CONNECTING ANALYTIC AND ASYMPTOTIC BEHAVIOUR THROUGH MULTIVARIATE DIAGONALS

Stephen Melczer
University of Waterloo and École normale supérieure de Lyon / LIP

Joint work with Mark Wilson, Marni Mishna, George Labahn, and Bruno Salvy

How many steps did I take?
PROLOGUE

A (Brief) Introduction to Analytic Combinatorics
Generating Functions

A priori, a generating function is a \textit{formal object} but it \textit{may} also be the expansion of an analytic function at the origin.
Generating Functions

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**Example**

\[ P(z) = \sum_{n \geq 0} n!z^n \]

does not converge for any \( z \neq 0 \).

**Example**

\[ B(z) = \sum_{n \geq 0} 2^n z^n = \frac{1}{1 - 2z} \]

converges for \( |z| < 1/2 \).
If \( S(z) = \sum_{k \geq 0} s_k z^k \) then

\[
\frac{1}{2\pi i} \oint_C S(z) \frac{dz}{z^{n+1}} = \frac{1}{2\pi i} \int_C \sum_{k \geq 0} s_k z^{k-n-1} \, dz
\]

\[
= \sum_{k \geq 0} \frac{1}{2\pi i} \oint_C s_k z^{k-n-1} \, dz
\]

\[
= s_n
\]

for a suitable closed curve \( C \) around the origin.
Theorem (Flajolet-Odlyzko / Darboux)

If \( f(z) \) is analytic in \( |z| \leq 1/R + \epsilon \), except at \( z = 1/R \) where

\[
f(z) = (1 - Rz)^\alpha + O \left( (1 - Rz)^A \right) \quad \alpha \notin \{0, 1, \ldots\}
\]

as \( z \to 1/R \), with \( \alpha < A \), then

\[
[z^n]f(z) = R^n \cdot \frac{n^{-\alpha-1}}{\Gamma(\alpha)} + O \left( R^n n^{-A-1} \right)
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\]

Exponentially smaller

Know local behaviour
Univariate AC

Theorem (Pringsheim)
If \( F(z) = \sum_{n \geq 0} a_n z^n \) is analytic at the origin with radius of convergence \( \rho > 0 \) and \( a_n \geq 0 \) then \( z = \rho \) is a singularity with minimal modulus.

Thus, the general method of AC (often) has one:

1. Find singularity closest to origin (using Pringsheim)
2. Calculate contribution of each singularity with this modulus
ACT I

A Combinatorial Problem
Lattice Paths in a Quadrant

**Given:** A finite set of steps / directions \( S \subset \{\pm 1, 0\}^2 \setminus \{(0,0)\} \)

**Count:** \( c_n = \# \) of walks staying in \( \mathbb{N}^2 \), starting at the origin
Lattice Paths in a Quadrant

Given: A finite set of steps / directions $\mathcal{S} \subset \{\pm 1, 0\}^2 \setminus \{(0, 0)\}$

Count: $c_n = \#$ of walks staying in $\mathbb{N}^2$, starting at the origin

For instance, given $\mathcal{S} = \{(1, -1), (1, 1), (-1, 1), (-1, -1)\} \subset \mathbb{Z}^2$

we have:

$c_1 = 1$
Lattice Paths in a Quadrant

**Given:** A finite set of steps / directions $S \subset \{\pm 1, 0\}^2 \setminus \{(0, 0)\}$

**Count:** $c_n = \# \text{ of walks staying in } \mathbb{N}^2$, starting at the origin

For instance, given $S = \{(1, -1), (1, 1), (-1, 1), (-1, -1)\} \subset \mathbb{Z}^2$

we have:

$c_2 = 4$
**Lattice Paths in a Quadrant**

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**Count:** $c_n = \#$ of walks staying in $\mathbb{N}^2$, starting at the origin

For instance, given $S = \{(1, -1), (1, 1), (-1, 1), (-1, -1)\} \subset \mathbb{Z}^2$

we have:

![Diagram of lattice paths]

$c_3 = 9$
Importance

Studied as far back as 18th and 19th centuries.

Applications to:

- Statistical mechanics (polymers in solution, Ising model, ...)
- Queueing theory / operations research
- Discrete Structures (trees, words, plane partitions, ...)
- Probability Theory (random walks, branching processes, ...)

Useful sandbox for developing methods for generating functions
A Reduction

A priori, there are $2^8 - 1 = 255$ different models. Some, like
diagram

are trivial. Some, like
diagram

live in a half-space and can be solved by methods of Banderier and Flajolet (2001).

Finally, some are equivalent.

diagram
Bousquet-Mélou and Mishna (2010) showed there are 79 non-isomorphic two dimensional models.
The Kernel Method

Bousquet-Mélou and Mishna (2010) were able to show that many of these walks have a \textit{D-finite} generating function.
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Let

\[ C(x, y, t) = \sum_{i, j, n \geq 0} c_{i,j,n} x^i y^j t^n \]

where \( c_{i,j,n} \) is the number of walks of length \( n \) staying in the quarter plane and ending at the point \((i, j)\).
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where \( c_{i,j,n} \) is the number of walks of length \( n \) staying in the quarter plane and ending at the point \((i, j)\).

Then

\[ C(1, 1, t) = \text{GF of all walks} \]
\[ C(1, 0, t) = \text{GF of walks ending at } y = 0 \]
\[ C(0, 1, t) = \text{GF of walks ending at } x = 0 \]
\[ C(0, 0, t) = \text{GF of walks ending at } (0, 0) \]
The Kernel Method

The recursive nature of a walk of length $n$ ending at $(i,j)$ implies that $C(x,y,t)$ satisfies a functional equation of the form

$$K(x,y,t) \cdot C(x,y,t) = 1 + A(x) \cdot C(x,0,t) + B(y) \cdot C(0,y,t)$$
The Kernel Method

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Bousquet-Mélou and Mishna use a group $\mathcal{G}$ of bi-rational transformations of the plane associated to this model.

When the group is finite it can usually be combined with the functional equation to give a nice representation of $C(1, 1, t)$. 
Finite Group

Infinite Group
$C(x, y, t)$ is D-Finite (Bousquet-Mélou & Mishna 2010)
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$C(x, y, t)$ is algebraic (Bostan & Kauers 2010)

$C(x, y, t)$ is not D-Finite (M. & Mishna 2014)

(Bostan, Raschel, Salvy 2014)
<table>
<thead>
<tr>
<th>$n$</th>
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<td>$\frac{4}{\pi} \cdot \frac{4^n}{n}$</td>
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**Tab. 1:** Asymptotics for the 23 D-finite models.

$A_n = 4(1-(-1)^n)+3\sqrt{2}(1+(-1)^n)$,

$B_n = \sqrt{3}(1-(-1)^n)+2(1+(-1)^n)$,

$C_n = 12/\sqrt{5}(1-(-1)^n)+\sqrt{30}(1+(-1)^n)$
Bostan and Kauers (2009) Guessed Asymptotics

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$C(1,1,t)$ is algebraic and has these asymptotics (Bousquet-Mélou Mishna & Bostan Kauers)
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<td>$\frac{7\sqrt{21}}{54\sqrt{\pi \cdot n^{3/2}}}$</td>
<td>$\frac{D}{285\pi \cdot n^2}$</td>
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<td>$\frac{2E}{1805\pi \cdot n^3}$</td>
<td>$\frac{768}{\pi \cdot n^5}$</td>
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<tr>
<td>$\sigma_n \frac{81\sqrt{3}}{\pi \cdot n^4}$</td>
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$A = 1 + \sqrt{2}$, \hspace{1cm} $B = 1 + \sqrt{3}$, \hspace{1cm} $C = 1 + \sqrt{6}$, \hspace{1cm} $D = (156 + 41\sqrt{6})\sqrt{23 - 3\sqrt{6}}$, \hspace{1cm} $E = (583 + 138\sqrt{6})\sqrt{23 - 3\sqrt{6}}$

$\delta_n = 1$ if $n \equiv 0 \mod 2$, $\sigma_n = 1$ if $n \equiv 0 \mod 3$, and $\epsilon_n = 1$ if $n \equiv 0 \mod 4$ – each is 0 otherwise.
Diagonals of Rational Functions

For an element

$$A(x, y, t) = \sum_{n \geq 0} \left( \sum_{i,j} a_{i,j,n} x^i y^j \right) t^n \in \mathbb{Q}[x, \overline{x}, y, \overline{y}][[t]]$$

define the diagonal operator

$$\Delta A(x, y, t) := \sum_{n \geq 0} a_{n,n,n} t^n \in \mathbb{Q}[[t]]$$
Diagonals of Rational Functions

For an element

\[ A(x, y, t) = \sum_{n \geq 0} \left( \sum_{i,j} a_{i,j,n} x^i y^j \right) t^n \in \mathbb{Q}[x, x, y, \bar{y}][[t]] \]

define the diagonal operator

\[ \Delta A(x, y, t) := \sum_{n \geq 0} a_{n,n,n} t^n \in \mathbb{Q}[[t]] \]

For a step set \( S \), define the \textit{characteristic} polynomial

\[ S(x, y) = \sum_{(i,j) \in S} x^{i} y^{j} \]

**Example**

If \( S = \{ (\pm 1, 0), (0, \pm 1) \} \) then \( S(x, y) = \bar{x} + x + \bar{y} + y \).
Diagonals of Rational Functions

Theorem
For each of the 23 D-finite models, except the algebraic models 5 — 8, there exists an explicit rational function $B_S(x, y)$ such that for $a, b \in \{0, 1\}$

$$C(a, b, t) = \Delta \left( \frac{B_S(x, y)}{(1 - x)^a(1 - y)^b(1 - txyS(x, y))} \right)$$
Theorem
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$$C(a, b, t) = \Delta \left( \frac{B_S(x, y)}{(1 - x)^a(1 - y)^b(1 - txyS(x, y))} \right)$$

Example
When $S = \{(\pm 1, 0), (0, \pm 1)\}$ then $B_S = (1 - x^2)(1 - y^2)$ so

$$C(1, 1, t) = \Delta \left( \frac{(1 + x)(1 + y)}{1 - txy(x + x + y + y)} \right)$$

$$C(0, 0, t) = \Delta \left( \frac{(1 - x^2)(1 - y^2)}{1 - txy(x + x + y + y)} \right)$$
Asymptotic Analysis

Can we link the analytic properties of these rational functions to the asymptotics of our objects?
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Yes! The field of analytic combinatorics in several variables has been developed over the last 10 - 20 years (see Pemantle and Wilson 2013)
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Theorem (M. and Wilson 2015+)
All of the guessed asymptotics on previous slides are true.

Furthermore, we can “read off” certain asymptotic properties from these diagonal equations.
Asymptotic Analysis

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Furthermore, we can read off certain asymptotic properties from these diagonal equations.

Theorem (M. and Wilson 2015+)
All of the guessed asymptotics on previous slides are true.
ACT II

Analytic Combinatorics in Several Variables
Theorem (CIF In Several Variables)
Let $f(x, y, t)$ be holomorphic on an open neighbourhood of $0$.
Then there is a unique series $F(x, y, t) = \sum a_{i,j,n} x^i y^j z^n$
converging to $f$ near the origin whose coefficients satisfy

$$c_n := a_{n,n,n} = \left( \frac{1}{2\pi i} \right)^3 \int_T \frac{f(x, y, t)}{(xyt)^n} \frac{dx dy dt}{xyt}$$

where $T$ is a polydisk sufficiently close to the origin.
**Theorem (CIF In Several Variables)**

Let $f(x, y, t)$ be holomorphic on an open neighbourhood of $0$. Then there is a unique series $F(x, y, t) = \sum a_{i,j,n} x^i y^j z^n$ converging to $f$ near the origin whose coefficients satisfy

$$c_n := a_{n,n,n} = \left( \frac{1}{2\pi i} \right)^3 \int_T \frac{f(x, y, t) \, dx \, dy \, dt}{(xyt)^n \, xyt}$$

where $T$ is a polydisk sufficiently close to the origin.

**Goal:** Deform $T$ without changing the value of the integral, until we can get asymptotic approximations.

**Idea:** Like the univariate case, find points where local behaviour determines asymptotics.
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where $T$ is a polydisk sufficiently close to the origin.

**Goal:** Deform $T$ without changing the value of the integral, until we can get asymptotic approximations.

**Idea:** Like the univariate case, find points where local behaviour determines asymptotics.

*In general this is very very hard!*
Minimal Points

Let $\mathcal{D}$ be the open domain of convergence of the power series. Points $\rho \in \partial \mathcal{D}$ are known as minimal points.

Minimal points give an ‘easy’ upper bound on asymptotics:

$$\sum_{i,j,n \geq 0} f_{i,j,n} x^i y^j t^n \text{ converges } \implies |f_{i,j,n}| \cdot |x^i y^j t^n| \to 0$$

$$\implies |f_{n,n,n}| \leq |xyt|^{-n}$$
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$$\implies \limsup |f_{n,n,n}|^{1/n} \leq |xyt|^{-1}$$
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Thus,

$$\limsup |c_n|^{1/n} \leq \inf_{(x,y,t) \in \partial \mathcal{D}} |xyt|^{-1}$$
Minimal Points

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$$

$$
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$$

$$
c_n \sim A^n \cdot \theta(n)
$$
Minimal Points

Let $\mathcal{D}$ be the open domain of convergence of the power series. Points $\rho \in \partial \mathcal{D}$ are known as *minimal points*.

Minimal points give an ‘easy’ upper bound on asymptotics:

$$
\sum_{i,j,n \geq 0} f_{i,j,n} x^i y^j t^n \text{ converges} \iff |f_{i,j,n}| \cdot |x^i y^j t^n| \to 0
$$

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\Rightarrow |f_{n,n,n}| \leq |x y t|^{-n}
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$$

Thus $c_n \sim A^n \cdot \theta(n)$.
The Singular Variety

Assume that \( f(x, y, t) = G(x, y, t)/H(x, y, t) \) is rational, and let \( \mathcal{V} \) be the set of zeroes of \( H(x, y, t) \) where no coordinate is zero.

Assume that \( \mathcal{V} \) is a smooth manifold; i.e., that at no point

\[
H_x = H_y = H_t = H = 0.
\]
The Singular Variety

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Assume that \( \mathcal{V} \) is a smooth manifold; i.e., that at no point

\[
H_x = H_y = H_t = H = 0.
\]

At each \( (x, y, t) \in \partial \mathcal{D} \) there is (at least) one point with the same coordinate-wise modulus in \( \mathcal{V} \).

Which points of \( \mathcal{V} \) give the best bound? When is it tight?
Critical Points

A point \((x, y, t) \in V\) is called critical if

\[xH_x = yH_y = tH_t\]
**Critical Points**

A point \((x, y, t) \in \mathcal{V}\) is called **critical** if

\[
x H_x = y H_y = t H_t
\]

At such a point,

\[
\int \frac{f(x, y, t)}{(x y t)^n} \frac{dxdydt}{x y t} \approx \int_{\mathcal{N}} A(u, v) e^{-n(a \cdot u^2 + b \cdot v^2) + O(|u+v|^3)} dudv
\]

**Idea:** We understand asymptotics of Gaussian integrals very well

\[
\int_{\mathcal{N}} A(z) e^{-n(z_1^2 + \cdots + z_d^2)} dz = A(0) \pi^{d/2} n^{-d/2} \left(1 + O(1/n)\right)
\]
Critical Minimal Points

Theorem

Let $W \subset \partial D$ be the set of minimal points maximizing $|xyt|$. Assume each point of $W$ has the same coordinate-wise modulus.

If the subset $C \subset W$ of critical points in $W$ is non-empty and finite then $C$ is the set of contributing points, in the sense that

$$a_{n,n,n} \sim \sum_{(x,y,t) \in C} \text{formula}_n(x, y, t)$$
Critical Minimal Points

Theorem

Let $\mathcal{W} \subset \partial \mathcal{D}$ be the set of minimal points maximizing $|x y t|$. Assume each point of $\mathcal{W}$ has the same coordinate-wise modulus.

If the subset $\mathcal{C} \subset \mathcal{W}$ of critical points in $\mathcal{W}$ is non-empty and finite then $\mathcal{C}$ is the set of contributing points, in the sense that

$$a_{n,n,n} \sim \sum_{(x,y,t) \in \mathcal{C}} \text{formula}_n(x, y, t)$$

There are extensions to more general cases with a finite number of critical minimal points.
Non-degenerate Smooth Formula

Near $\zeta$ parametrize $z_d = g(z_1, \ldots, z_{d-1})$

$$\psi(z_1, \ldots, z_{d-1}) = (z_1 \cdots z_{d-1}) \cdot g(z_1, \ldots, z_{d-1})$$

The diagonal coefficients $A_n$ behave like

$$A_n = (\zeta_1 \cdots \zeta_d)^{-n} n^{\frac{1-d}{2}} \left( \frac{(2\pi)^{(1-d)/2}}{\sqrt{\zeta^{3-d} |\mathcal{H}(\zeta)| / \zeta_d^2}} \frac{-G(\zeta)}{\zeta_d \frac{\partial H}{\partial z_d}(\zeta)} + O\left(\frac{1}{n}\right) \right)$$

where $|\mathcal{H}|$ is the determinant of the Hessian of $\psi$ above.
Critical Minimal Points

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$$a_{n,n,n} \sim \sum_{(x,y,t) \in \mathcal{C}} \text{formula}_n(x, y, t)$$

There are extensions to more general cases with a finite number of critical minimal points.

Problem: critical minimal points may not exist!
“Combinatorial” Case

Luckily, there are important cases where we are guaranteed critical minimal points.

**Theorem**

Suppose that \( G(x, y, t)/H(x, y, t) \) has a power series expansion

\[
\sum_{i,j,n \geq 0} a_{i,j,n} x^i y^j t^n
\]

where each \( a_{i,j,n} \geq 0 \). Then there is at least one critical minimal point, and \( (x, y, t) \in \partial D \) only if \( (|x|, |y|, |t|) \in \partial D \cap V \).
The Method of ACSV

1) Find contributing points. Can be done:
   - when dealing with minimal critical points* (if they exist)
   - when the denominator has only linear factors
   - in dimension 2 (DeVries, van der Hoeven, Pemantle)

2) Calculate the contribution of each, using pre-calculated formulae (can get very messy)
The Method of ACSV

1) Find contributing points. Can be done:
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   - when the denominator has only linear factors
   - in dimension 2 (DeVries, van der Hoeven, Pemantle)

2) Calculate the contribution of each, using pre-calculated formulae (can get very messy)

Work in Progress (with Bruno Salvy and George Labahn):
   Make this effective and give complexity (see ISSAC 2016?)
ACT III
Back to Lattice Paths
The 23 D-Finite Models
The 23 D-Finite Models

Highly Symmetric — M. and Mishna 2014/15

\[ c_n \sim C \cdot n^{-1} \cdot |S|^n \]
Highly Symmetric Walks

\[ C(1, 1, t) = \Delta \left( \frac{(1 + x)(1 + y)}{1 - txyS(x, y)} \right) \]
Highly Symmetric Walks

\[ C(1, 1, t) = \Delta \left( \frac{(1 + x)(1 + y)}{1 - txyS(x, y)} \right) \]

**Theorem (M. and Mishna 2014)**

Let \( S \subseteq \{-1, 0, 1\}^d \setminus \{0\} \) be symmetric with respect to each axis and take a positive step in each direction. Then

\[ c_n \sim \left[ \left( s^{(1)} \ldots s^{(d)} \right)^{-1/2} \pi^{-d/2} |S|^{d/2} \right] \cdot n^{-d/2} \cdot |S|^n, \]

where \( s^{(k)} = \# \) of steps which have \( k^{th} \) coordinate 1.
The 23 D-Finite Models

Highly Symmetric — M. and Mishna 2014/15

\[ c_n \sim C \cdot n^{-1} \cdot |S|^n \]
The 23 D-Finite Models

Positive Drift — M. and Wilson 2015

\[ c_n \sim C \cdot n^{-3/2} \cdot |S|^n \]
The 23 D-Finite Models

Negative Drift — M. and Wilson 2015

\[ c_n \sim C_n \cdot n^{-2} \cdot S(1, \tau)^n \]
Negative Drift

\[ C(1, 1, t) = \Delta \left( \frac{(1 + x)(1 - y^2 S_{-1}(x)/S_1(x))}{(1 - y)(1 - txyS(x, y))} \right) \]

\[ S_{-1}(x) = [y^1]S(x, y) \]
\[ S_1(x) = [y^1]S(x, y) \]
Positive Drift

\[ C(1, 1, t) = \Delta \left( \frac{(1 + x)(1 - y^2S_{-1}(x)/S_1(x))}{(1 - y)(1 - txyS(x, y))} \right) \]

\[ S_{-1}(x) = [y^{-1}]S(x, y) \]
\[ S_1(x) = [y^1]S(x, y) \]
The 23 D-Finite Models

Negative Drift — M. and Wilson 2015

\[ c_n \sim C_n \cdot n^{-2} \cdot S(1, \tau)^n \]
The 23 D-Finite Models

Algebraic Zero Orbit Sum Cases
Mishna / Bousquet-Mélou and Mishna / Bostan and Kauers
The 23 D-Finite Models

Three Sporadic Cases
M. and Wilson 2015 (and Bousquet-Mélou and Mishna)
Example: Negative Drift

Example

Let

\[ S = \{(0, 1), (-1, -1), (0, -1), (1, -1)\} = \]

Then

\[ c_n = \left[ t^n \right] \Delta \left( \frac{(1 + x)(1 - y^2(1 + x + x^2))}{x(1 - y)(1 - t(x + y^2 + xy^2 + x^2y^2))} \right) \]
Example: Negative Drift

Example

Let

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\[ c_n = [t^n] \Delta \left( \frac{(1 + x)(1 - y^2(1 + x + x^2))}{x(1 - y)(1 - t(x + y^2 + xy^2 + x^2y^2))} \right) \]

\[ = [t^{n+1}] \Delta \left( \frac{yt(1 + x)(1 - y^2(1 + x + x^2))}{(1 - y)(1 - t(x + y^2 + xy^2 + x^2y^2))} \right) \]
Example: Negative Drift

Example

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\[
c_n = [t^n] \Delta \left( \frac{(1 + x)(1 - y^2(1 + x + x^2))}{x(1 - y)(1 - t(x + y^2 + xy^2 + x^2y^2))} \right)
\]

\[
= [t^{n+1}] \Delta \left( \frac{yt(1 + x)(1 - y^2(1 + x + x^2))}{(1 - y)(1 - t(x + y^2 + xy^2 + x^2y^2))} \right)
\]

The contributing points turn out to be

\[ \rho_1 = (1, 1/\sqrt{3}, 1/2) \text{ and } \rho_2 = (1, -1/\sqrt{3}, 1/2) \]
Example: Negative Drift

Example

The Sage package of Raichev calculates (after shifting)

\[
\Psi_n^{(\rho_1)} = \frac{3\sqrt{3}(2 + \sqrt{3})}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2} + O\left(\frac{(2\sqrt{3})^n}{n^3}\right)
\]

\[
\Psi_n^{(\rho_2)} = \frac{3\sqrt{3}(2 - \sqrt{3})}{\pi} \cdot \frac{(-2\sqrt{3})^n}{n^2} + O\left(\frac{(2\sqrt{3})^n}{n^3}\right)
\]
Example: Negative Drift

Example

The Sage package of Raichev calculates (after shifting)

\[ \Psi_n^{(\rho_1)} = \frac{3\sqrt{3}(2 + \sqrt{3})}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2} + O \left( \frac{(2\sqrt{3})^n}{n^3} \right) \]

\[ \Psi_n^{(\rho_2)} = \frac{3\sqrt{3}(2 - \sqrt{3})}{\pi} \cdot \frac{(-2\sqrt{3})^n}{n^2} + O \left( \frac{(2\sqrt{3})^n}{n^3} \right) \]

Thus,

\[ c_n = \frac{(2\sqrt{3})^n}{n^2} \cdot \frac{3\sqrt{3}}{\pi} \left( \sqrt{3}(1 - (-1)^n) + 2(1 + (-1)^n) + O(n^{-1}) \right) \]

\[ = \begin{cases} 
\frac{(2\sqrt{3})^n}{n^2} \cdot \left( \frac{12\sqrt{3}}{\pi} + O(n^{-1}) \right) & : n \text{ even} \\
\frac{(2\sqrt{3})^n}{n^2} \cdot \left( \frac{18}{\pi} + O(n^{-1}) \right) & : n \text{ odd}
\end{cases} \]
CONCLUSION
Conclusion

We are able to

- Prove (and re-discover) conjectured asymptotics of Bostan and Kauers from 2009
- Prove (and re-discover) conjectured asymptotics for returns to boundaries
- Derive compact representations for these combinatorial objects
- Explain combinatorial properties analytically

Extensions to:

- Step sets in $d$ variables symmetric over every axis (M. & Mishna 2015)
- Step sets in $d$ variables missing one symmetry* (M. & Wilson 2015+)
Conclusion

Other on-going work:

- Computational complexity of ACSV methods (for combinatorial case and beyond) — use Computer Algebra
- Comparison of ACSV methods to creative telescoping algorithms
- Dealing with weights
- Dealing with longer steps (joint work with Alin Bostan and Mireille Bosquet-Méléou)
- Walks in other regions (Weyl Chambers)
- Applications to other areas of Combinatorics / CS / Number Theory
FIN
Example: A Sporadic Case

Example

Let

\[ S = \{(1, 0), (-1, 0), (-1, 1), (1, -1)\} = \]

Then

\[ c_n = [t^{n+2}] \Delta \left( \frac{yt^2(x + 1)(y - x^2)(x - y)(x + y)}{1 - t(x^2y + x^2 + y^2 + y)} \right) \]
Example: A Sporadic Case

Example

Let

\[ S = \{(1, 0), (-1, 0), (-1, 1), (1, -1)\} = \]

Then

\[ c_n = [t^{n+2}] \Delta \left( \frac{yt^2(x + 1)(y - x^2)(x - y)(x + y)}{1 - t(x^2y + x^2 + y^2 + y)} \right) \]

This rational function has a smooth denominator and is combinatorial! We easily get contributing points:

\[ \rho_1 = (1, 1, 1/4) \text{ and } \rho_2 = (-1, 1, 1/4) \]
Example: A Sporadic Case

Example

The Sage package of Raichev calculates (after shifting)

\[
\Psi_n^{(\rho_1)} = \frac{8}{\pi} \cdot \frac{4^n}{n^2} + O \left( \frac{4^n}{n^3} \right)
\]

\[
\Psi_n^{(\rho_2)} = \frac{4}{\pi} \cdot \frac{4^n}{n^3} + O \left( \frac{4^n}{n^4} \right)
\]

Thus,

\[
c_n = \frac{4^n}{n^2} \cdot \frac{8}{\pi} + O \left( \frac{4^n}{n^3} \right)
\]

Bousquet-Mélou and Mishna used Gosper-summability of a related sequence to prove this.
The MP Asymptotic Formula

\[ \Psi_{n,N}^{(p)} := p^{-n_1} \cdot n^{(k-d-1)/2} \cdot (2\pi)^{(k-d-1)/2} \cdot (\text{det} \tilde{\Phi}''(0))^{-1/2} \]

\[
(11) \quad \times \left[ \sum_{q=0}^{N-1} n^{-q} \sum_{0 \leq j \leq \min(k-1,q)}^{\max(0, q-k) \leq i \leq q} L_i(\tilde{A}_j, \tilde{\Phi}) \binom{k-1}{j} \binom{k-j}{k+i-q} (-1)^{q-j-i} \right] + O(n^{-N})
\]

where

\[
L_i(\tilde{A}_j, \tilde{\Phi}) = \sum_{l=0}^{2i} \frac{\mathcal{H}^{i+l}(\tilde{A}_j \tilde{\Phi}^l)(0)}{(-1)^i 2^i i! l! (i+l)!}
\]

with \( \mathcal{H} \) the differential operator

\[
\mathcal{H} = - \sum_{1 \leq a, b \leq d+k-1} (\tilde{\Phi}''(0)^{-1})_{a,b} \partial_a \partial_b
\]
The MP Asymptotic Formula

\[ H_j(z, t) = U_j(z, t) \left( t - \frac{1}{h_j(z)} \right), \]

\[ \tilde{G}(z, t) = \frac{G(z, t)}{U_1(z, t) \cdots U_k(z, t)} \prod_{j=1}^{k} \frac{-h_j(z)}{t} \]

\[ h(z, \sigma) = \sigma_1 h_1 + \cdots + \sigma_{k-1} h_{k-1} + \left( 1 - \sum_{j=1}^{k-1} \sigma_j \right) h_k \]

\[ A_j(z, t) = (-1)^{k-1} t^{-k+j} \left( \frac{\partial}{\partial t} \right)^j \tilde{G}(z, t^{-1}) \]

\[ e(\theta) = (p_1 \exp(i\theta_1), \ldots, p_d \exp(i\theta_d)) \]

\[ \tilde{h}(\theta, \sigma) = h(e(\theta), s) \]

\[ \tilde{A}_j(\theta, \sigma) = A_j(e(\theta), \tilde{h}(\theta, \sigma)) \]

\[ \tilde{\Phi}(\theta, \sigma) = -\log(p_i \tilde{h}(\theta, \sigma)) + i \sum_{m=1}^{d} \theta_m \]

\[ \tilde{\Phi}(\theta, \sigma) = \tilde{\Phi}(\theta, \sigma) - \frac{1}{2} (\theta, \sigma) \tilde{\Phi}''(0)(\theta, \sigma)^T, \]