

Partition identities and q -difference equations

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 - Schur's theorem
 - Schur's theorem for overpartitions
 - Andrews' generalisations of Schur's theorem
 - Generalisation of Andrews' theorems to overpartitions
 - Deducing $(r = 2, N = 3)$ from $(r = 1, N = 3)$ in the first theorem
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Integer partitions

Definition

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \dots, \lambda_m$ such that $\lambda_1 + \dots + \lambda_m = n$. The integers $\lambda_1, \dots, \lambda_m$ are called the *parts* of the partition.

Example

There are 5 partitions of 4:

$$4, 3 + 1, 2 + 2, 2 + 1 + 1 \text{ and } 1 + 1 + 1 + 1.$$

Let $p(n)$ denote the number of partitions of n .

Generating functions

Question of Naudé (1740): How many partitions of 50 into 7 distinct parts?

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Solution of Euler: generating functions

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Solution of Euler: generating functions

Let n, k be positive integers and let $Q(n, k)$ denote the number of partitions of n into k distinct parts. Then

$$\begin{aligned}
 1 + \sum_{n \geq 1} \sum_{k \geq 1} Q(n, k) z^k q^n &= (1 + zq)(1 + zq^2)(1 + zq^3)(1 + zq^4) \cdots \\
 &= \prod_{n \geq 1} (1 + zq^n).
 \end{aligned}$$

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Recurrence relation: $Q(n, k) = Q(n - k, k) + Q(n - k, k - 1)$.

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Recurrence relation: $Q(n, k) = Q(n - k, k) + Q(n - k, k - 1)$.

\Rightarrow There are 522 partitions of 50 into 7 distinct parts.

Generating functions

Let $p(n, k)$ denote the number of partitions of n into k parts. Then

$$\begin{aligned}
 1 + \sum_{n \geq 1} \sum_{k \geq 1} p(n, k) z^k q^n &= (1 + zq + z^2q^2 + \cdots)(1 + zq^2 + z^2q^4 + \cdots) \cdots \\
 &= \prod_{n \geq 1} (1 + zq^n + z^2q^{2n} + \cdots) \\
 &= \prod_{n \geq 1} \frac{1}{(1 - zq^n)}.
 \end{aligned}$$

Partition identities

Theorem (Euler 1748)

For every integer n , the number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

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Proof.

$$\begin{aligned}
 \prod_{n \geq 1} (1 + q^n) &= \prod_{n \geq 1} \frac{(1 + q^n)(1 - q^n)}{1 - q^n} \\
 &= \prod_{n \geq 1} \frac{1 - q^{2n}}{1 - q^n} \\
 &= \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}}.
 \end{aligned}$$



The Rogers-Ramanujan identities

The first Rogers-Ramanujan identity is the following q -series identity:

Theorem (Rogers 1894, Rogers-Ramanujan 1919)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})},$$

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Theorem (Partition version)

For every positive integer n , the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.

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Theorem (Partition version)

For every positive integer n , the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.

Rogers-Ramanujan type identity: “for all n , the number of partitions of n satisfying some difference conditions is equal to the number of partitions of n satisfying some congruence conditions.”

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Theorem (Schur 1926)

For any positive integer n , let $A(n)$ denote the number of partitions of n into distinct parts congruent to 1 or 2 modulo 3 and $B(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_m$ of n such that

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3 & \text{if } \lambda_{i+1} \equiv 1, 2 \pmod{3}, \\ 4 & \text{if } \lambda_{i+1} \equiv 0 \pmod{3}. \end{cases}$$

Then $A(n) = B(n)$.

Example

The partitions counted by $A(10)$ are 10 , $8 + 2$, $7 + 2 + 1$ and $5 + 4 + 1$.

The partitions counted by $B(10)$ are 10 , $9 + 1$, $8 + 2$ and $7 + 3$.

There are 4 partitions in both cases.

Several proofs: Schur, Andrews, Bressoud, Bessenrodt, Alladi-Gordon, ...

Idea of Andrews' first proof

- $b_i(m, n)$: number of partitions $\lambda_1 + \cdots + \lambda_m$ of n counted by $B(n)$, having m parts, such that $\lambda_m \geq i$.

$$f_i(x) = f_i(x, q) := 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_i(m, n) x^m q^n.$$

- f_1 satisfies the following q -difference equation

$$f_1(x) = (1 + xq + xq^2)f_1(xq^3) + xq^3(1 - xq^3)f_1(xq^6).$$

- We solve it and obtain:

$$\sum_{n \geq 0} B(n)q^n = f_1(1) = \prod_{j=0}^{\infty} (1 + q^{3j+1})(1 + q^{3j+2}) = \sum_{n \geq 0} A(n)q^n.$$

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Overpartitions

Definition

Let n be a positive integer. An *overpartition* of n is a partition of n in which the first occurrence of a number may be overlined.

Example

There are 8 overpartitions of 3:

$$3, \overline{3}, 2 + 1, \overline{2} + 1, 2 + \overline{1}, \overline{2} + \overline{1}, 1 + 1 + 1, \text{ and } \overline{1} + 1 + 1.$$

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Example

There are 8 overpartitions of 3:

$$3, \bar{3}, 2 + 1, \bar{2} + 1, 2 + \bar{1}, \bar{2} + \bar{1}, 1 + 1 + 1, \text{ and } \bar{1} + 1 + 1.$$

Let $\bar{p}(n, k)$ denotes the number of overpartitions of n with k non-overlined parts. Then

$$1 + \sum_{n \geq 1} \sum_{k \geq 0} \bar{p}(n, k) d^k q^n = \prod_{n \geq 1} \frac{1 + q^n}{1 - dq^n}.$$

Theorem (Lovejoy 2005)

Let $A(k, n)$ denote the number of overpartitions of n into parts congruent to 1 or 2 modulo 3, with k non-overlined parts. Let $B(k, n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of n , having k non-overlined parts and satisfying the difference conditions

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 + 3\chi(\overline{\lambda_{i+1}}) & \text{if } \lambda_{i+1} \equiv 1, 2 \pmod{3}, \\ 1 + 3\chi(\overline{\lambda_{i+1}}) & \text{if } \lambda_{i+1} \equiv 0 \pmod{3}, \end{cases}$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise.

Then for all k, n , $A(k, n) = B(k, n)$.

The case $k = 0$ corresponds to Schur's theorem.

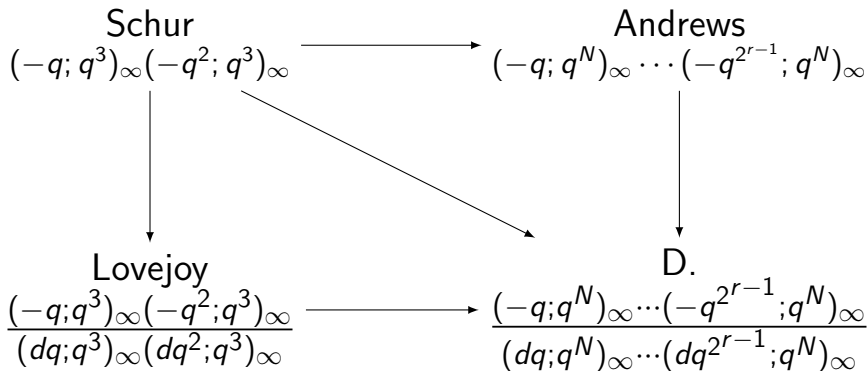
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Generalisations of Schur's theorem

Notation :

$$(a; q)_{\infty} = \prod_{k \geq 0} (1 - aq^k).$$



Notation

- In the following, r is a positive integer and $N \geq 2^r - 1$.
- $\beta_N(m) :=$ the least positive residue of $m \pmod N$.
- For $\alpha \in \{1, 2, \dots, 2^r - 1\}$,
 $w(\alpha) :=$ the number of powers of 2 appearing in the binary expansion of α ,
 $v(\alpha) :=$ the smallest power of 2 appearing in this expansion.

Andrews' first theorem

Theorem (Andrews)

Let $D(r, N; n)$ denote the number of partitions of n into distinct parts congruent to $2^0, 2^1, \dots, 2^{r-1}$ modulo N .

Let $E(r, N; n)$ denote the number of partitions $\lambda_1 + \dots + \lambda_s$ of n into parts congruent to $1, 2, \dots, 2^r - 1$ modulo N such that

$$\lambda_i - \lambda_{i+1} \geq Nw(\beta_N(\lambda_{i+1})) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Then for all n , $D(r, N; n) = E(r, N; n)$.

Schur's theorem corresponds to the case $r = 2$, $N = 3$.

Andrews' second theorem

Theorem (Andrews)

Let $F(r, N; n)$ denote the number of partitions of n into distinct parts congruent to $-2^0, -2^1, \dots, -2^{r-1}$ modulo N .

Let $G(r, N; n)$ denote the number of partitions $\lambda_1 + \dots + \lambda_s$ of n into parts congruent to $-1, -2, \dots, -2^r + 1$ modulo N such that

$$\lambda_i - \lambda_{i+1} \geq Nw(\beta_N(-\lambda_i)) + v(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

and $\lambda_s \geq N(w(\beta_N(-\lambda_s)) - 1)$.

Then for all n , $F(r, N; n) = G(r, N; n)$.

Again, Schur's theorem corresponds to the case $r = 2$, $N = 3$. But for other values, the two theorems are different.

The case $r = 3, N = 7$

Theorem (Andrews 1)

Let $A(n)$ denote the number of partitions of n into distinct parts congruent to 1, 2 or 4 modulo 7. Let $B(n)$ denote the number of partitions of n of the form $n = \lambda_1 + \dots + \lambda_s$, where

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 7 & \text{if } \lambda_{i+1} \equiv 1, 2, 4 \pmod{7}, \\ 12 & \text{if } \lambda_{i+1} \equiv 3 \pmod{7}, \\ 10 & \text{if } \lambda_{i+1} \equiv 5, 6 \pmod{7}, \\ 15 & \text{if } \lambda_{i+1} \equiv 0 \pmod{7}. \end{cases}$$

Then for all n , $A(n) = B(n)$.

The case $r = 3, N = 7$

Theorem (Andrews 2)

Let $C(n)$ denote the number of partitions of n into distinct parts congruent to 3, 5 or 6 modulo 7. Let $D(n)$ denote the number of partitions of n of the form $n = \lambda_1 + \cdots + \lambda_s$, where

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 7 & \text{if } \lambda_i \equiv 3, 5, 6 \pmod{7}, \\ 12 & \text{if } \lambda_i \equiv 4 \pmod{7}, \\ 10 & \text{if } \lambda_i \equiv 1, 2 \pmod{7}, \\ 15 & \text{if } \lambda_i \equiv 0 \pmod{7}, \end{cases}$$

and $\lambda_s \neq 1, 2, 4, 7$.

Then for all n , $C(n) = D(n)$.

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The first theorem

Theorem (D. 2014)

Let $D(r, N; k, n)$ denote the number of overpartitions of n into parts congruent to $2^0, 2^1, \dots, 2^{r-1}$ modulo N , having k non-overlined parts.

Let $E(r, N; k, n)$ denote the number of overpartitions $\lambda_1 + \dots + \lambda_s$ of n into parts congruent to $1, 2, \dots, 2^r - 1$ modulo N , having k non-overlined parts, such that

$$\lambda_i - \lambda_{i+1} \geq N \left(w(\beta_N(\lambda_{i+1})) - 1 + \chi(\overline{\lambda_{i+1}}) \right) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Then for all $k, n \geq 0$, $D(r, N; k, n) = E(r, N; k, n)$.

The case $k = 0$ corresponds to Andrews' first theorem.

The case $N = 3$, $r = 2$ corresponds to Schur's theorem for overpartitions.

The second theorem

Theorem (D. 2014)

Let $F(r, N; k, n)$ denote the number of overpartitions of n into parts congruent to $-2^0, -2^1, \dots, -2^{r-1}$ modulo N , having k non-overlined parts.

Let $G(r, N; k, n)$ denote the number of overpartitions $\lambda_1 + \dots + \lambda_s$ of n into parts congruent to $-1, -2, \dots, -2^r + 1$ modulo N , having k non-overlined parts, such that

$$\lambda_i - \lambda_{i+1} \geq N \left(w(\beta_N(-\lambda_i)) - 1 + \chi(\overline{\lambda_{i+1}}) \right) + v(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

$$\lambda_s \geq N \left(w(\beta_N(-\lambda_s)) - 1 \right),$$

Then for all $k, n \geq 0$, $F(r, N; k, n) = G(r, N; k, n)$.

The case $k = 0$ corresponds to Andrews' second theorem.

Again, the case $N = 3, r = 2$ corresponds to Lovejoy's theorem.

Proof of the first theorem

- Define $b_i^r(k, m, n)$ as the number of overpartitions $\lambda_1 + \dots + \lambda_m$ counted by $E(r, N; k, n)$, having m parts, such that $\lambda_m \geq i$, and

$$f_i^r(x) = f_i^r(x, q, d) := 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} b_i^r(k, m, n) x^m d^k q^n.$$

- Find the q -difference equation ($eq_{r,N}$) satisfied by $f_1^r(x)$.
- Show by induction on r that a function f satisfying ($eq_{r,N}$) and $f(0) = 1$ satisfies

$$f(1) = \prod_{k=0}^{r-1} \frac{(-q^{2^k}; q^N)_{\infty}}{(dq^{2^k}; q^N)_{\infty}},$$

which is the generating function for overpartitions counted by $D(r, N; k, n)$.

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The theorem for $r = 1, N = 3$

Theorem

Let $A(k, n)$ denote the number of overpartitions of n into parts congruent to 1 modulo 3, having k non-overlined parts. Let $B(k, n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_m$ of n into parts congruent to 1 modulo 3, having k non-overlined parts, such that

$$\lambda_i - \lambda_{i+1} \geq 0 + 3\chi(\overline{\lambda_{i+1}}).$$

Then $A(k, n) = B(k, n)$.

We have

$$\begin{aligned} b_1^1(k, m, n) - b_4^1(k, m, n) &= b_1^1(k-1, m-1, n-1) \\ &\quad + b_1^1(k, m-1, n-3(m-1)-1). \end{aligned}$$

Thus

$$b_1^1(k, m, n) = b_1^1(k, m, n - 3m) + b_1^1(k - 1, m - 1, n - 1) \\ + b_1^1(k, m - 1, n - 3m + 2),$$

and

$$(1 - dxq)f_1^1(x) = (1 + xq)f_1^1(xq^3).$$

Iterating gives

$$f_1^1(x) = \prod_{k \geq 0} \frac{(1 + xq^{3k+1})}{(1 - dxq^{3k+1})} f_1^1(0) = \prod_{k \geq 0} \frac{(1 + xq^{3k+1})}{(1 - dxq^{3k+1})}.$$

So

$$f_1^1(1) = \prod_{k \geq 0} \frac{(1 + q^{3k+1})}{(1 - dq^{3k+1})}.$$

Deducing Lovejoy's theorem

In the same way as before, we obtain a q -difference equation:

$$(1 - dxq)(1 - dxq^2)f_1^2(x) = (1 + xq + xq^2 + dxq^3 - dx^2q^3 - dx^2q^6)f_1^2(xq^3) + xq^3(1 - xq^3)f_1^2(xq^6),$$

and $f_1^2(0) = 1$.

Let

$$F(x) := f_1^2(x) \prod_{k=0}^{\infty} \frac{1 - dxq^{3k+2}}{1 - xq^{3k}}.$$

Then $F(0) = 1$ and

$$(1 - dxq)(1 - x)F(x) = (1 + xq + xq^2 + dxq^3 - dx^2q^3 - dx^2q^6)F(xq^3) + xq^3(1 - dxq^5)F(xq^6).$$

Deducing Lovejoy's theorem

Let

$$F(x) =: \sum_{n \geq 0} A_n x^n.$$

Then $A_0 = 1$ and

$$\begin{aligned} (1 - q^{3n})A_n &= (1 + dq + q^{3n-2})(1 + q^{3n-1})A_{n-1} \\ &\quad - dq(1 + q^{3n-1})(1 + q^{3n-4})A_{n-2}. \end{aligned}$$

Now

$$a_n := \frac{A_n}{\prod_{k=0}^{n-1} (1 + q^{3k+2})}.$$

Then $a_0 = 1$ and

$$(1 - q^{3n})a_n = (1 + dq + q^{3n-2})a_{n-1} - dqa_{n-2}.$$

Deducing Lovejoy's theorem

Let

$$G(x) := \sum_{n \geq 0} a_n x^n.$$

Then $G(0) = 1$ and

$$(1-x)(1-dxq)G(x) = (1+xq)G(xq^3).$$

Now

$$g(x) := G(x) \prod_{k \geq 0} (1 - xq^{3k}).$$

Then $g(0) = 1$ and

$$(1-dxq)g(x) = (1+xq)g(xq^3).$$

This is (eq_{1,3}), so

$$g(1) = f_1^1(1) = \prod_{k \geq 0} \frac{(1+q^{3k+1})}{(1-dq^{3k+1})}.$$

Deducing Lovejoy's theorem

Theorem (Appell's Lemma)

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \rightarrow \infty} a_n$ is finite. Then

$$\lim_{x \rightarrow 1} (1-x) \sum_{n \geq 0} a_n x^n = \lim_{n \rightarrow \infty} a_n.$$

Then

$$\begin{aligned} \frac{g(1)}{\prod_{k \geq 1} (1 - q^{3k})} &= \lim_{x \rightarrow 1} (1-x) G(x) \\ &= \lim_{x \rightarrow 1} (1-x) \sum_{n \geq 0} a_n x^n \\ &= \lim_{n \rightarrow \infty} a_n. \end{aligned}$$

Deducing Lovejoy's theorem

Thus

$$\lim_{n \rightarrow \infty} a_n = \prod_{k \geq 0} \frac{(1 + q^{3k+1})}{(1 - dq^{3k+1})(1 - q^{3k+3})},$$

so

$$\lim_{n \rightarrow \infty} A_n = \prod_{k \geq 0} \frac{(1 + q^{3k+1})(1 + q^{3k+2})}{(1 - dq^{3k+1})(1 - q^{3k+3})},$$

and

$$\begin{aligned} \lim_{x \rightarrow 1} (1 - x)F(x) &= \lim_{n \rightarrow \infty} A_n \\ &= f_1^2(1) \prod_{k \geq 0} \frac{1 - dq^{3k+2}}{1 - q^{3k+3}}. \end{aligned}$$

Thus

$$f_1^2(1) = \prod_{k \geq 0} \frac{(1 + q^{3k+1})(1 + q^{3k+2})}{(1 - dq^{3k+1})(1 - dq^{3k+2})}. \quad \square$$

The q -difference equation ($eq_{r,N}$) in the general case

$$\begin{aligned}
 & \prod_{j=0}^{r-1} \left(1 - dxq^{2^j}\right) f_1^r(x) = f_1^r(xq^N) \\
 & + \sum_{j=1}^r \left(\sum_{m=0}^{r-j} d^m \sum_{\substack{\alpha < 2^r \\ w(\alpha)=j+m}} xq^\alpha \left((-x)^{m-1} \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N} \right. \right. \\
 & \left. \left. + (-x)^m \begin{bmatrix} j+m \\ m \end{bmatrix}_{q^N} \right) \right) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_1^r(xq^{jN}).
 \end{aligned}$$

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The theorem

Theorem (Siladić 2005)

The number of partitions of an integer n into distinct odd parts equals the number of partitions $\lambda_1 + \dots + \lambda_s$ of n into parts different from 2 such that the difference between two consecutive parts is at least 5 (ie. $\lambda_i - \lambda_{i+1} \geq 5$) and

$$\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 1, \pm 5, \pm 7 \pmod{16},$$

$$\lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 2, \pm 6 \pmod{16},$$

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The theorem

Theorem (Siladić 2005)

The number of partitions of an integer n into distinct odd parts equals the number of partitions $\lambda_1 + \dots + \lambda_s$ of n into parts different from 2 such that the difference between two consecutive parts is at least 5 (ie. $\lambda_i - \lambda_{i+1} \geq 5$) and

$$\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 1, \pm 5, \pm 7 \pmod{16},$$

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Originally proved by studying representations of the twisted affine Lie algebra $A_2^{(2)}$.

The refinement

Theorem (D. 2013)

For $k, n \in \mathbb{N}$, let $C(k, n)$ denote the number of partitions of n into k distinct odd parts. For $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$, let $D(k, n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n such that k equals the number of odd part plus twice the number of even parts, satisfying the following conditions:

- ① $\forall i \geq 1, \lambda_i \neq 2,$
- ② $\forall i \geq 1, \lambda_i - \lambda_{i+1} \geq 5,$
- ③ $\forall i \geq 1,$

$$\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i \equiv 1, 4 \pmod{8},$$

$$\lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i \equiv 1, 3, 5, 7 \pmod{8},$$

$$\lambda_i - \lambda_{i+1} = 7 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 6, 7 \pmod{8},$$

$$\lambda_i - \lambda_{i+1} = 8 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 5, 7 \pmod{8}.$$

Then for all $k, n \in \mathbb{N}$, $C(k, n) = D(k, n)$.

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 - ▶ Let $d_N(k, n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ counted by $D(k, n)$ such that the largest part λ_1 is at most N , and

$$G_N(t, q) = 1 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} d_N(k, n) t^k q^n.$$

By a combinatorial reasoning, we establish eight q -difference equations satisfied by $G_N(t, q)$.

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- ▶ By induction, we show that for all $m \in \mathbb{N}^*$,

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- ▶ Letting $m \rightarrow \infty$ and iterating leads to

$$\lim_{N \rightarrow \infty} G_N(t, q) = \prod_{k=0}^{\infty} (1 + tq^{2k+1}).$$

The q -difference equations

For all $N \in \mathbb{N}^*$,

$$G_{8N}(t, q) = G_{8N-1}(t, q) + t^2 q^{8N} G_{8N-7}(t, q),$$

$$G_{8N+1}(t, q) = G_{8N}(t, q) + tq^{8N+1} G_{8N-4}(t, q),$$

$$G_{8N+2}(t, q) = G_{8N+1}(t, q) + t^2 q^{8N+2} G_{8N-7}(t, q),$$

$$G_{8N+3}(t, q) = G_{8N+2}(t, q) + tq^{8N+3} G_{8N-3}(t, q),$$

$$G_{8N+4}(t, q) = G_{8N+3}(t, q) + t^2 q^{8N+4} G_{8N-3}(t, q) + t^3 q^{16N+3} G_{8N-7}(t, q),$$

$$G_{8N+5}(t, q) = G_{8N+4}(t, q) + tq^{8N+5} G_{8N-3}(t, q) + t^2 q^{16N+4} G_{8N-7}(t, q),$$

$$G_{8N+6}(t, q) = G_{8N+5}(t, q) + t^2 q^{8N+6} G_{8N-3}(t, q) + t^3 q^{16N+5} G_{8N-7}(t, q),$$

$$G_{8N+7}(t, q) = G_{8N+6}(t, q) + tq^{8N+7} G_{8N+1}(t, q).$$

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Example of Schur's theorem

Reminder: Schur's theorem

The number of partitions of n into distinct parts congruent to 1 or 2 modulo 3 equals the number of partitions $\lambda_1 + \cdots + \lambda_m$ of n such that

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3 & \text{if } \lambda_{i+1} \equiv 1, 2 \pmod{3}, \\ 4 & \text{if } \lambda_{i+1} \equiv 0 \pmod{3}. \end{cases}$$

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The method of weighted words consists of finding a refinement of the theorem by assigning a color to each part according to its value modulo 3.

color a : $1 \pmod{3}$,

color b : $2 \pmod{3}$,

color ab : $0 \pmod{3}$.

The non-dilated version

Consider the positive integers in three colors, a , b , and ab , with the order

$$1_{ab} < 1_a < 1_b < 2_{ab} < 2_a < 2_b < 3_{ab} < 3_a < 3_b < \dots .$$

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Consider the positive integers in three colors, a , b , and ab , with the order

$$1_{ab} < 1_a < 1_b < 2_{ab} < 2_a < 2_b < 3_{ab} < 3_a < 3_b < \dots .$$

Schur's theorem non-dilated [Alladi-Gordon 1995]

Let $S(u, v, n)$ denote the number of partitions of n with u parts colored a or ab and v parts colored b or ab such that there is no part 1_{ab} , and the difference $\lambda_i - \lambda_{i+1} \geq 2$ if $c(\lambda_i) = ab$ or $c(\lambda_i) < c(\lambda_{i+1})$. Then we have

$$\sum S(u, v, n) a^u b^v q^n = \prod_{n \geq 1} (1 + aq^n)(1 + bq^n).$$

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$$\sum S(u, v, n) a^u b^v q^n = \prod_{n \geq 1} (1 + aq^n)(1 + bq^n).$$

The dilation $q \rightarrow q^3, a \rightarrow aq^{-2}, b \rightarrow bq^{-1}$ implies Schur's theorem.

Non-dilated version of Siladić's theorem

Let us consider the following ordered set of colored integers:

$$1_{ab} < 1_a < 1_{b^2} < 1_b < 2_{ab} < 2_a < 3_{a^2} < 2_b < 3_{ab} < 3_a < 3_{b^2} < 3_b < \dots$$

The colors a^2 and b^2 only appear for odd integers.

Non-dilated version of Siladić's theorem

Let us consider the following ordered set of colored integers:

$$1_{ab} < 1_a < 1_{b^2} < 1_b < 2_{ab} < 2_a < 3_{a^2} < 2_b < 3_{ab} < 3_a < 3_{b^2} < 3_b < \dots$$

The colors a^2 and b^2 only appear for odd integers.

Consider partitions $\lambda_1 + \dots + \lambda_s$ where the entry (x, y) in the matrix A gives the minimal difference between λ_i of color x and λ_{i+1} of color y :

$$A = \begin{matrix} & a_{\text{odd}} & b^2 & b_{\text{odd}} & ab_{\text{even}} & a_{\text{even}} & a^2 & b_{\text{even}} & ab_{\text{odd}} \\ \begin{matrix} a \\ b \\ ab \\ a^2 \\ b^2 \end{matrix} & \left(\begin{array}{ccccccccc} 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 3 & 3 & 4 & 3 & 3 & 4 \\ 2 & 4 & 4 & 3 & 3 & 2 & 3 & 3 & 2 \end{array} \right) \end{matrix}.$$

Theorem (D. 2016)

Let $D(u, v, n)$ denote the number of partitions $\lambda_1 + \dots + \lambda_s$ of n , with no part 1_{ab} or 1_{b^2} , satisfying the difference conditions given by the matrix, such that u equals the number of parts a or ab plus twice the number of parts a^2 and v equals the number of parts b or ab plus twice the number of parts b^2 .

Then for all $u, v, n \in \mathbb{N}$,

$$\sum D(u, v, n) a^u b^v q^n = \prod_{n \geq 1} (1 + aq^n)(1 + bq^n).$$

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The dilation $q \rightarrow q^4, a \rightarrow aq^{-3}, b \rightarrow bq^{-1}$ implies Siladić's theorem.

If we keep the same order and difference conditions but do the dilation $q \rightarrow q^4$, $a \rightarrow aq^{-1}$, $b \rightarrow bq^{-3}$, we obtain a companion of Siladic's theorem.

Theorem (D. 2016)

Let $C(k, n)$ denote the number of partitions of n into k distinct odd parts. Let $E(k, n)$ denote the number of partitions of n , where 2 is not a part, such that k equals the number of odd part plus twice the number of even parts, satisfying the following conditions:

$$\lambda_i - \lambda_{i+1} \begin{cases} = 5, 6, 8, 9 \text{ or } \geq 11 \text{ if } \lambda_i \equiv 0 \pmod{8}, \\ = 2 \text{ or } \geq 5 \text{ if } \lambda_i \equiv 1 \pmod{8}, \\ = 11 \text{ or } \geq 13 \text{ if } \lambda_i \equiv 2 \pmod{8}, \\ \geq 7 \text{ if } \lambda_i \equiv 3 \pmod{8}, \\ = 5 \text{ or } \geq 7 \text{ if } \lambda_i \equiv 4 \pmod{8}, \\ = 2, 3, 5, 6 \text{ or } \geq 8 \text{ if } \lambda_i \equiv 5 \pmod{8}, \\ = 3, 4, 6, 7 \text{ or } \geq 9 \text{ if } \lambda_i \equiv 6 \pmod{8}, \\ = 8 \text{ or } \geq 10 \text{ if } \lambda_i \equiv 7 \pmod{8}. \end{cases}$$

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Perspectives

- Method of weighted words on the generalisations of Andrews' theorems
- Generalise other Rogers-Ramanujan type partition identities to overpartitions (Capparelli, Siladić, ...)
- Combinatorial proof of other partition identities coming from Lie algebras