Survol de la Mécanique Quantique

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Outline:

- Quantum mechanics: a bit of history
- Double slits Young experiments
- Some principles
- Einstein-Podolski-Rosen: the analysis approach

Some references

For those interested and who want to know more:

1. Video of the conference of Julien Bobroff "Suprémacie Quantique" sur le site de la SMF.
   https://smf.emath.fr/smf-dossiers-et-ressources/suprematie-quantique-julien-bobroff-video-2020
   Thanks Yvan!
2. *Mes premiers pas en Mécanique Quantique* by Christos Gougoussis and Nicolas Poilvert. Technical level fits last year of highschool.
3. *Mécanique Quantique* from "Les cours de physique de Feynman".

Quantum mechanics: a bit of history

During the 19^{th} century, Physics was solved and the world secrets fully understood.

Classical Newton mechanics would predict successfully the motion of bodies in our universe with the expected amount of precision.

Thermodynamics would allow us to build machines and understand heat transfers and so on...

Maxwell equations would describe light, radio waves, magnets, electricity and so on. The electromagnetic fields unification was incredibly successful and beautiful.

All of that up-to a few technical details, that would resist to analysis....

- The Maxwell equations did not agree with the classical Galilean relativity. Indeed they predicted that the speed of light was constant in any referential, the suckers! But, wait it's fine, Maxwell equations are new, therefore probably not completely correct yet, it's just a matter of putting dots on the i's and bars on the t's...
- The motion of Mercury is a bit weird, but no big deal, surely some hidden planet somewhere would explain everything...
- Atoms are super small so we can't see how they work. But, pff, we can't even break them, the Greek knew that already.
- Oh! And there is this black body radiation thing going on. Relax, we understood the high energy part already so the low energy is going to come soon.

Putting dot on the i's and bars on the t's, no wait, the h's....
So what about those points??

For a start, just forget about the first two points. They are only mildly relevant minor points of science history anyway. Someone named Einstein did something about those, but, quite frankly, all he had really was a great haircut.

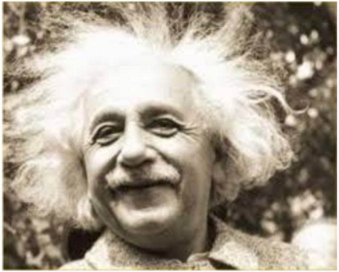

Figure: Einstein's haircut, the most overly paid hairdresser of all times.

Putting dot on the i's and bars on the t's, no wait, the h's....

Okay, cool! First two points solved, that was easy!

What about the last two points?

Well, as I said, atoms are small. That's maybe why they are so discreet......
Or rather why their emission/absorption spectrum is discrete:

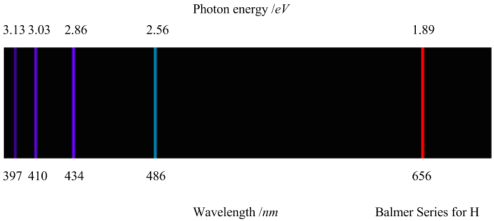

Figure: Emission spectrum Hydrogen

Putting dot on the i's and bars on the t's, no wait, the h's....

So, those atoms? Lot of debates, they are either:

- made of positive jelly with negative raisins in it (Thomson's, aka *The Cook*, model).
- made of a positive nucleus, around which negative charges are orbiting (Rutherford, aka *I-don't-have-imagination-so-I'm-gonna-copy-the-solar-system*, model).

Putting dot on the i's and bars on the t's, no wait, the h's....

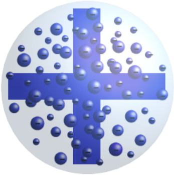

Figure: Negative raisins in positive jelly

Well, almost. In principle Thomson's model could give discrete spectra if everything is fine in "the best of all possible world" (we just left the 19^{th} century to visit our friend Gottfried). But Thomson is struggling with this.

Plus, everyone one knows that God hates jelly, so it's definitely not that, as Rutherford showed! Hmm... More mystery.

Putting dot on the i's and bars on the t's, no wait, the h's....

Hey, remember! We had this no-imagination model. What about it?

Hmm, ok, well. Remember these brand new Maxwell equations? Let's say they don't agree. These new interns, they always put such a mess... Why that?
$\rightarrow$ According to Maxwell equations every accelerated charge radiates energy in electromagnetic form ("light"). And the problem with electrons orbiting is, they are accelerated... So they radiate their energy, so their velocity diminishes, so they radiate more energy and so on, until they crash on the positive nucleus. And pfut! Atoms don't exist anymore. And let's say the process is quite fast...

Putting dot on the i's and bars on the t's, no wait, the h's....

So Thomson-Rutherford-Nature: $0 - 0 - 2$. Nature has an edge.

So, in 1913, Bohr decided that one should allow only discrete sets of orbits. Why? Because! More or less. How do we do so? We quantize the angular momentum of electrons orbiting the nucleus. Doing so only some orbits are allowed and transition between them require a discrete amount of energy. Perfect, we get our discrete spectrum, for hydrogen.

That actually fall quite well in place, since Planck just found out this new constant h in investigating the black body problem. It's called, Planck's constant, and it solves the black body problem. A very nice peculiarity of this constant is, it has the dimensions of angular momentum! Plus the guy with funny hair advocates for it. So you know what, just put a bar on the h by deciding $\hbar = \frac{h}{2\pi}$ (things are orbiting, so dividing by 2π is natural).

Putting dot on the i's and bars on the t's, no wait, the h's....

Bohr atom energy levels: $E_n = -\frac{k^2 m q_e^4}{2n^2\hbar^2} \propto -\frac{1}{n^2}$.

So, solved? Well, no. It's more complicated than that. It's a process. First we did not talk about the black body radiation. Plus the Bohr model does not work when there is more than one electron (it works with hydrogen, and ionized helium, but honestly who cares about those??).

Something to take on is that, in this process, discretizing accessible states for matter has been successful. This quantization idea here did not come out of nowhere. This was first used by Planck to solve the black body problem. But Planck used that idea as a pure mathematical trick and did not propose a physical origin for it. Einstein tried to argue that it is physical, but really everyone was very confused.

After a long historical developments we start to have a better view of what can happen. So I am going to jump forward and tell you a bit more about what can happen in a classical experiment. We'll do it in several flavors, with:

1. Bullets, from a gun (the idea comes from a guy from the US.... No surprise here.)
2. Water waves.
3. Electrons.

Generic feature of this experiment, take a wall, dig two slits in it, throw stuff at it and then look at what happens on the other side of the wall. Simple, no?

The case with bullets:

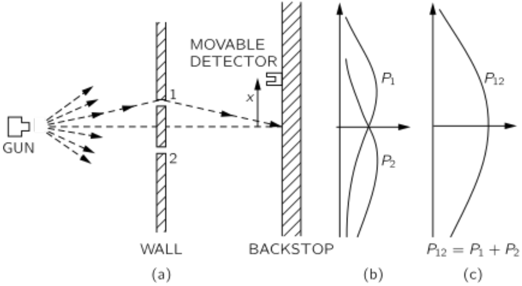

Figure: The double slits experiment with bullets

The idea is as follows, one has a very bad gun shooting bullets evenly over a widespread angle. Some of these bullets pass through the slits in the wall and are detected with a bullet detector which counts the number of bullets arriving at each point.

The case with water waves:

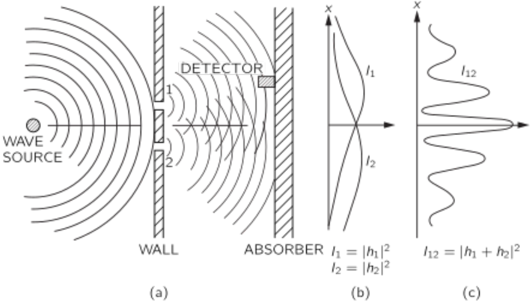

Figure: The double slits experiment with waves

Oh! Wait! That looks different!
If you want to know whether something is a wave or a particle, just throw said something at a wall with two slits.

Particles $\Rightarrow$ Previous slide
Waves $\Rightarrow$ Current slide.

The case with electrons:

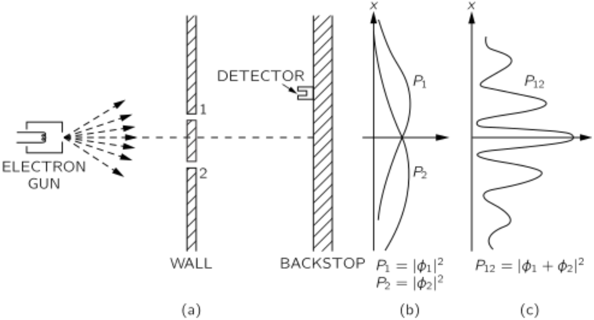

Figure: The double slits experiment with electrons

Ok, so we are done, electrons are waves. Well, no! First we can count electrons coming one after the other, and they come entire electrons after entire electrons, not by half or quarter of electron.

Spying on the electrons:

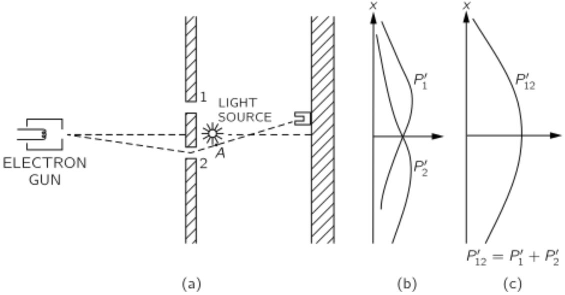

Figure: Watching the electrons

And if you watch by using a light which slit they are passing through, interference disappear. This does not happen with waves.

So what? What have we learned.

1. Electrons are neither waves nor particles, they are something else.
2. Electrons seem to be described mathematically like waves by using complex numbers for both amplitude and phase. These complex numbers are called **probability amplitude**.
   This is true at least as long as we don't watch them. Why? Really they follow the Schrödinger equation which is a linear equation and so different solutions can be added together to form a new solution. Solution through hole 1 + solution through hole 2 is a solution. This is the origin of interference pattern.

Some principles

1. Probability of an event is given by the square of the modulus of a complex ϕ called probability amplitude:

$$P = \text{Probability} \tag{1}$$
$$\phi = \text{Probability amplitude} \tag{2}$$
$$P = |\phi|^2. \tag{3}$$

2. When an event can occur in alternative ways, the probability amplitude for the event is the sum of the probability amplitudes for each way. This leads to interference.

$$\phi = \phi_1 + \phi_2 \tag{4}$$
$$P = |\phi_1 + \phi_2|^2 \tag{5}$$

3. If an experiment is capable of determining whether one or another alternative is taken, the probability of the event is the sum of the probabilities for each alternative. No interference:

$$P = P_1 + P_2 \tag{6}$$

Mathematically, we can keep track of the probability amplitude for each possible event i in a big vector $|\phi\rangle = (\phi_1, \phi_2, \ldots ...)$.
For instance electrons can be oriented. Either up or down.if the probability amplitude for an electron to be oriented up is $\frac{1}{\sqrt{2}}$ and the probability amplitude for the electron to be oriented down is $\frac{i}{\sqrt{2}}$, then the corresponding vector is

$$|\phi\rangle = (\phi_\uparrow = \frac{1}{\sqrt{2}}, \phi_\downarrow = \frac{i}{\sqrt{2}}). \tag{7}$$

But, in Quantum Mechanics, the events form a basis for a vector space. So each event is itself a vector. Therefore

$$|\phi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\rangle. \tag{8}$$

One then notice that $\phi_\uparrow = \langle\uparrow|\phi\rangle$ and $\phi_\downarrow = \langle\downarrow|\phi\rangle$.

Einstein-Podolski-Rosen

That's the difficult part of this talk. We've probably not reached this stage anyway. But if we do, please don't panic. Electrons are oriented. This orientation is called their spin.

If you measure their orientation along a direction, then they are either oriented one way (positively, up, or $\uparrow$) or the other way (negatively, down, or $\downarrow$ along this direction).

The troubling fact is that this is true whatever the direction you pick. The orientation of the electron will always project along this direction, either positively or negatively.

Now consider that we have two electrons (or, rather a pair electron-positron) together, not interacting. The possible events if we measure their orientation along two directions surpported by two vectors $\vec{a}, \vec{b}$ are

$$\uparrow_{\vec{a}}\uparrow_{\vec{b}},\ \uparrow_{\vec{a}}\downarrow_{\vec{b}},\ \downarrow_{\vec{a}}\uparrow_{\vec{b}},\ \downarrow_{\vec{a}}\downarrow_{\vec{b}}\ . \tag{9}$$

With the ket notation, these events are vectors

$$|\uparrow_{\vec{a}}\uparrow_{\vec{b}}\rangle,\ |\uparrow_{\vec{a}}\downarrow_{\vec{b}}\rangle,\ |\downarrow_{\vec{a}}\uparrow_{\vec{b}}\rangle,\ |\downarrow_{\vec{a}}\downarrow_{\vec{b}}\rangle. \tag{10}$$

Each event comes with a probability amplitude, and, a priori, the state of the pair is a vector of the form

$$|\phi\rangle = \phi_{\uparrow_{\vec{a}}\uparrow_{\vec{b}}}|\uparrow_{\vec{a}}\uparrow_{\vec{b}}\rangle + \phi_{\uparrow_{\vec{a}}\downarrow_{\vec{b}}}|\uparrow_{\vec{a}}\downarrow_{\vec{b}}\rangle + \phi_{\downarrow_{\vec{a}}\uparrow_{\vec{b}}}|\downarrow_{\vec{a}}\uparrow_{\vec{b}}\rangle + \phi_{\downarrow_{\vec{a}}\downarrow_{\vec{b}}}|\downarrow_{\vec{a}}\downarrow_{\vec{b}}\rangle \tag{11}$$

However we can choose another additional direction $\vec{z}$ and measure both orientations along this direction. One then has, a priori,

$$|\phi\rangle = \phi_{\uparrow\uparrow}|\uparrow\uparrow\rangle + \phi_{\uparrow\downarrow}|\uparrow\downarrow\rangle + \phi_{\downarrow\uparrow}|\downarrow\uparrow\rangle + \phi_{\downarrow\downarrow}|\downarrow\downarrow\rangle \tag{12}$$

Now I start with a particular experiment. I have one atom whose nucleus undergoes a nuclear reaction which releases one electron and positron.

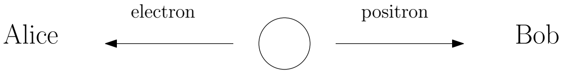

Figure: Desintegration of a nucleus of spin 0 into a pair electron-positron

I cheat a first time: Electron and positron have to go into opposite directions.

I cheat a second time: the total spin, measured in the same direction $\vec{z}$, of the pair must be zero because the one of the nucleus is.

Finally I cheat even more, by telling you that the state of the pair is given by

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{13}$$

There are reasons for all the cheating, but I can't explain in such a short time.

Now consider that Alice measure the spin of her electron along the direction $\vec{a} = \vec{z}$ and Bob along a direction $\vec{b}$.
The action of measuring projects the state over the result of the measurement. That is if Alice finds a positive orientation, the resulting state after the measurement is

$$|\psi\rangle = |\uparrow\downarrow\rangle. \tag{14}$$

If $\vec{b} \neq \vec{a}$ then calling θ, ν the polar angles between $\vec{b}$ and $\vec{a}$ one has

$$|\uparrow_{\vec{b}}\rangle = \sin(\theta/2)e^{i\nu/2}|\downarrow + \cos(\theta/2)e^{-i\nu/2}|\uparrow\rangle. \tag{15}$$

So after the measurement of Alice, the probability for Bob to find respectively $\uparrow_{\vec{b}}$ or $\downarrow_{\vec{b}}$ are

$$P_B(\uparrow_{\vec{b}}) = \sin^2(\theta/2), \quad P_B(\downarrow_{\vec{b}}) = \cos^2(\theta/2). \tag{16}$$

EPR "paradox" is represented by the question: Why is the result that Bob gets somehow depends on the result that Alice gets?

EPR said, this strange result show that there should be some information we do not know, that encode the result beforehand. Some hidden variables explain this.

Bell's inequality:

One can compute the order 2 joint moment of Alice and Bob measurement following Quantum Mechanics predictions. In that case one finds

$$\mathbb{E}(\sigma_A \sigma_B) = -\frac{1}{4}\cos(\theta) = -\frac{1}{4}\vec{a}\cdot\vec{b} \tag{17}$$

What if we do the same in a general theory in which there exists hidden variables?

In the hidden variables theory, there exists some functions $\sigma_A(v, \vec{a}) = \pm\frac{1}{2}$ and $\sigma_B(v, \vec{b}) = \pm\frac{1}{2}$ which are deterministic function. They are only made random by the fact that we do not know the hidden variables v!

We only know that $\sigma_A(v, \vec{b}) = -\sigma_B(v, \vec{b})$

What is the joint moment for these two functions?

$$\mathbb{E}(\sigma_A(\vec{a})\sigma_B(\vec{b})) = \int \mathrm{d}^n v \rho(v)\sigma_A(v,\vec{a})\sigma_B(v,\vec{b}) \tag{18}$$
$$= -\int \mathrm{d}^n v \rho(v)\sigma_A(v,\vec{a})\sigma_A(v,\vec{b}) \tag{19}$$

Consider yet another direction supported by the vector $\vec{c}$. Then one has

$$\mathbb{E}(\sigma_A(\vec{a})\sigma_B(\vec{b})) - \mathbb{E}(\sigma_A(\vec{a})\sigma_B(\vec{c})) =$$
$$-\int \mathrm{d}^n v \rho(v)\sigma_A(v,\vec{a})(\sigma_A(v,\vec{b}) - \sigma_A(v,\vec{c}))$$
$$= -\int \mathrm{d}^n v \rho(v)\sigma_A(v,\vec{a})\sigma_A(v,\vec{b})(1 - 4\sigma_A(v,\vec{b})\sigma_A(v,\vec{c})) \tag{20}$$

Using the remark that $(1 - 4\sigma_A(v, \vec{b})\sigma_A(v, \vec{c})) \geq 0$ and that $|\sigma_A(v, \vec{a})\sigma_A(v, \vec{b})| \leq \frac{1}{4}$ one has **Bell's inequality**

$$|\mathbb{E}(\sigma_A(\vec{a})\sigma_B(\vec{b})) - \mathbb{E}(\sigma_A(\vec{a})\sigma_B(\vec{c}))| \leq \frac{1}{4}(1 + 4\mathbb{E}_v(\sigma_A(\vec{b})\sigma_A(\vec{c}))). \tag{21}$$

Insert QM predictions in the LHS leads to:

$$\frac{1}{4}|\vec{a} \cdot (\vec{b} - \vec{c})| \tag{22}$$

Then insert QM prediction in the RHS this leads to:

$$\frac{1}{4}(1 - \vec{b} \cdot \vec{c}) \tag{23}$$

Is it always true that LHS≤RHS? Just pick $\vec{a} \cdot \vec{b} = 0$ and $\vec{c} = \sin(\varphi)\vec{a} + \cos(\varphi)\vec{b}$. In this case

$$\mathrm{LHS} = \frac{1}{4}|\sin\varphi| \tag{24}$$

and

$$\mathrm{RHS} = \frac{1}{4}(1 - \cos\varphi). \tag{25}$$

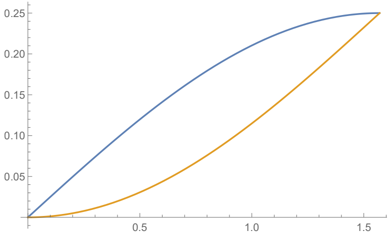

Figure: Blue curve: LHS. Orange curve: RHS

Bell's inequality is violated almost everywhere by QM!

The END

That's all for today. Thank you. Hopefully it was understandable.