (w(\beta_N(-\lambda_s)) - 1)$.

Then for all $n$, $F(r, N; n) = G(r, N; n)$.

Again, Schur’s theorem corresponds to the case $r = 2, N = 3$. But for other values, the two theorems are different.
The case $r = 3, N = 7$

**Theorem (Andrews 1)**

Let $A(n)$ denote the number of partitions of $n$ into distinct parts congruent to 1, 2 or 4 modulo 7. Let $B(n)$ denote the number of partitions of $n$ of the form $n = \lambda_1 + \cdots + \lambda_s$, where

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 
7 & \text{if } \lambda_{i+1} \equiv 1, 2, 4 \mod 7, \\
12 & \text{if } \lambda_{i+1} \equiv 3 \mod 7, \\
10 & \text{if } \lambda_{i+1} \equiv 5, 6 \mod 7, \\
15 & \text{if } \lambda_{i+1} \equiv 0 \mod 7.
\end{cases}$$

Then for all $n$, $A(n) = B(n)$. 
The case $r = 3, N = 7$

**Theorem (Andrews 2)**

Let $C(n)$ denote the number of partitions of $n$ into distinct parts congruent to 3, 5 or 6 modulo 7. Let $D(n)$ denote the number of partitions of $n$ of the form $n = \lambda_1 + \cdots + \lambda_s$, where

\[
\lambda_i - \lambda_{i+1} \geq \begin{cases} 
7 & \text{if } \lambda_i \equiv 3, 5, 6 \pmod{7}, \\
12 & \text{if } \lambda_i \equiv 4 \pmod{7}, \\
10 & \text{if } \lambda_i \equiv 1, 2 \pmod{7}, \\
15 & \text{if } \lambda_i \equiv 0 \pmod{7}, 
\end{cases}
\]

and $\lambda_s \neq 1, 2, 4, 7$.

Then for all $n$, $C(n) = D(n)$. 
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**The first theorem**

**Theorem (D. 2014)**

Let $D(r, N; k, n)$ denote the number of overpartitions of $n$ into parts congruent to $2^0, 2^1, \ldots, 2^{r-1}$ modulo $N$, having $k$ non-overlined parts.

Let $E(r, N; k, n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of $n$ into parts congruent to $1, 2, \ldots, 2^r - 1$ modulo $N$, having $k$ non-overlined parts, such that

$$\lambda_i - \lambda_{i+1} \geq N \left( w(\beta_N(\lambda_{i+1})) - 1 + \chi(\lambda_{i+1}) \right) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Then for all $k, n \geq 0$, $D(r, N; k, n) = E(r, N; k, n)$.

The case $k = 0$ corresponds to Andrews’ first theorem.

The case $N = 3, r = 2$ corresponds to Schur’s theorem for overpartitions.
The second theorem

Theorem (D. 2014)

Let $F(r, N; k, n)$ denote the number of overpartitions of $n$ into parts congruent to $-2^0, -2^1, \ldots, -2^{r-1}$ modulo $N$, having $k$ non-overlined parts. Let $G(r, N; k, n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of $n$ into parts congruent to $-1, -2, \ldots, -2^r + 1$ modulo $N$, having $k$ non-overlined parts, such that

$$
\lambda_i - \lambda_{i+1} \geq N \left( w(\beta_N(-\lambda_i)) - 1 + \chi(\lambda_{i+1}) \right) + v(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),
$$

$$
\lambda_s \geq N \left( w(\beta_N(-\lambda_s)) - 1 \right),
$$

Then for all $k, n \geq 0$, $F(r, N; k, n) = G(r, N; k, n)$.

The case $k = 0$ corresponds to Andrews’ second theorem. Again, the case $N = 3, r = 2$ corresponds to Lovejoy’s theorem.
Proof of the first theorem

- Define $b_i^r(k, m, n)$ as the number of overpartitions $\lambda_1 + \cdots + \lambda_m$ counted by $E(r, N; k, n)$, having $m$ parts, such that $\lambda_m \geq i$, and

$$f_i^r(x) = f_i^r(x, q, d) := 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} b_i^r(k, m, n)x^md^kq^n.$$ 

- Find the $q$-difference equation $(eq_{r, N})$ satisfied by $f_1^r(x)$.
- Show by induction on $r$ that a function $f$ satisfying $(eq_{r, N})$ and $f(0) = 1$ satisfies

$$f(1) = \prod_{k=0}^{r-1} \frac{(-q^{2^k}; q^N)_\infty}{(dq^{2^k}; q^N)_\infty},$$

which is the generating function for overpartitions counted by $D(r, N; k, n)$. 

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The theorem for \( r = 1, N = 3 \)

**Theorem**

Let \( A(k, n) \) denote the number of overpartitions of \( n \) into parts congruent to 1 modulo 3, having \( k \) non-overlined parts. Let \( B(k, n) \) denote the number of overpartitions \( \lambda_1 + \cdots + \lambda_m \) of \( n \) into parts congruent to 1 modulo 3, having \( k \) non-overlined parts, such that

\[
\lambda_i - \lambda_{i+1} \geq 0 + 3\chi(\lambda_{i+1}).
\]

Then \( A(k, n) = B(k, n) \).

We have

\[
b_1^1(k, m, n) - b_4^1(k, m, n) = b_1^1(k - 1, m - 1, n - 1) + b_1^1(k, m - 1, n - 3(m - 1) - 1).
\]
Thus
\[ b_1^1(k, m, n) = b_1^1(k, m, n - 3m) + b_1^1(k - 1, m - 1, n - 1) \]
\[ + b_1^1(k, m - 1, n - 3m + 2), \]
and
\[ (1 - dxq)f_1^1(x) = (1 + xq)f_1^1(xq^3). \]
Iterating gives
\[ f_1^1(x) = \prod_{k \geq 0} \frac{(1 + xq^{3k+1})}{(1 - dxq^{3k+1})} f_1^1(0) = \prod_{k \geq 0} \frac{(1 + xq^{3k+1})}{(1 - dxq^{3k+1})}. \]
So
\[ f_1^1(1) = \prod_{k \geq 0} \frac{(1 + q^{3k+1})}{(1 - dq^{3k+1})}. \]
Deducing Lovejoy’s theorem

In the same way as before, we obtain a $q$-difference equation:

$$(1 - dxq)(1 - dxq^2)f_1^2(x) = (1 + xq + xq^2 + dxq^3 - dx^2 q^3 - dx^2 q^6)f_1^2(xq^3)$$

$$+ xq^3(1 - xq^3)f_1^2(xq^6),$$

and $f_1^2(0) = 1$.

Let

$$F(x) := f_1^2(x) \prod_{k=0}^{\infty} \frac{1 - dxq^{3k+2}}{1 - xq^{3k}}.$$ 

Then $F(0) = 1$ and

$$(1 - dxq)(1 - x)F(x) = (1 + xq + xq^2 + dxq^3 - dx^2 q^3 - dx^2 q^6)F(xq^3)$$

$$+ xq^3(1 - dxq^5)F(xq^6).$$
Deducing Lovejoy’s theorem

Let

\[ F(x) =: \sum_{n \geq 0} A_n x^n. \]

Then \( A_0 = 1 \) and

\[
(1 - q^{3n})A_n = (1 + dq + q^{3n-2})(1 + q^{3n-1})A_{n-1} - dq(1 + q^{3n-1})(1 + q^{3n-4})A_{n-2}.
\]

Now

\[
a_n := \frac{A_n}{\prod_{k=0}^{n-1}(1 + q^{3k+2})}.
\]

Then \( a_0 = 1 \) and

\[
(1 - q^{3n})a_n = (1 + dq + q^{3n-2})a_{n-1} - dq a_{n-2}.
\]
Deducing Lovejoy’s theorem

Let
\[ G(x) := \sum_{n \geq 0} a_n x^n. \]

Then \( G(0) = 1 \) and
\[ (1 - x)(1 - dxq)G(x) = (1 + xq)G(xq^3). \]

Now
\[ g(x) := G(x) \prod_{k \geq 0} (1 - xq^{3k}). \]

Then \( g(0) = 1 \) and
\[ (1 - dxq)g(x) = (1 + xq)g(xq^3). \]

This is (eq1,3), so
\[ g(1) = f_1^1(1) = \prod_{k \geq 0} \frac{1 + q^{3k+1}}{(1 - dq^{3k+1})}. \]
Deducing Lovejoy’s theorem

Theorem (Appell’s Lemma)

Let \((a_n)_{n \in \mathbb{N}}\) be a sequence such that \(\lim_{n \to \infty} a_n\) is finite. Then

\[
\lim_{x \to 1} (1 - x) \sum_{n \geq 0} a_n x^n = \lim_{n \to \infty} a_n.
\]

Then

\[
\frac{g(1)}{\prod_{k \geq 1} (1 - q^{3k})} = \lim_{x \to 1} (1 - x) G(x) = \lim_{x \to 1} (1 - x) \sum_{n \geq 0} a_n x^n = \lim_{n \to \infty} a_n.
\]
Deducing Lovejoy’s theorem

Thus

$$\lim_{n \to \infty} a_n = \prod_{k \geq 0} \frac{(1 + q^{3k+1})}{(1 - dq^{3k+1})(1 - q^{3k+3})},$$

so

$$\lim_{n \to \infty} A_n = \prod_{k \geq 0} \frac{(1 + q^{3k+1})(1 + q^{3k+2})}{(1 - dq^{3k+1})(1 - q^{3k+3})},$$

and

$$\lim_{x \to 1} (1 - x)F(x) = \lim_{n \to \infty} A_n$$

$$= f_1^2(1) \prod_{k \geq 0} \frac{1 - dq^{3k+2}}{1 - q^{3k+3}}.$$
The $q$-difference equation ($eq_{r,N}$) in the general case

\[
\prod_{j=0}^{r-1} \left(1 - dxq^{2j}\right) f_1'(x) = f_1'(xq^N)
\]

\[
+ \sum_{j=1}^{r} \left( \sum_{m=0}^{r-j} d^m \sum_{\alpha<2^r} \sum_{w(\alpha)=j+m} xq^\alpha \left((-x)^{m-1} \binom{j+m-1}{m-1} \right)_{q^N} \right. \\
\left. + (-x)^m \binom{j+m}{m}_{q^N} \right) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_1'(xq^{jN}).
\]
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The theorem

Theorem (Siladić 2005)

The number of partitions of an integer \( n \) into distinct odd parts equals the number of partitions \( \lambda_1 + \cdots + \lambda_s \) of \( n \) into parts different from 2 such that the difference between two consecutive parts is at least 5 (ie. \( \lambda_i - \lambda_{i+1} \geq 5 \)) and

\[
\begin{align*}
\lambda_i - \lambda_{i+1} &= 5 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm1, \pm5, \pm7 \pmod{16}, \\
\lambda_i - \lambda_{i+1} &= 6 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm2, \pm6 \pmod{16}, \\
\lambda_i - \lambda_{i+1} &= 7 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm3 \pmod{16}, \\
\lambda_i - \lambda_{i+1} &= 8 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm4 \pmod{16}.
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\end{align*}
$$

Originally proved by studying representations of the twisted affine Lie algebra $A_2^{(2)}$. 
The refinement

Theorem (D. 2013)

For \( k, n \in \mathbb{N} \), let \( C(k, n) \) denote the number of partitions of \( n \) into \( k \) distinct odd parts. For \( n \in \mathbb{N} \) and \( k \in \mathbb{N}^* \), let \( D(k, n) \) denote the number of partitions \( \lambda_1 + \cdots + \lambda_s \) of \( n \) such that \( k \) equals the number of odd part plus twice the number of even parts, satisfying the following conditions:

1. \( \forall i \geq 1, \lambda_i \neq 2 \),
2. \( \forall i \geq 1, \lambda_i - \lambda_{i+1} \geq 5 \),
3. \( \forall i \geq 1, \lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i \equiv 1, 4 \mod 8 \),
   \( \lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i \equiv 1, 3, 5, 7 \mod 8 \),
   \( \lambda_i - \lambda_{i+1} = 7 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 6, 7 \mod 8 \),
   \( \lambda_i - \lambda_{i+1} = 8 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 5, 7 \mod 8 \).

Then for all \( k, n \in \mathbb{N} \), \( C(k, n) = D(k, n) \).
Idea of the proof

- Show that the two formulations are equivalent
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- The generating function for $C(k, n)$ is easy to obtain. We need to show that the generating function for $D(k, n)$ is the same:
Idea of the proof

- Show that the two formulations are equivalent
- The generating function for \( C(k, n) \) is easy to obtain. We need to show that the generating function for \( D(k, n) \) is the same:
  - Let \( d_N(k, n) \) denote the number of partitions \( \lambda_1 + \cdots + \lambda_s \) counted by \( D(k, n) \) such that the largest part \( \lambda_1 \) is at most \( N \), and

\[
G_N(t, q) = 1 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} d_N(k, n) t^k q^n.
\]

By a combinatorial reasoning, we establish eight \( q \)-difference equations satisfied by \( G_N(t, q) \).
Siladić’s theorem and the method of weighted words

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- By induction, we show that for all $m \in \mathbb{N}^*$,
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  G_{2m}(t, q) = (1 + tq) G_{2m-3}(tq^2, q).
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Idea of the proof

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  By a combinatorial reasoning, we establish eight $q$-difference equations satisfied by $G_N(t, q)$.
- By induction, we show that for all $m \in \mathbb{N}^*$,
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  G_{2m}(t, q) = (1 + tq) G_{2m-3}(tq^2, q).
  \]
- Letting $m \to \infty$ and iterating leads to
  \[
  \lim_{N \to \infty} G_N(t, q) = \prod_{k=0}^{\infty} (1 + tq^{2k+1}).
  \]
The $q$-difference equations

For all $N \in \mathbb{N^*}$,

\[
G_{8N}(t, q) = G_{8N-1}(t, q) + t^2 q^{8N} G_{8N-7}(t, q),
\]

\[
G_{8N+1}(t, q) = G_{8N}(t, q) + tq^{8N+1} G_{8N-4}(t, q),
\]

\[
G_{8N+2}(t, q) = G_{8N+1}(t, q) + t^2 q^{8N+2} G_{8N-7}(t, q),
\]

\[
G_{8N+3}(t, q) = G_{8N+2}(t, q) + tq^{8N+3} G_{8N-3}(t, q),
\]

\[
G_{8N+4}(t, q) = G_{8N+3}(t, q) + t^2 q^{8N+4} G_{8N-3}(t, q) + t^3 q^{16N+3} G_{8N-7}(t, q),
\]

\[
G_{8N+5}(t, q) = G_{8N+4}(t, q) + tq^{8N+5} G_{8N-3}(t, q) + t^2 q^{16N+4} G_{8N-7}(t, q),
\]

\[
G_{8N+6}(t, q) = G_{8N+5}(t, q) + t^2 q^{8N+6} G_{8N-3}(t, q) + t^3 q^{16N+5} G_{8N-7}(t, q),
\]

\[
G_{8N+7}(t, q) = G_{8N+6}(t, q) + tq^{8N+7} G_{8N+1}(t, q).
\]
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Example of Schur’s theorem

Reminder: Schur’s theorem

The number of partitions of $n$ into distinct parts congruent to 1 or 2 modulo 3 equals the number of partitions $\lambda_1 + \cdots + \lambda_m$ of $n$ such that

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3 & \text{if } \lambda_{i+1} \equiv 1, 2 \mod 3, \\ 4 & \text{if } \lambda_{i+1} \equiv 0 \mod 3. \end{cases}$$
Example of Schur’s theorem

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The method of weighted words consists of finding a refinement of the theorem by assigning a color to each part according to its value modulo 3.

- color $a : 1 \pmod{3}$,
- color $b : 2 \pmod{3}$,
- color $ab : 0 \pmod{3}$. 
The non-dilated version

Consider the positive integers in three colors, \(a\), \(b\), and \(ab\), with the order

\[1_{ab} < 1_a < 1_b < 2_{ab} < 2_a < 2_b < 3_{ab} < 3_a < 3_b < \cdots.\]
The non-dilated version

Consider the positive integers in three colors, $a$, $b$, and $ab$, with the order

$$1_{ab} < 1_a < 1_b < 2_{ab} < 2_a < 2_b < 3_{ab} < 3_a < 3_b < \cdots .$$

**Schur’s theorem non-dilated [Alladi-Gordon 1995]**

Let $S(u, v, n)$ denote the number of partitions of $n$ with $u$ parts colored $a$ or $ab$ and $v$ parts colored $b$ or $ab$ such that there is no part $1_{ab}$, and the difference $\lambda_i - \lambda_{i+1} \geq 2$ if $c(\lambda_i) = ab$ or $c(\lambda_i) < c(\lambda_{i+1})$. Then we have

$$\sum S(u, v, n) a^u b^v q^n = \prod_{n \geq 1} (1 + a q^n)(1 + b q^n).$$
The non-dilated version

Consider the positive integers in three colors, $a$, $b$, and $ab$, with the order

$$1_{ab} < 1_a < 1_b < 2_{ab} < 2_a < 2_b < 3_{ab} < 3_a < 3_b < \cdots .$$

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$$\sum S(u, v, n) a^u b^v q^n = \prod_{n \geq 1} (1 + aq^n) (1 + bq^n).$$

The dilation $q \rightarrow q^3, a \rightarrow aq^{-2}, b \rightarrow bq^{-1}$ implies Schur’s theorem.
Non-dilated version of Siladić’s theorem

Let us consider the following ordered set of colored integers:

\[ 1_{ab} < 1_a < 1_{b^2} < 1_b < 2_{ab} < 2_a < 3_{a^2} < 2_b < 3_{ab} < 3_a < 3_{b^2} < 3_b < \cdots \]

The colors \( a^2 \) and \( b^2 \) only appear for odd integers.
Non-dilated version of Siladić’s theorem

Let us consider the following ordered set of colored integers:

\[ 1_{ab} < 1_a < 1_{b^2} < 1_b < 2_{ab} < 2_a < 3_{a^2} < 2_b < 3_{ab} < 3_a < 3_{b^2} < 3_b < \cdots \]

The colors \( a^2 \) and \( b^2 \) only appear for odd integers.

Consider partitions \( \lambda_1 + \cdots + \lambda_s \) where the entry \((x, y)\) in the matrix \( A \) gives the minimal difference between \( \lambda_i \) of color \( x \) and \( \lambda_{i+1} \) of color \( y \):

\[
A = \begin{pmatrix}
a_{\text{odd}} & b^2 & b_{\text{odd}} & ab_{\text{even}} & a_{\text{even}} & a^2 & b_{\text{even}} & ab_{\text{odd}} \\
a & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\
b & 1 & 2 & 2 & 1 & 1 & 2 & 1 \\
ab & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 2 \\
a^2 & 4 & 4 & 4 & 3 & 3 & 4 & 3 & 4 \\
b^2 & 2 & 4 & 4 & 3 & 3 & 2 & 3 & 2
\end{pmatrix}.
\]
Theorem (D. 2016)

Let $D(u, v, n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of $n$, with no part $1_{ab}$ or $1_{b^2}$, satisfying the difference conditions given by the matrix, such that $u$ equals the number of parts $a$ or $ab$ plus twice the number of parts $a^2$ and $v$ equals the number of parts $b$ or $ab$ plus twice the number of parts $b^2$.

Then for all $u, v, n \in \mathbb{N}$,

$$
\sum D(u, v, n) a^u b^v q^n = \prod_{n \geq 1} (1 + a q^n)(1 + b q^n).
$$
Theorem (D. 2016)

Let \( D(u, v, n) \) denote the number of partitions \( \lambda_1 + \cdots + \lambda_s \) of \( n \), with no part \( 1_{ab} \) or \( 1_{b^2} \), satisfying the difference conditions given by the matrix, such that \( u \) equals the number of parts \( a \) or \( ab \) plus twice the number of parts \( a^2 \) and \( v \) equals the number of parts \( b \) or \( ab \) plus twice the number of parts \( b^2 \).

Then for all \( u, v, n \in \mathbb{N} \),

\[
\sum D(u, v, n) a^u b^v q^n = \prod_{n \geq 1} (1 + a q^n) (1 + b q^n).
\]

The dilation \( q \to q^4, a \to a q^{-3}, b \to b q^{-1} \) implies Siladić’s theorem.
If we keep the same order and difference conditions but do the dilation $q \rightarrow q^4, a \rightarrow aq^{-1}, b \rightarrow bq^{-3}$, we obtain a companion of Siladic’s theorem.

**Theorem (D. 2016)**

Let $C(k, n)$ denote the number of partitions of $n$ into $k$ distinct odd parts. Let $E(k, n)$ denote the number of partitions of $n$, where 2 is not a part, such that $k$ equals the number of odd part plus twice the number of even parts, satisfying the following conditions:

\[
\begin{align*}
\lambda_i - \lambda_{i+1} &= 5, 6, 8, 9 \text{ or } \geq 11 \text{ if } \lambda_i \equiv 0 \mod 8, \\
&= 2 \text{ or } \geq 5 \text{ if } \lambda_i \equiv 1 \mod 8, \\
&= 11 \text{ or } \geq 13 \text{ if } \lambda_i \equiv 2 \mod 8, \\
&\geq 7 \text{ if } \lambda \equiv 3 \mod 8, \\
&= 5 \text{ or } \geq 7 \text{ if } \lambda_i \equiv 4 \mod 8, \\
&= 2, 3, 5, 6 \text{ or } \geq 8 \text{ if } \lambda_i \equiv 5 \mod 8, \\
&= 3, 4, 6, 7 \text{ or } \geq 9 \text{ if } \lambda_i \equiv 6 \mod 8, \\
&= 8 \text{ or } \geq 10 \text{ if } \lambda_i \equiv 7 \mod 8.
\end{align*}
\]
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4 Perspectives
Perspectives

- Method of weighted words on the generalisations of Andrews’ theorems
- Generalise other Rogers-Ramanujan type partition identities to overpartitions (Capparelli, Siladić, ...)
- Combinatorial proof of other partition identities coming from Lie algebras