

Visualization of Regular Maps

Jarke J. van Wijk
Eindhoven University of Technology

JCB 2016, Bordeaux

Visualization Eindhoven

information visualization – software visualization – perception – geographic visualization
scientific visualization – **math vis** – human computer interaction – visual analytics – parallel



Data: flow fields – trees – graphs – tables – mobile data – events – ...
Applications: software analysis – business cases – bioinformatics – ...



Eindhoven 2004

Starting MathVis



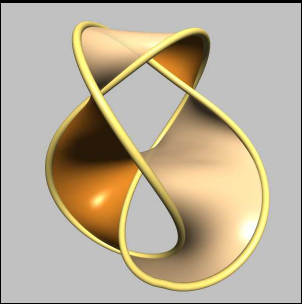
Can you draw a Seifert surface?

Huh?

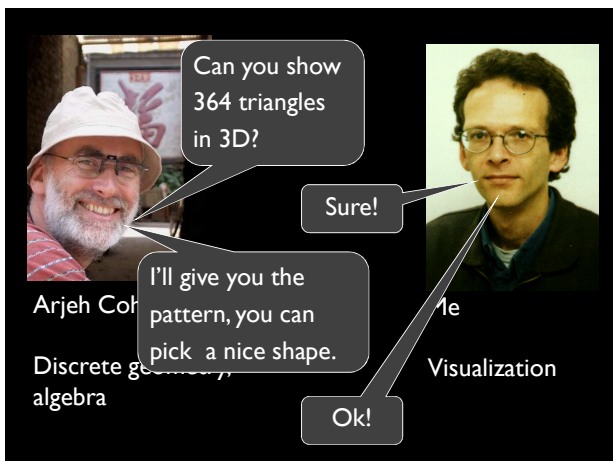
Arjeh Cohen
Discrete geometry,
algebra

Me
Visualization

Seifert surface



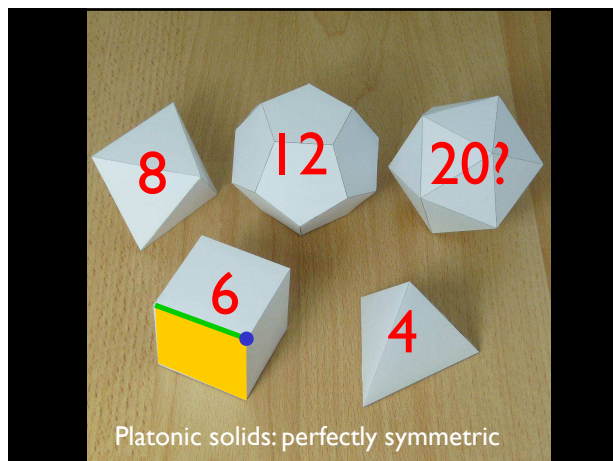
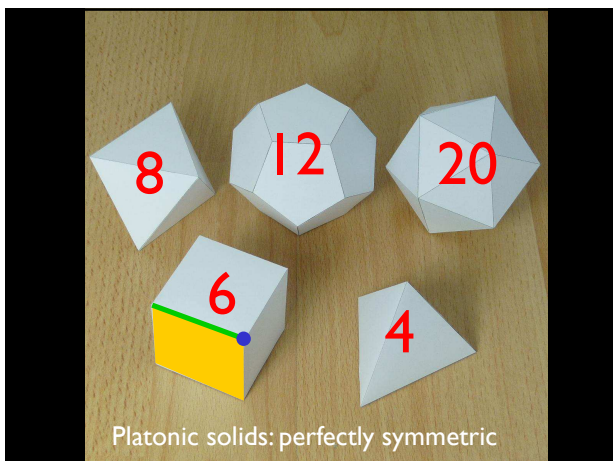
Eindhoven 2006



Aberdeenshire
2000 BC



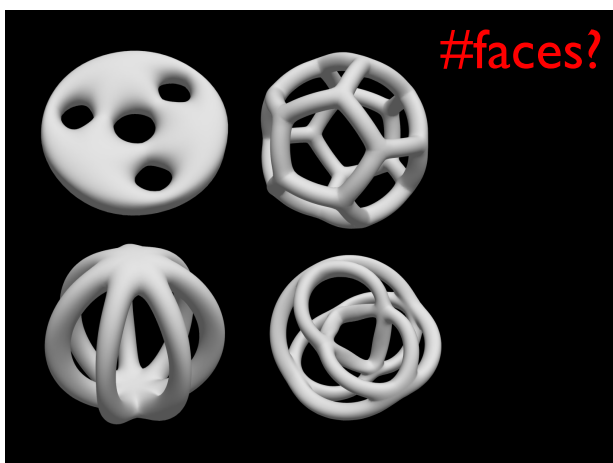
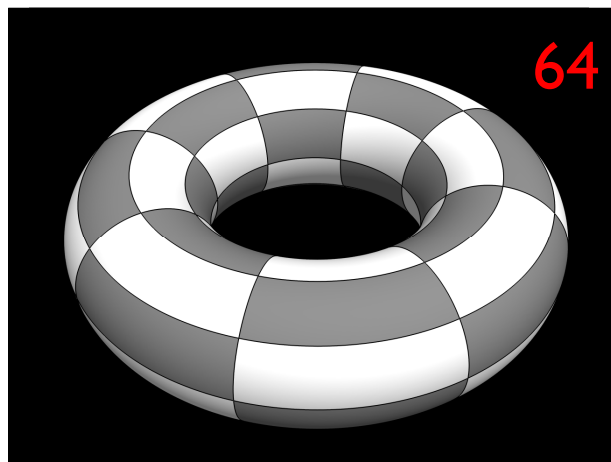
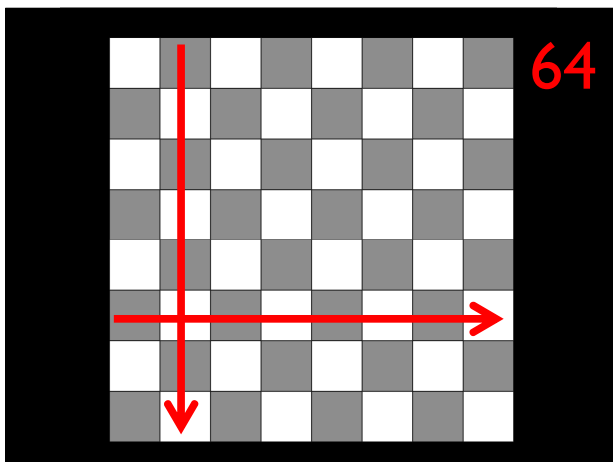
Athens
450 BC




How to get more faces,
all perfectly symmetric?

How to get more faces,
all perfectly symmetric?

Use shapes with holes
Aim only at topological symmetry

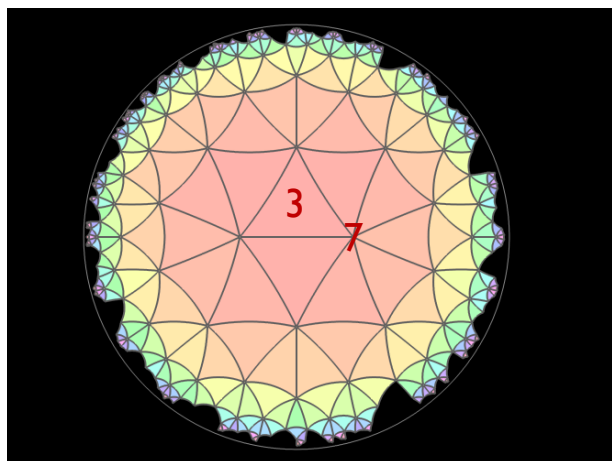


Königsberg
1893



Adolf Hurwitz
1859-1919

max. #faces:
 $28(\text{genus} - 1)$



Genus Faces

3	56
7	168
14	364
...	...

Genus Faces

3	56
7	168
14	364
...	...

Arjeh Cohen: Can you show 364 triangles in 3D?

Me: Sure!

Arjeh Cohen: I'll give you the pattern, you can pick a nice shape.

Me: Ok!

New Orleans
2009
three years later

Symmetric Tiling of Closed Surfaces: Visualization of Regular Maps

Jarke van Wijk
TU Eindhoven

ACM SIGGRAPH 2009, August 3-7, New Orleans

26

The general puzzle

Construct space models of *regular maps*

- Surface topology, combinatorial group theory, graph theory, algebraic geometry, hyperbolic geometry, physics, chemistry, ...

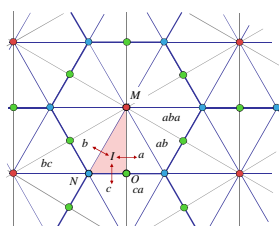
27

Triangle group

$$T(p, q, r) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = I \rangle$$

a, b, c : reflections edges

ab, bc, ca : rotations vertices

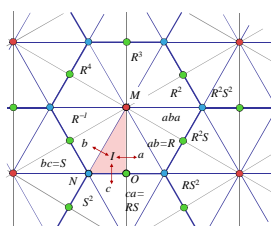


28

Regular maps

Quotient group of triangle group $T(p, q, 2)$:

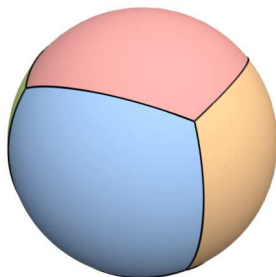
$$G(p, q) = \langle R, S \mid R^p = S^q = (RS)^2 = \dots = I \rangle$$



R : ab , rotation center polygon
 S : bc , rotation vertex
 RS : ca , rotation edge

29

Regular maps



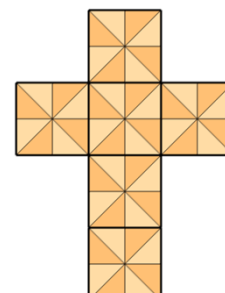
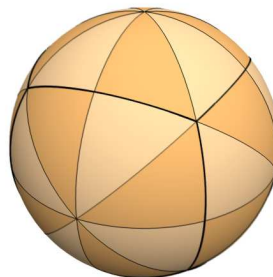
Regular map:
Embedding of a graph in
a closed surface, such
that topologically

- faces are identical
- vertices are identical
- edges are identical

Tiling of closed surface with *maximal symmetry*

30

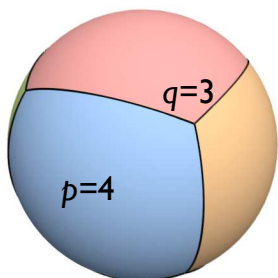
Symmetries of the cube



- 48-fold symmetry ($2pN = 2 \times 4 \times 6 = 48$)

31

Symmetric tiling sphere



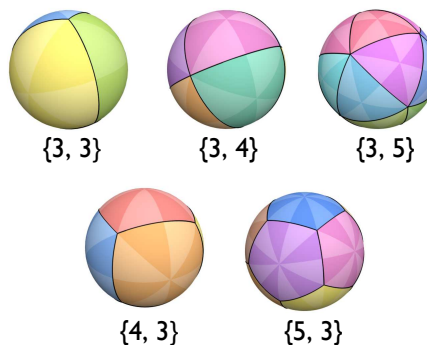
Symmetric tiling:
Covering of surface with regular polygons

$\{p, q\}$: Schläfli symbol
At each vertex:
 q p -gons meet

$\{4, 3\}$: cube

32

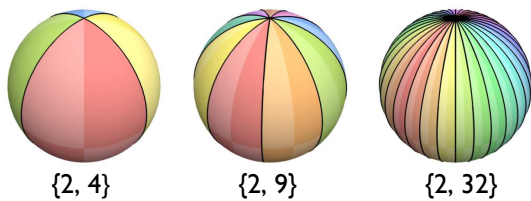
Genus 0: Platonic solids



33

Genus 0: hosohedra

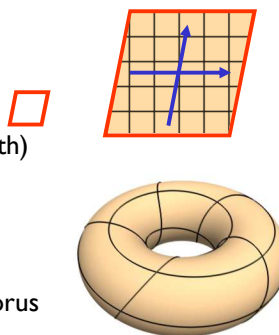
- hosohedron: faces with two edges



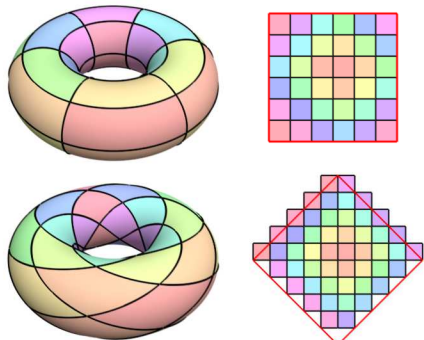
34

Genus 1: tori

- Tile the plane
- Define a rhombus (all sides same length)
- Project tiling
- Fold rhombus to torus

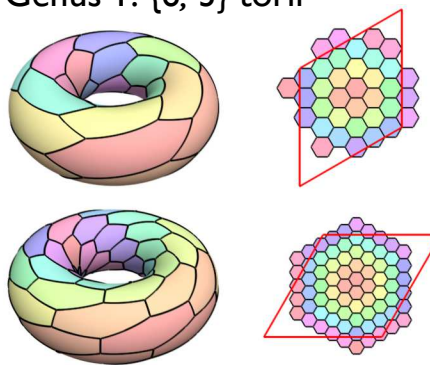


Genus 1: $\{4, 4\}$ torii



36

Genus 1: $\{6, 3\}$ torii



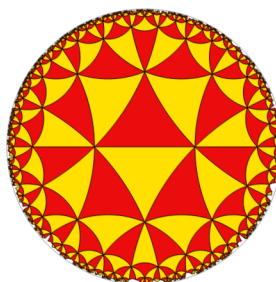
37

Genus $g \geq 2$

g	shape	geometry	transf.	tilings
0	sphere	spherical	3D rotation	$\{3,3\}, \{3,4\}, \{4,3\}, \{3,5\}, \{5,3\}, \{2,n\}$
1	torus	planar	2D Euclidean	$\{4,4\}, \{3,6\}, \{6,3\}$
≥ 2	?	hyperbolic	Möbius	$\{3,7\}, \{4,5\}, \{5,4\}, \{4,6\}, \{6,4\}, \{5,5\}, \dots$

38

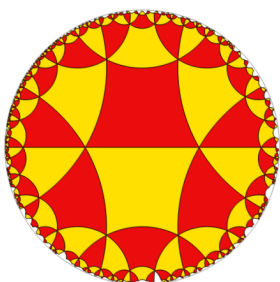
Tiling hyperbolic plane



$\{3, 8\}$ tiling
hyperbolic plane

39

Tiling hyperbolic plane



$\{4, 6\}$ tiling
hyperbolic plane

40

Regular maps

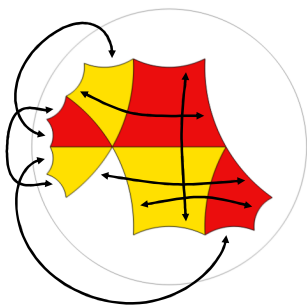


Regular map:
Cut out part of tiling hyperbolic plane

For instance:
6 quads

41

Regular maps

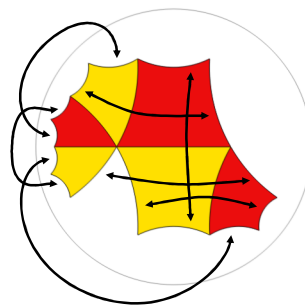


Regular map:
Cut out part of tiling hyperbolic plane

For instance:
6 quads,
and match edges

42

Regular maps



Regular map:
Cut out part of tiling hyperbolic plane

M. Conder (2006):
enumerated all
regular maps for
 $g \leq 101$

43

Conder's list

R2.1 : Type {3,8}_12 Order 96 mV = 2 mF = 1
 Defining relations for automorphism group:
 [T^2, R^3, (R * S)^2, (R * T)^2, (S * T)^2, (R * S^-3)^2]

R2.2 : Type {4,6}_12 Order 48 mV = 3 mF = 2
 Defining relations for automorphism group:
 [T^2, R^4, (R * S)^2, (R * S^-1)^2, (R * T)^2, (S * T)^2, S^6]

R2.3 : Type {4,8}_8 Order 32 mV = 8 mF = 2
 Defining relations for automorphism group:
 [T^2, R^4, (R * S)^2, (R * S^-1)^2, (R * T)^2, (S * T)^2, S^2 * R^2 * S^2]

R2.4 : Type {5,10}_2 Order 20 mV = 10 mF = 5
 Defining relations for automorphism group:
 [T^2, S * R^2 * S, (R, S), (R * T)^2, (S * T)^2, R^5]

.....

R101.55 : Type {204,204}_2 Order 816 mV = 204 mF = 204 Self-dual
 Defining relations for automorphism group:
 [T^2, S * R^2 * S, (R, S), (R * T)^2, (S * T)^2, R^92 * S^-1 * R^3 * T * S^2 * T * R^16 * S^-89 * R]

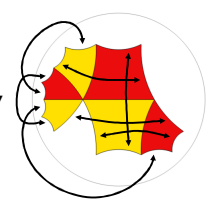
R101.56 : Type {404,404}_2 Order 808 mV = 404 mF = 404 Self-dual
 Defining relations for automorphism group:
 [T^2, S * R^2 * S, (R, S), (R * T)^2, (S * T)^2, R^98 * T * S^2 * T * R^10 * T * R^3 * T * R^4 * S^85]

Total number of maps in list above: 3378

Conder's list

R2.2 : Type {4,6}_12 Order 48 mV = 3 mF = 2
 Defining relations for automorphism group:
 [T^2, R^4, (R * S)^2, (R * S^-1)^2, (R * T)^2, (S * T)^2, S^6]

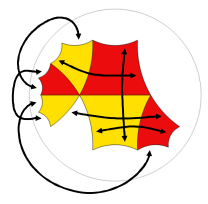
- Rg.i: genus g, member i
- complete definition topology (connectivity)
- No cue on possible 3D geometry



Conder's list

R2.2 : Type {4,6}_12 Order 48 mV = 3 mF = 2
 Defining relations for automorphism group:
 [T^2, R^4, (R * S)^2, (R * S^-1)^2, (R * T)^2, (S * T)^2, S^6]

The challenge:
 Given the complete topology,
 find a *space model*: an
 embedding of faces, edges and
 vertices in 3D space



A perfect puzzle

Find a space model of a regular map:

- Easy to understand
- Scalable
- Many approaches possible
- Fascinating
- Many puzzles to be cracked
- Few known solutions

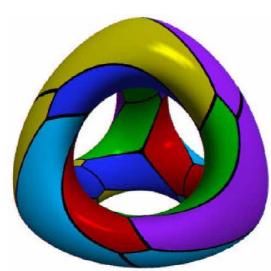
Genus 3

- Helaman Ferguson, 1993
- The Eightfold Way
- Hurwitz genus 3
- Klein's surface
- 24 heptagons



Genus 3

- Carlo Séquin, 2006
- Klein's surface
- 24 heptagons
- tetrahedral frame



Carlo Séquin, 2009:
 many more, found manually
 by exploiting symmetries

Work of Carlo Séquin



50

An addictive puzzle



Carlo Séquin
Berkeley

“During the last few nights I woke up at 3am with some ideas, and sometimes they worked, and sometimes they evaporated in daylight!”

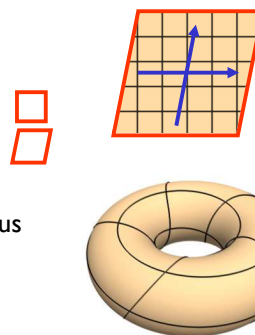
“... , that is why I had to physically remove all signs of these puzzles from my desk and 'lock them up in a vault', so that I would not be constantly distracted from the duties that I HAVE to fulfill ...”

51



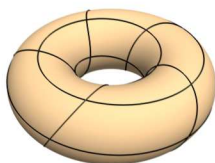
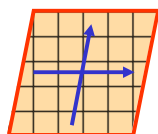
Tori (genus 1) (reprise)

- Tile the plane
- Take a torus
- Unfold to square
- Warp to a rhombus
- Project tiling
- Map rhombus to torus



Approach for $g \geq 2$

- Tile the **hyperbolic** plane
- Take a **nice genus g** shape
- Unfold to **cut out**
- Warp to **match shape**
- Project tiling
- Map **cut out** to **nice shape**



54

Nice genus g shape?



Solid shape with holes?



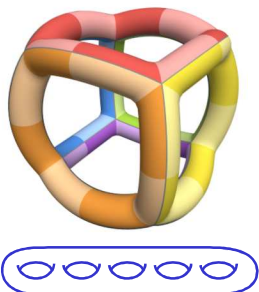
Sphere with handles?

*Where to place holes or handles to get maximal symmetry?
For $g = 6, 13, 17, \dots$?*

55

Tubified regular maps

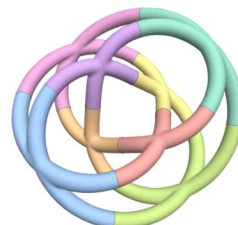
- Take a regular map
- Turn edges into tubes
- Remove faces
- edges → tubes
- vertices → junctions
- faces → holes
- triangles → 1/4 tubes



56

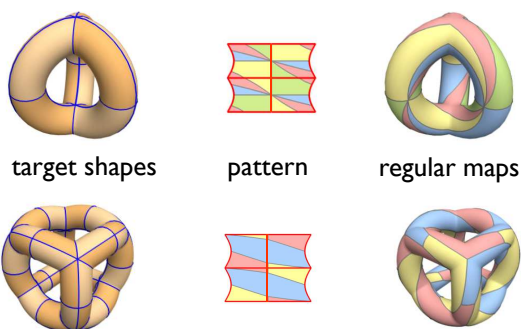
Tubified regular maps

- Take a regular map
- Turn edges into tubes
- Remove faces
- edges → tubes
- vertices → junctions
- faces → holes
- triangles → 1/4 tubes

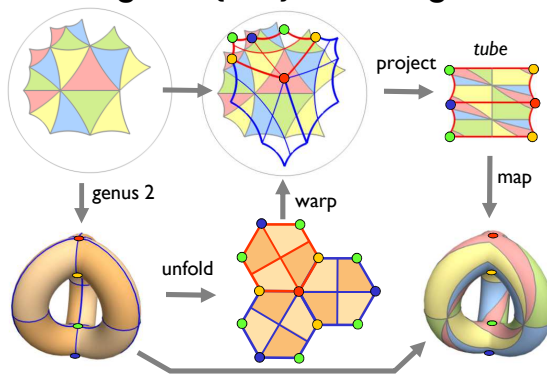


57

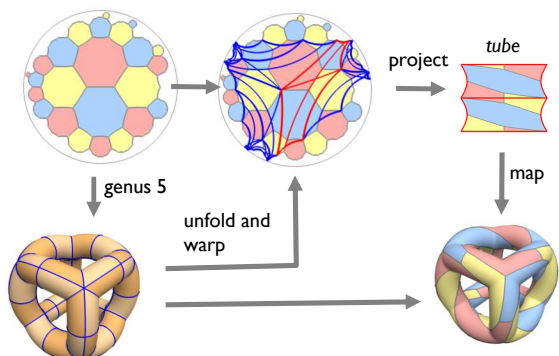
Basic idea



Solving R2.1{3, 8}, 16 triangles



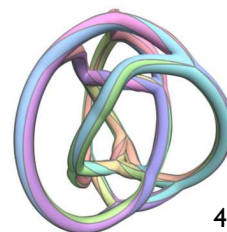
Solving R5.1{3, 8}, 24 octagons



60

Tubified regular maps, reprise

1. Take a regular map
2. Turn edges into tubes
3. Remove faces
4. Map a regular map
5. Goto step 2 again



4.

61

Results

- About 50 different space models for regular maps found automatically

62

Symmetric Tiling of Closed Surfaces: Visualization of Regular Maps

ACM SIGGRAPH 2009

Paris
2014

Five years later

Visualization of Regular Maps: *The Chase Continues*

Jarke J. van Wijk
Eindhoven University of Technology

IEEE SciVis, 2014, Paris

Results 2014

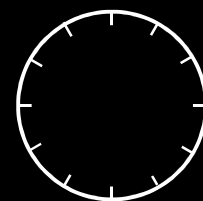
- More generic approach for regular shapes
 - *45 new space models for regular maps found*
- New smoothing approach
 - *Better quality of models*

(Lots of) details: see paper

66

Visualization of cyclic groups

- Subdivide a circle into 18 intervals, given a circle, subdivided in 12 intervals
- Source: C_{18}
- Target: C_{12}



67

Visualization of cyclic groups

Source C_{18} :

- $C_9 \times \{1, 2\}$
- $C_6 \times \{1, 2, 3\}$
- $C_3 \times \{1, 2, 3, 4, 5, 6\}$
- $C_2 \times \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Target: C_{12} :

- $C_6 \times \{1, 2\}$
- $C_4 \times \{1, 2, 3\}$
- $C_3 \times \{1, 2, 3, 4\}$
- $C_2 \times \{1, 2, 3, 4, 5, 6\}$

68

Visualization of cyclic groups

Source C_{18} :

- $C_9 \times \{1, 2\}$
- $C_6 \times \{1, 2, 3\}$
- $C_3 \times \{1, 2, 3, 4, 5, 6\}$
- $C_2 \times \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Target: C_{12} :

- $C_6 \times \{1, 2\}$
- $C_4 \times \{1, 2, 3\}$
- $C_3 \times \{1, 2, 3, 4\}$
- $C_2 \times \{1, 2, 3, 4, 5, 6\}$

• C_6 : greatest common subgroup

• Match sets:

– Source $\{1, 2, 3\}$



– Target $\{1, 2\}$:



69

Visualization of cyclic groups

• Subdivide a circle into 18 intervals, given a circle, subdivided in 12 intervals

- Source: C_{18}
- Target: C_{12}



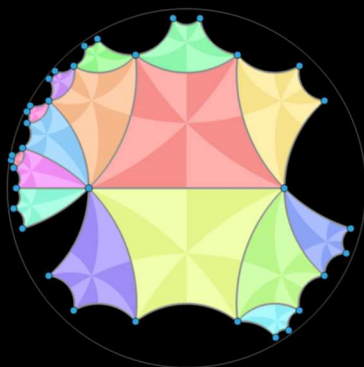
70

Approach

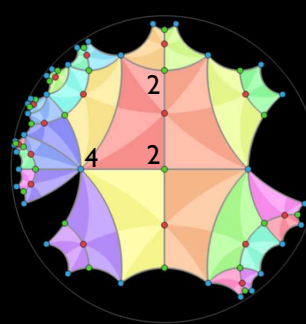
- Take regular maps. Produce face-transitive variations by enumerating subgroups;
- Make target shapes. Produce face-transitive maps by duplicating regular maps, enumerating subgroups, punching, gluing
- Match variations regular map and target.

71

Face-transitive variations



Face-transitive variations



G_S : given source group

Factorization:

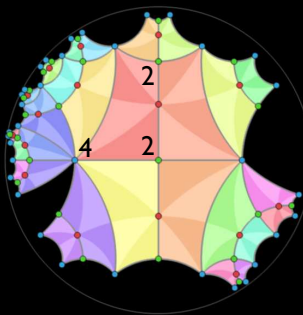
$$G_S = H_S A_S$$

H_S : subgroup G_S

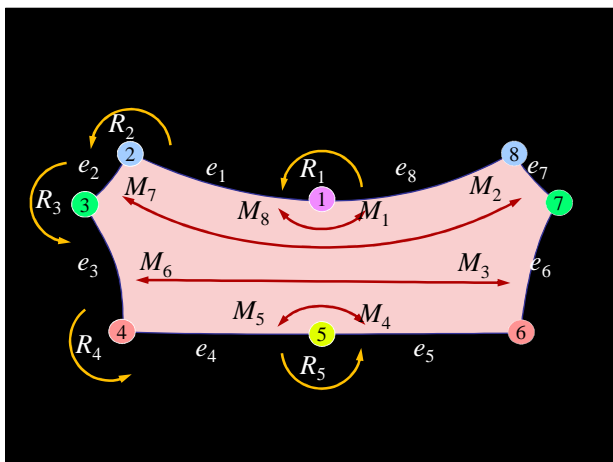
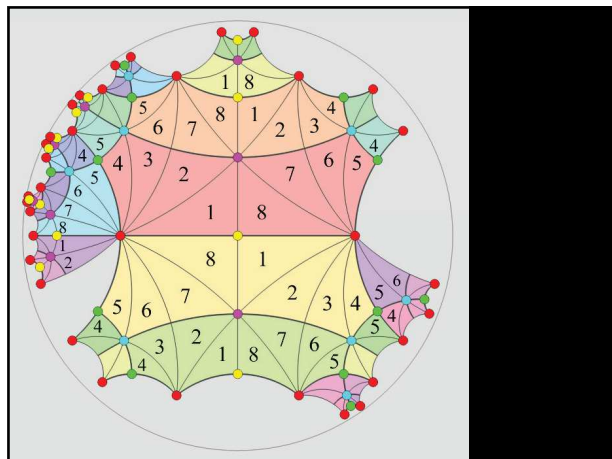
A_S : subset G_S ,

tile of Fuchsian map here (0; 4, 2, 2)

Face-transitive variations



Fuchsian groups:
 Hyperbolic plane can be tessellated with face-transitive polygons, characterized by genus and rotational symmetry boundary points (Poincaré, 1882)

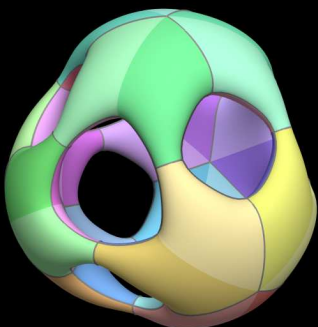


Approach

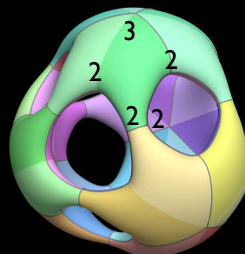
- Take regular maps. Produce face-transitive variations by enumerating subgroups;
- Make target shapes. Produce face-transitive maps by duplicating regular maps, enumerating subgroups, punching, gluing;
- Match variations regular map and target.

77

Producing target surface



Producing target surfaces



G_T : group target surface

Factorization:

$$G_T = H_T A_T$$

H_T : subgroup G_T

A_T : subset G_T ,

tile of Fuchsian map here (0; 3, 2, 2, 2, 2)

Approach

- Take regular maps. Produce face-transitive variations by enumerating subgroups;
- Make target shapes. Produce face-transitive maps by duplicating regular maps, enumerating subgroups, punching, gluing
- Match variations regular map and target:
 - same type of face, same group
 - geometric match in hyperbolic plane

80

Approach

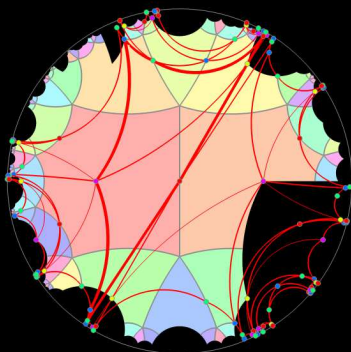
- Match variations regular map and target:

Given lists of factorizations of regular maps and alternative target surfaces:

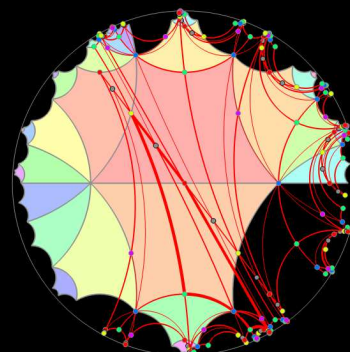
- Find matches of $G_S = H_S A_S$ and $G_T = H_T A_T$, such that $H_S = H_T$ and $A_S = A_T$
- Find corresponding polygons in the hyperbolic plane

81

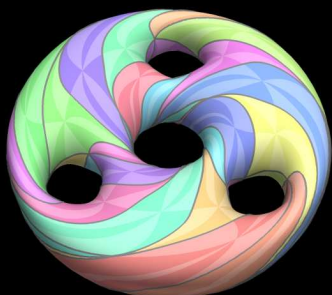
R9.16 {5, 6} on tetrahedron



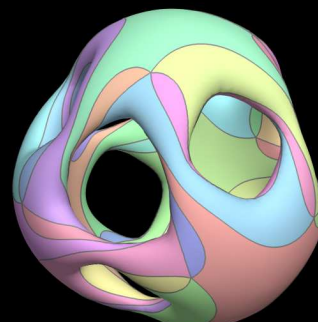
R9.6 {4, 8} on torus 2,2



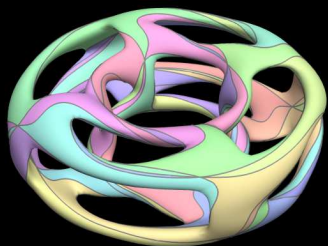
R4.2 {4, 5} on hosohedron-3



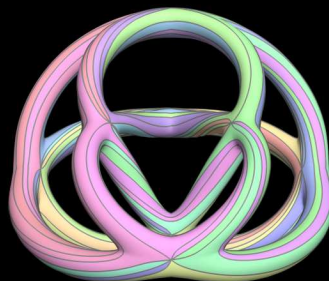
R9.16 {5, 6} on tetrahedron



R17.20 {6, 6} on torus 2,0

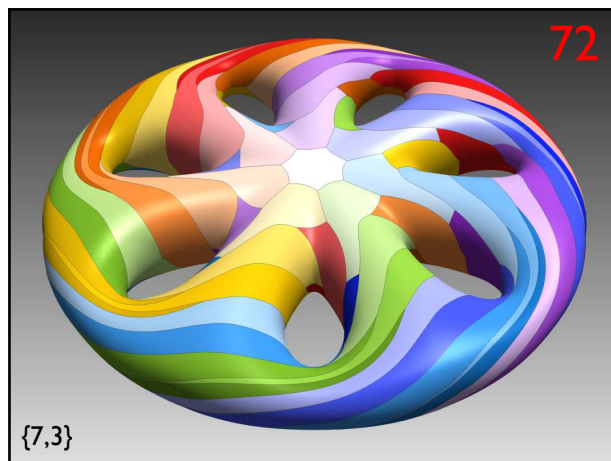
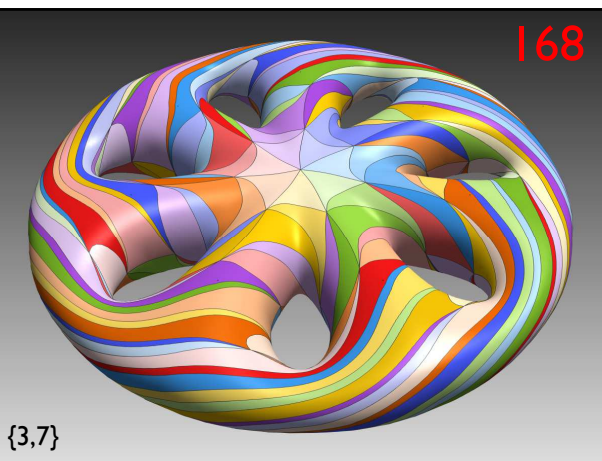
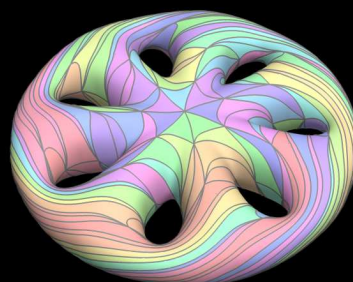


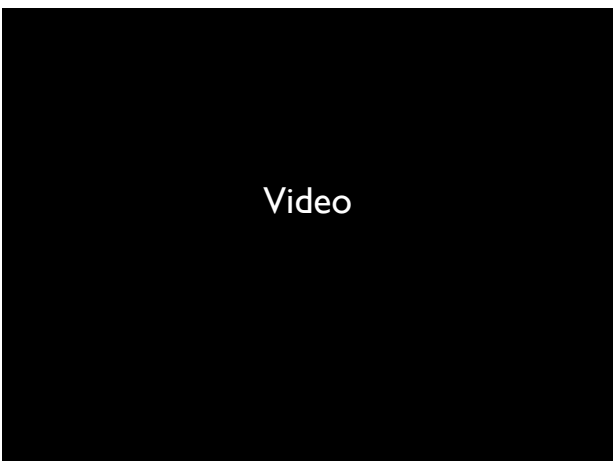
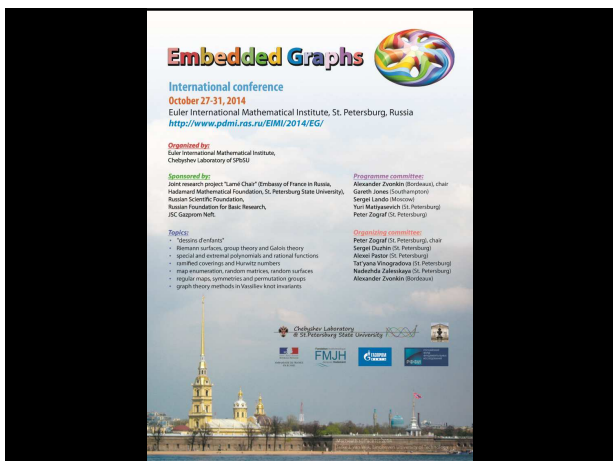
R9.6 {4, 8} on torus 2,2



The genus 7 Hurwitz/MacBeath surface...

R7.1 {3, 7} on 7-hosohedron





Holes	Faces	
3	56	done
7	168	<i>done</i>
14	364	...
...

Holes	Faces	
3	56	done
7	168	done
14	364	<i>todo</i>
...

Some place
 20XX

The chase ain't over yet

Thank you!