

**Pattern avoiding permutations,  
interval orders  
and  
involutions with no neighbour nesting**

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joint work with...

## The Icelanders:



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Anders Claesson<sup>2</sup>



Sergey Kitaev<sup>2</sup>

1. University of Iceland
2. University of Reykjavík

# I. L'histoire

## Pattern avoiding permutations

### Classical pattern avoidance

Consider  $\mathcal{S}_n(2-3-1)$ ; the collection of all permutations  $\pi \in \mathcal{S}_n$  such that there do not exist indices  $1 \leq i < j < k \leq n$  with  $\pi_k < \pi_i < \pi_j$ .

$$\mathcal{S}_n(2-3-1) = \mathcal{S}_n \left( \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right)$$

### Generalized pattern avoidance

Consider  $\mathcal{S}_n(23-1)$ ; the collection of all permutations  $\pi \in \mathcal{S}_n$  such that there do not exist indices  $1 \leq i < i+1 < k \leq n$  with  $\pi_k < \pi_i < \pi_{i+1}$ .

$$\mathcal{S}_n(23-1) = \mathcal{S}_n \left( \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right)$$

The ‘—’ represents an elastic length.

- E. Babson and E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, *Sém. Lothar. Combin.* **44** (2000) Art. B44b, 18 pp.

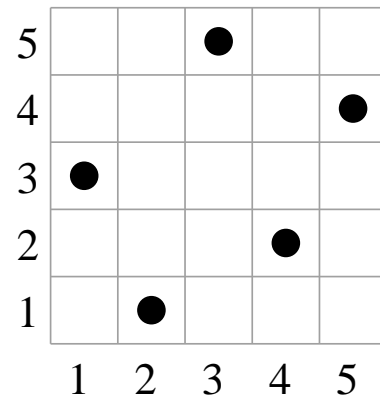
# A new class of pattern avoiding permutations

What about

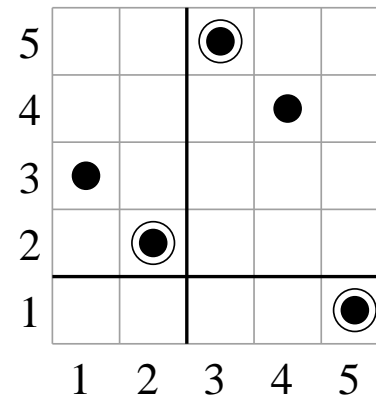
$$\mathcal{R}_n = \mathcal{S}_n \left( \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right) ?$$

A permutation is in  $\mathcal{R}_n$  if there do not exist indices  $i$  and  $k$  satisfying  $1 \leq i < i + 1 < k \leq n$  and such that

$$\pi_k + 1 = \pi_i < \pi_{i+1}.$$



$31524 \in \mathcal{R}_5$



$32541 \notin \mathcal{R}_5$

# Enumeration

Some known results:

- $\mathcal{S}_n \left( \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right) \Rightarrow C_n$ , Catalan numbers

- $\mathcal{S}_n \left( \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right) \Rightarrow B_n$ , Bell numbers

What about

- $\mathcal{R}_n = \mathcal{S}_n \left( \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right) \Rightarrow 1, 1, 2, 5, 15, 53, 217, 1014...?$

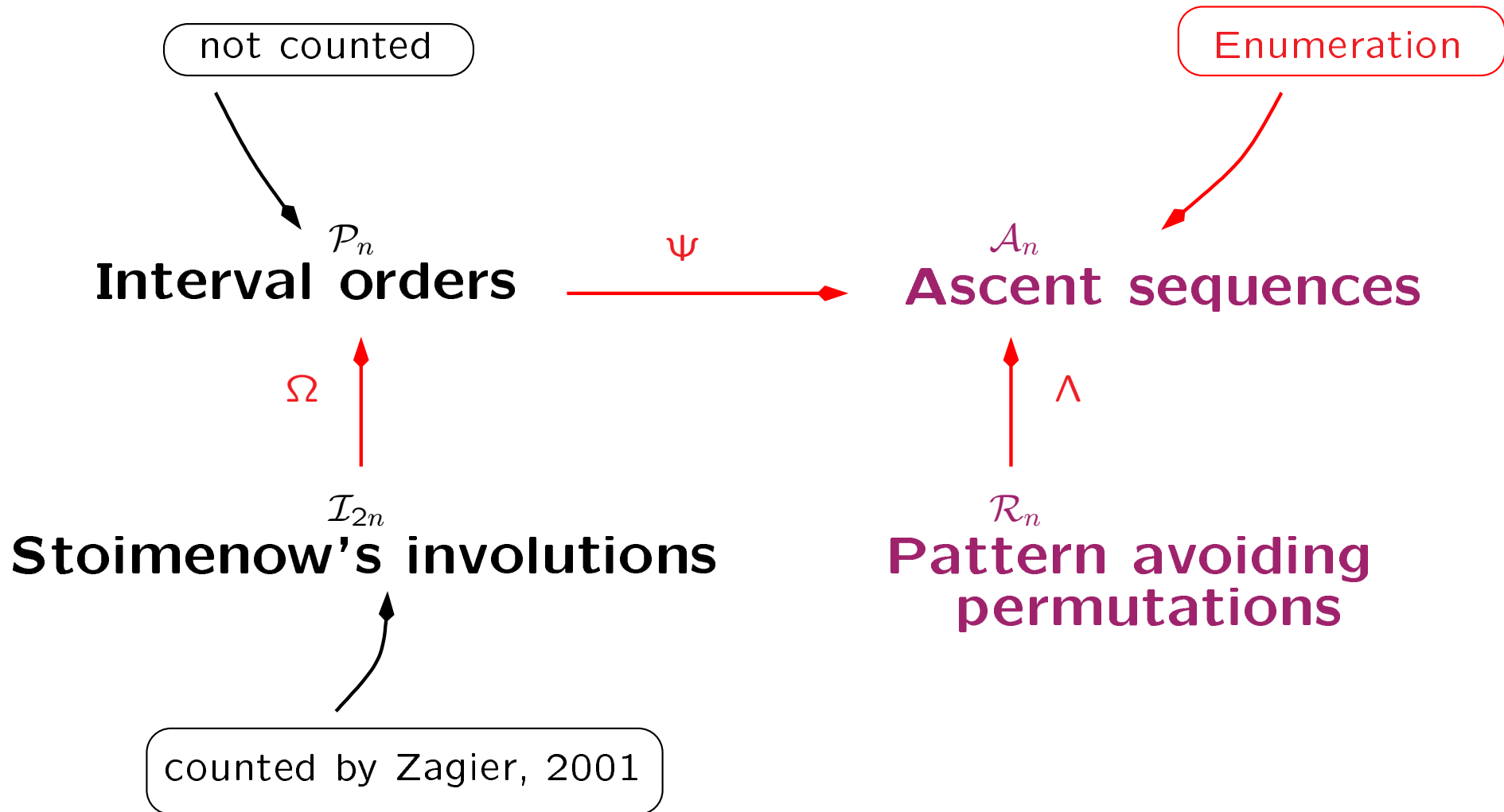
In the On-Line Encyclopedia of Integer Sequences: A022493

- Unlabelled interval orders on  $n$  points

- Linearized chord diagrams of degree  $n$ , introduced by [Stoimenow, 1998](#)

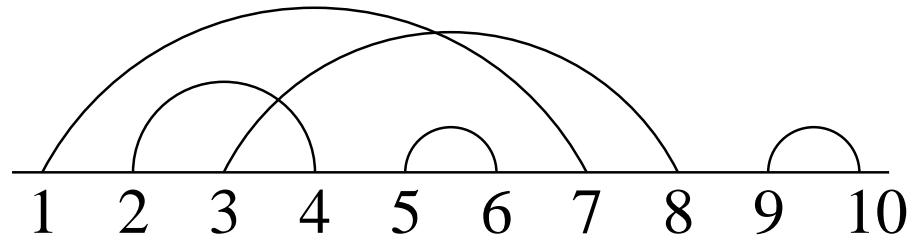
(here: involutions with no neighbour nesting)

# In this talk



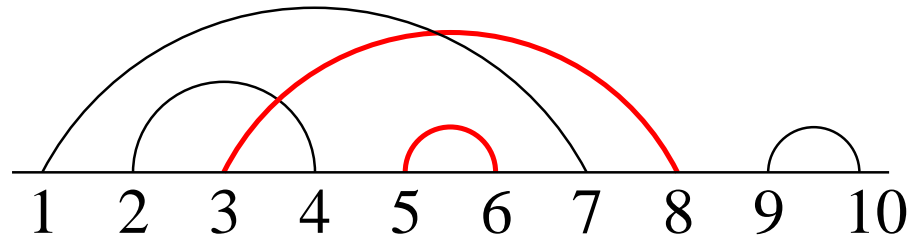
## **II. Involutions with no neighbour nestings and interval orders**

## Involutions with no neighbour nestings



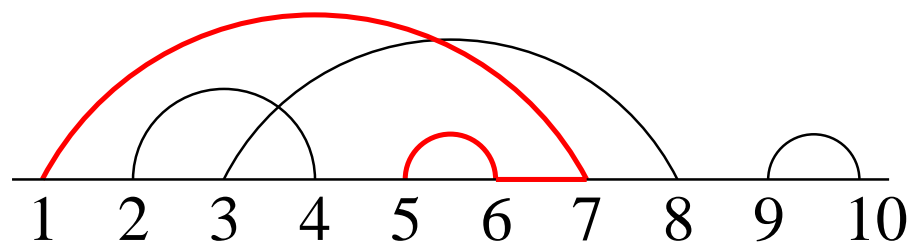
A (fixed-point free) involution

## Involutions with no neighbour nestings



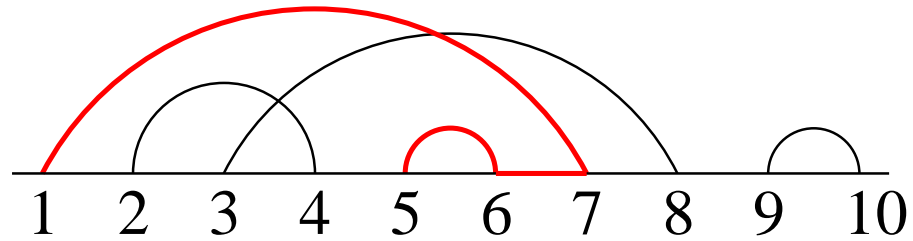
A **nesting** between two arches

## Involutions with no neighbour nestings

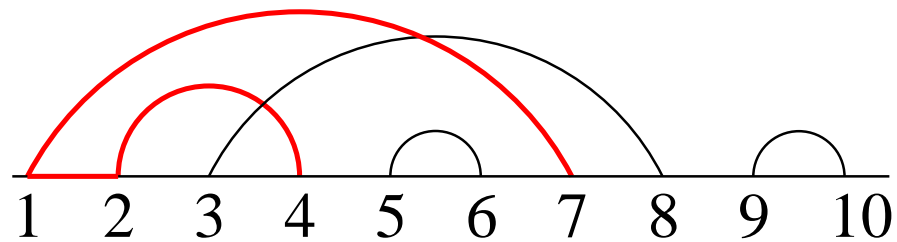


A neighbour nesting

## Involutions with no neighbour nestings

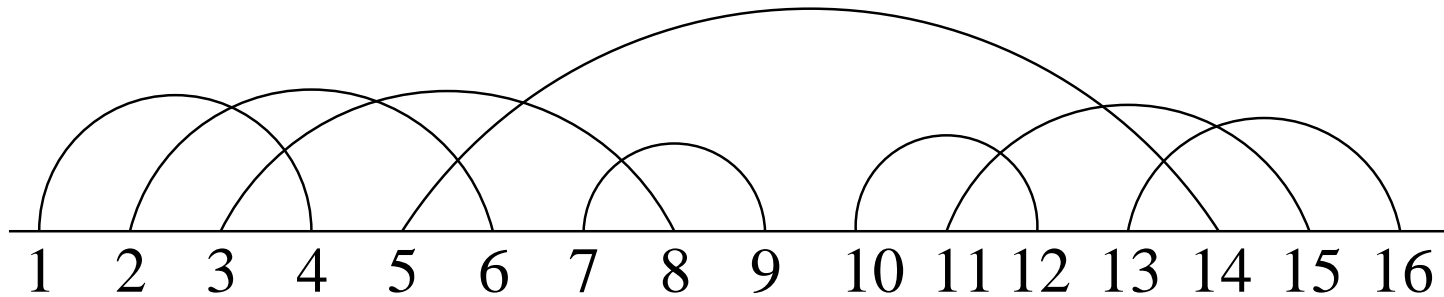


A neighbour nesting



Another one

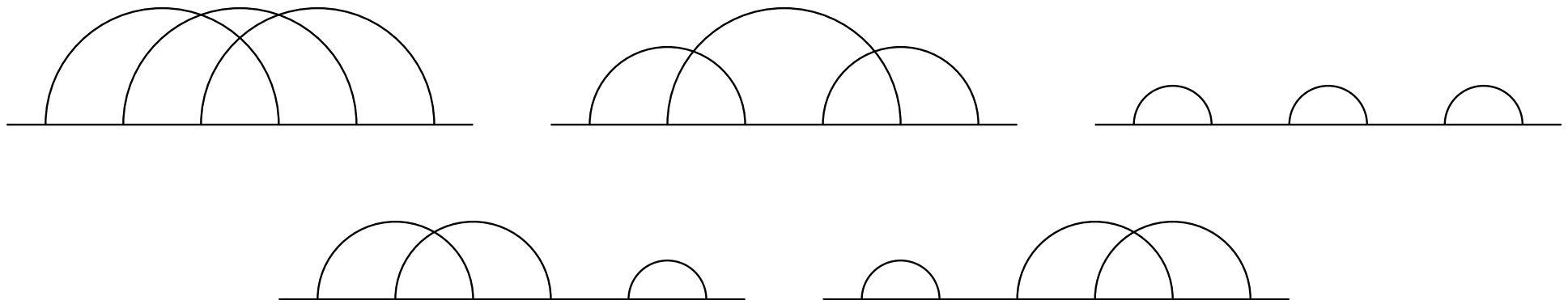
## Involutions with no neighbour nestings



[Stoimenow 98]: bounds on the dimension of the vector space of Vassiliev invariants of degree  $n$

Equivalently: all descents cross the diagonal

Example:  $n = 3$

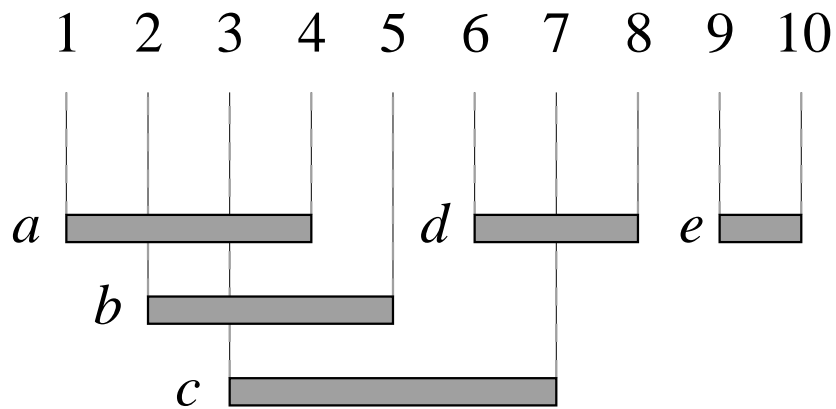


## Interval orders

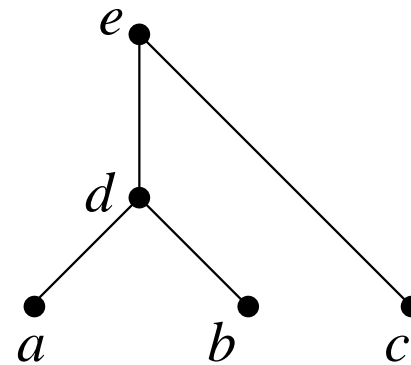
**Definition:** The relative order of a collection of intervals of the real line under the order relation:

$$[x, y] < [z, t] \iff y < z.$$

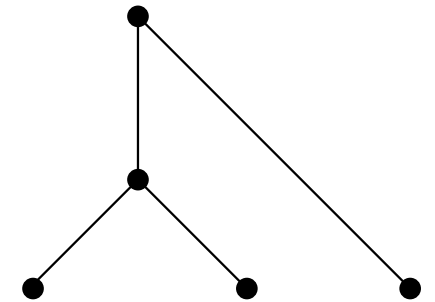
### Example



Intervals



(Labelled) Poset



Unlabelled Poset

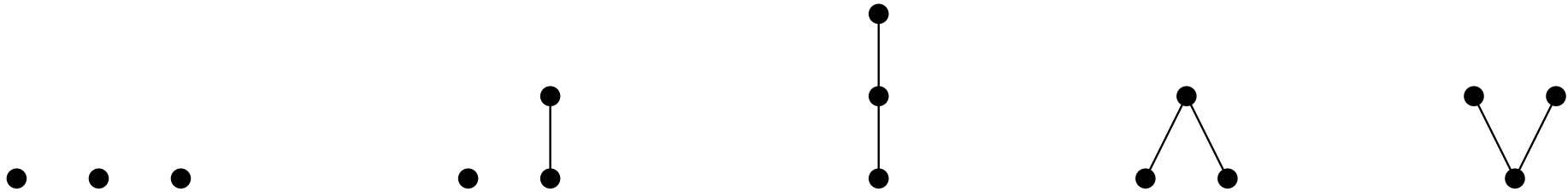
(Hasse diagram)

## Remarks

- Different collections of intervals may give the same order:



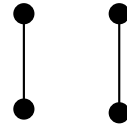
- Example:  $n = 3$



- For the number of unlabelled posets, only 16 terms are known!  
(18 in the labelled case)

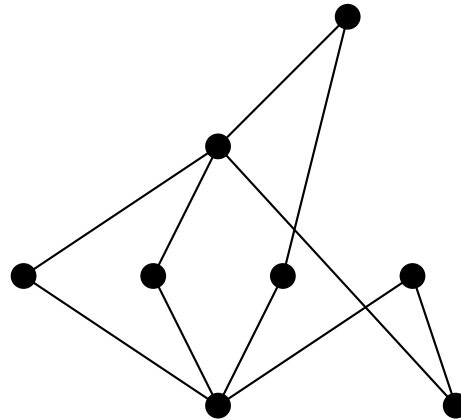
## Interval orders: two characterizations

- **First characterization:** The poset  $P$  has no induced  $(2 + 2)$  sub-poset



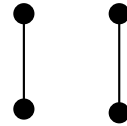
[Fishburn 70]

Example:



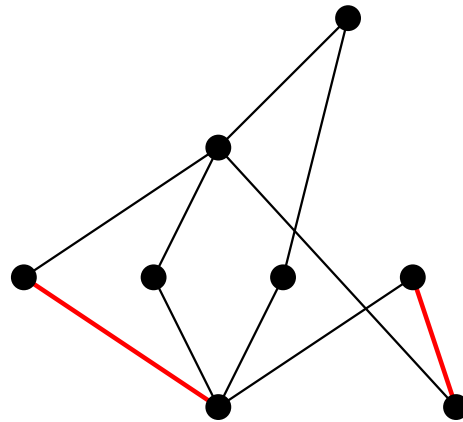
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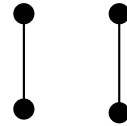
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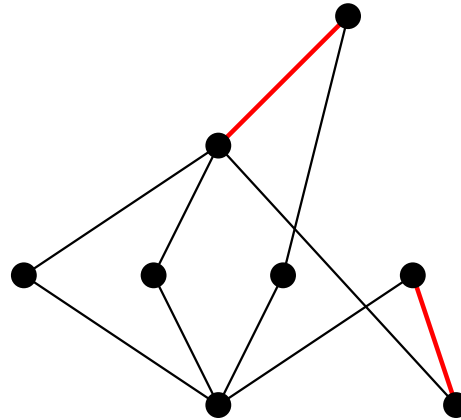
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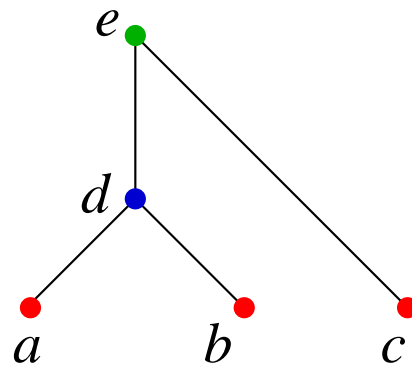
## Interval orders: two characterizations

- **Second characterization [Fishburn 70].** The predecessor sets (or: down-sets)

$$D(x) = \{y \in P : y < x\}, \quad \text{for } x \in P$$

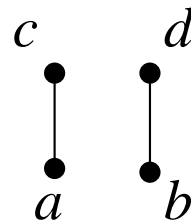
can be totally ordered by inclusion.

Example:



$$\emptyset = D(a) = D(b) = D(c) \subset D(d) = \{a, b\} \subset D(e) = \{a, b, c, d\}$$

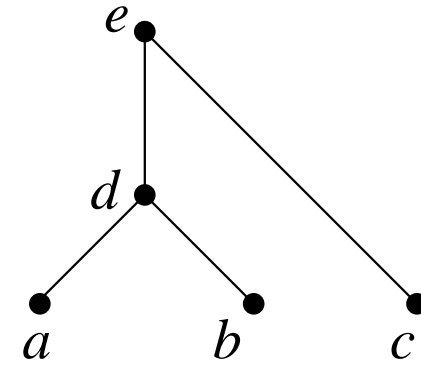
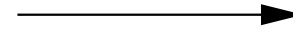
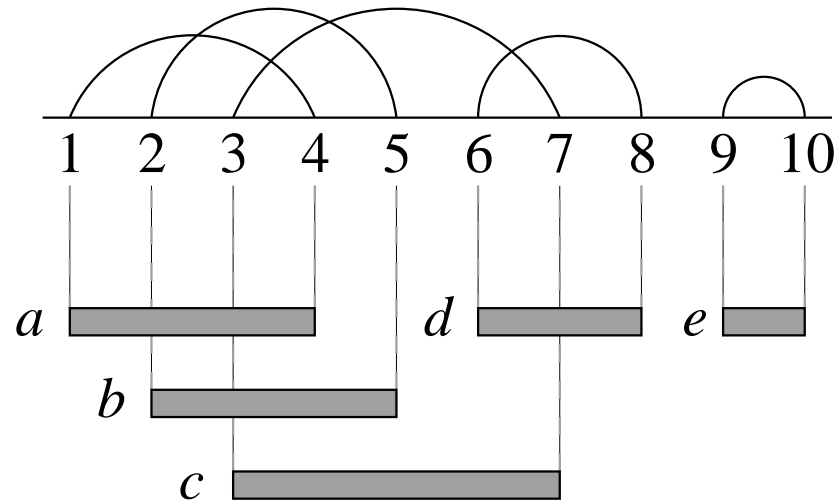
Counter-example: In



$D(c) = \{a\}$  is not comparable to  $D(d) = \{b\}$ .

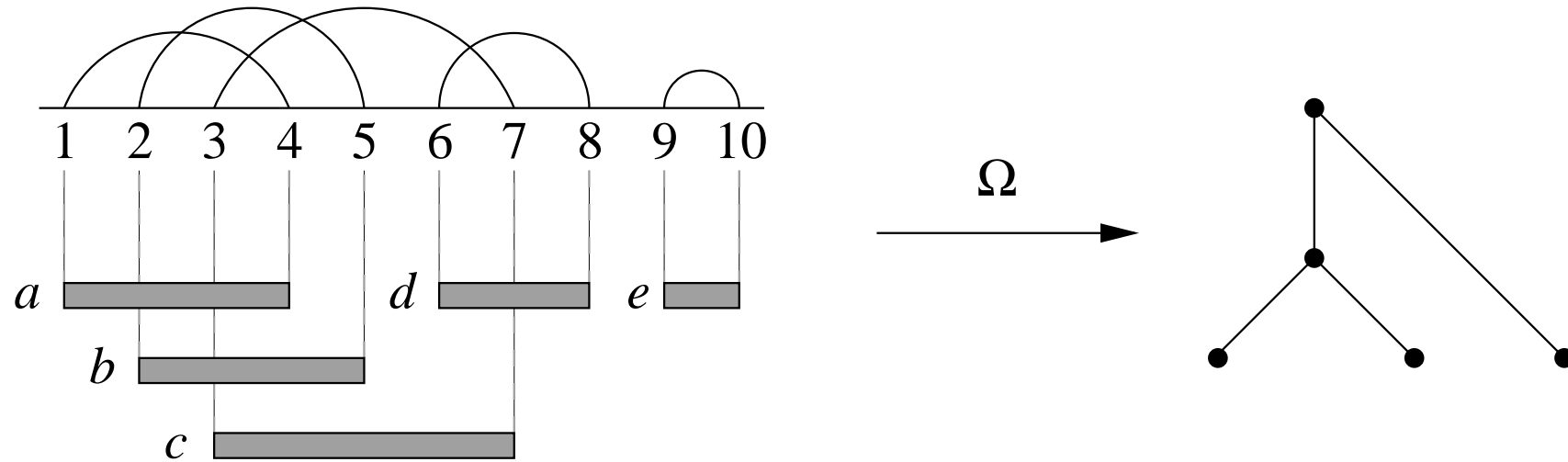
# First bijection:

From involutions with no neighbour nesting to interval orders



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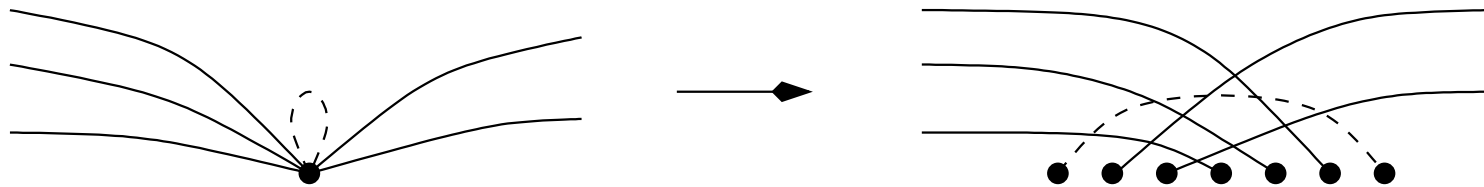
**Proposition:**  $\Omega$  sends bijectively involutions on  $2n$  points with no neighbour nesting onto interval orders on  $n$  unlabelled elements.

**Remark.** Reflecting  $\sigma$  across a vertical line  $\iff$  Taking the dual poset.

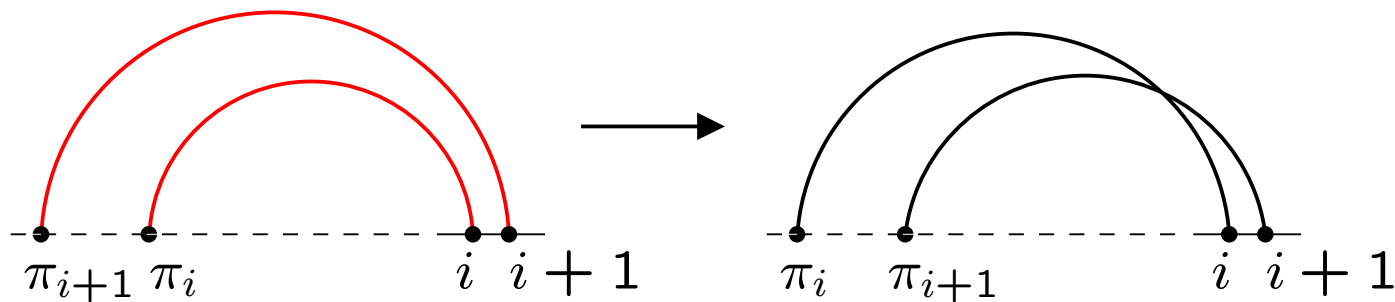
## First bijection:

### From involutions with no neighbour nesting to interval orders

- $\Omega$  is surjective. Let  $P$  be an interval order on  $n$  elements.
  - Take a collection of  $n$  intervals whose order is  $P$
  - One can assume that the endpoints of these intervals are  $2n$  distinct points



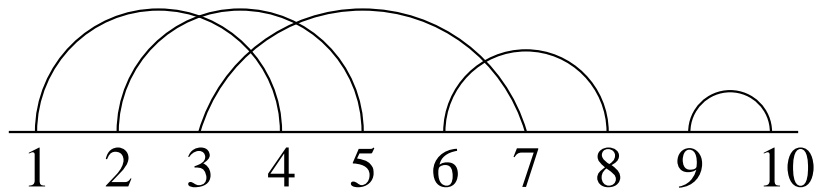
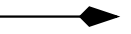
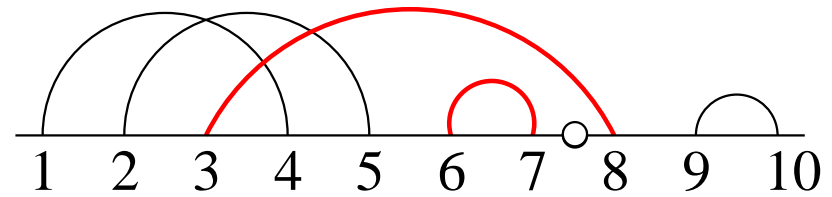
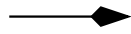
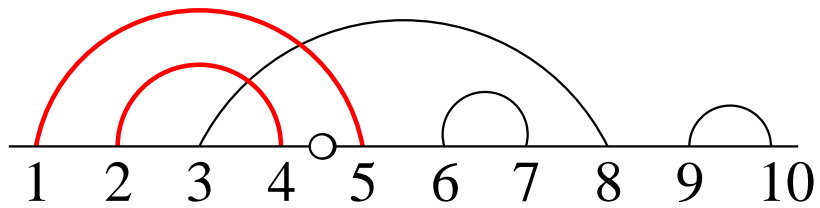
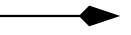
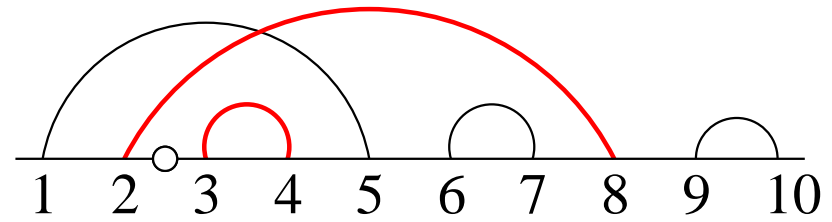
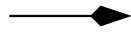
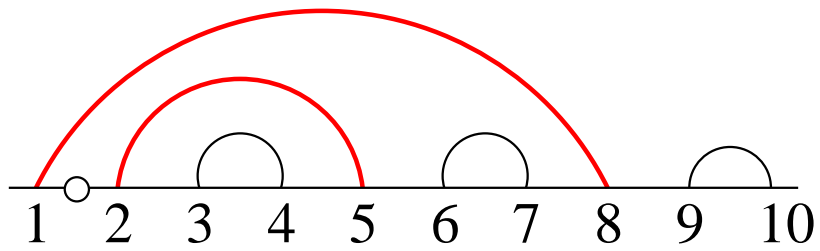
- One can assume that these points are  $1, 2, \dots, 2n \Rightarrow$  a fixed point free involution
- Delete neighbour nestings without changing the order



(and symmetrically)

## Surjectivity: an example

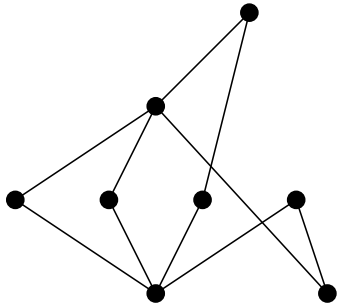
Collection of intervals:  $[1, 8]$ ,  $[2, 5]$ ,  $[3, 4]$ ,  $[6, 7]$ ,  $[9, 10]$



No neighbour nesting

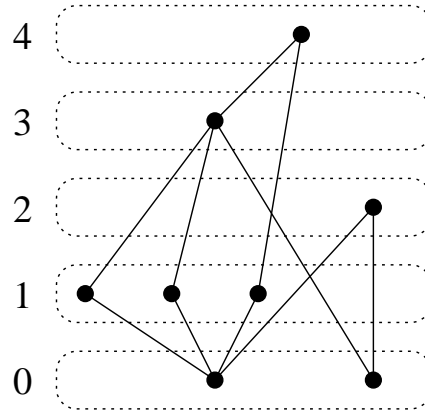
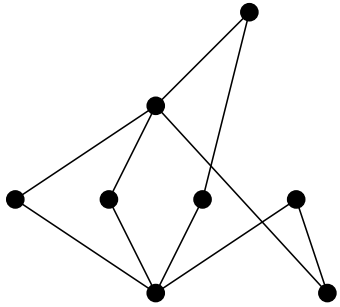
## From involutions with no neighbour nesting to interval orders

- $\Omega$  is injective. Let  $\sigma \in \mathcal{I}_{2n}$ , and let  $P = \Omega(\sigma)$ . How can we reconstruct  $\sigma$  from  $P$ ?



## From involutions with no neighbour nesting to interval orders

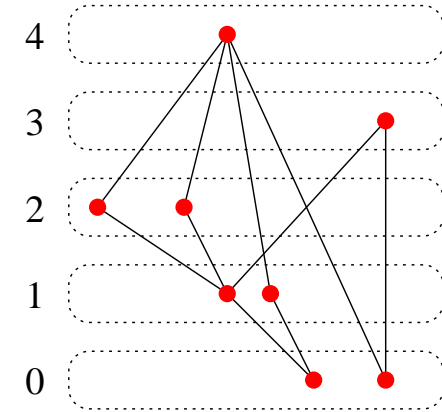
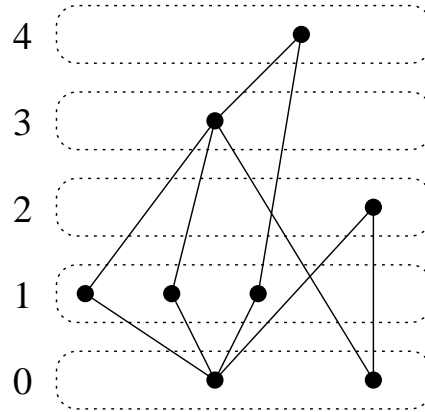
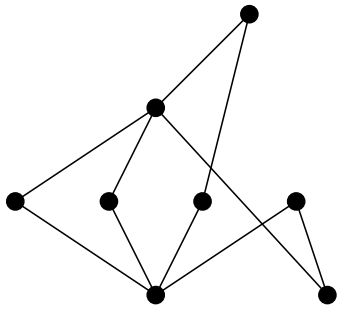
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- Display  $P$  in **canonical form**: if  $\{D(x), x \in P\} = \{D_0, D_1, \dots, D_k\}$  with  $\emptyset = D_0 \subset D_1 \subset \dots \subset D_k$ , put  $x$  at level  $i$  if  $D(x) = D_i$ .

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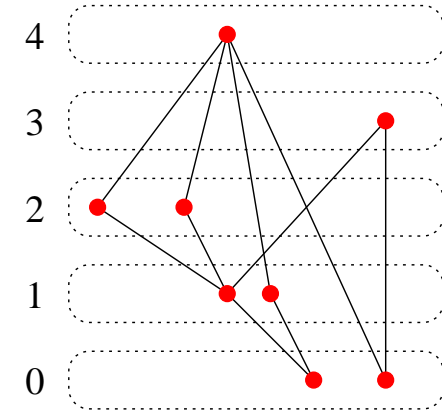
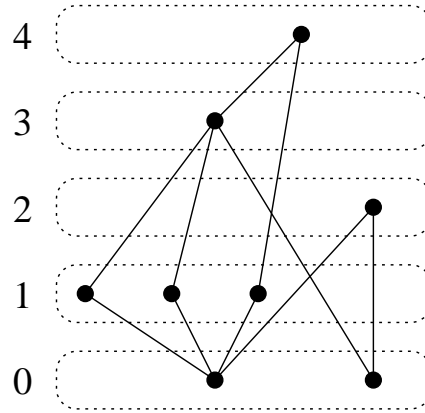
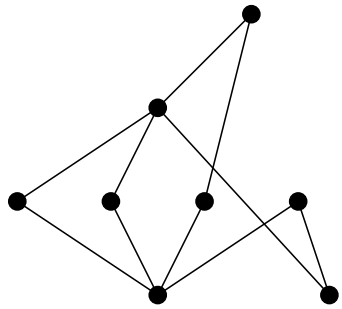
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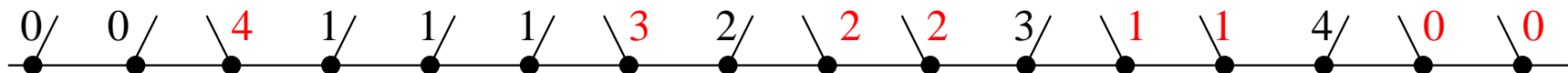
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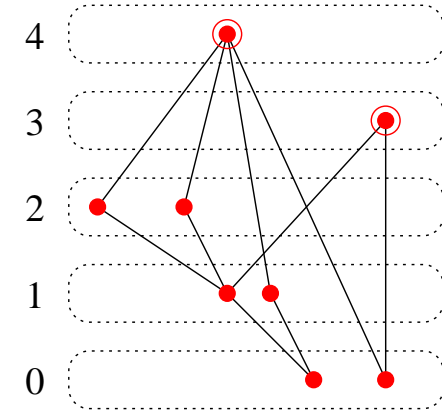
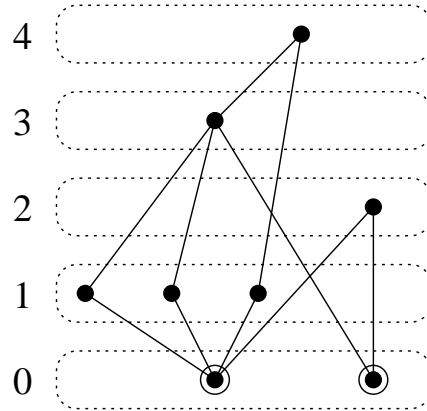
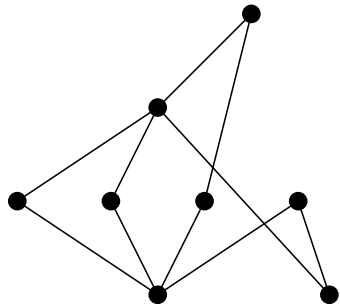


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- Form runs of opening/**closing** arches by reading the levels in  $P$  (resp.  $P^*$ )

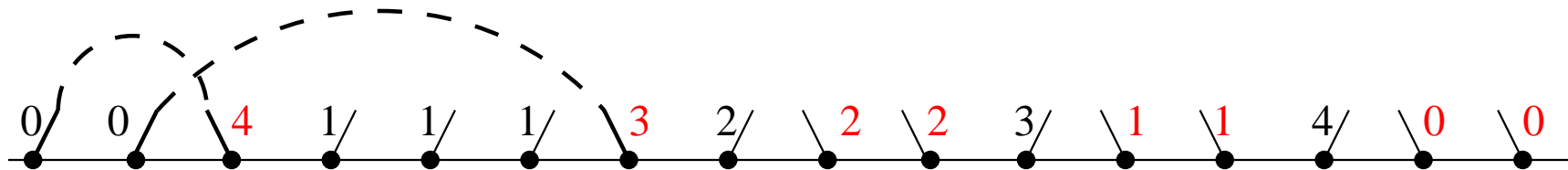


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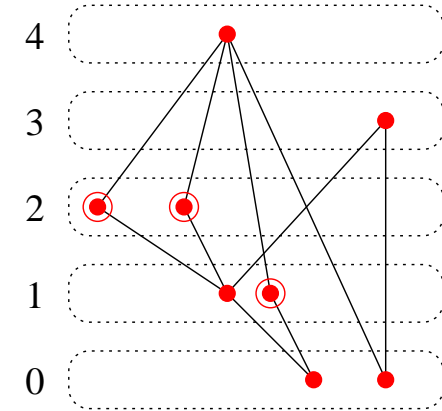
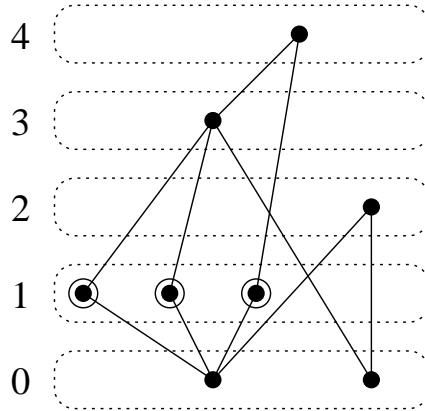
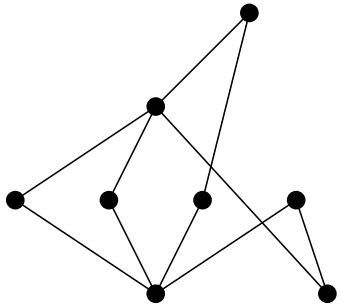
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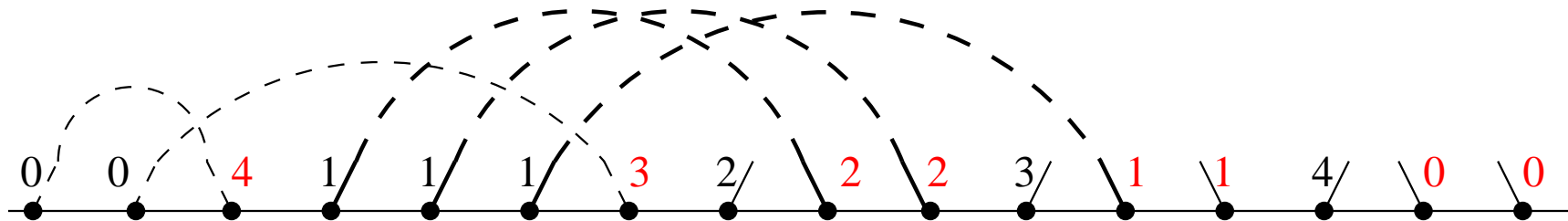
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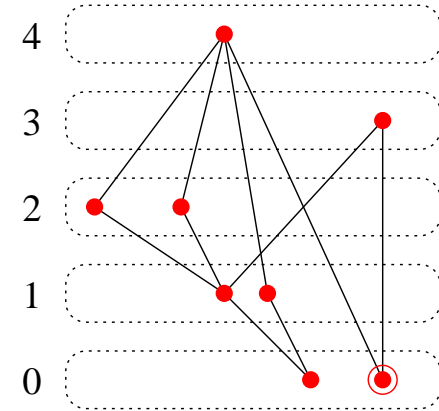
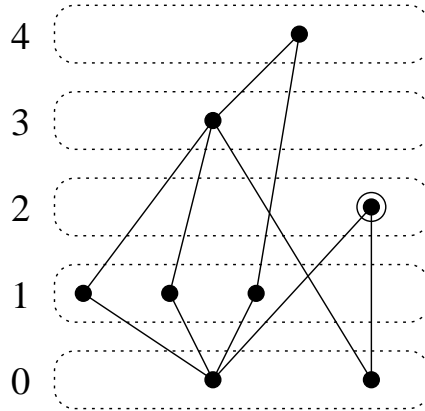
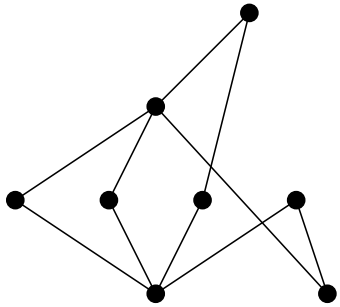
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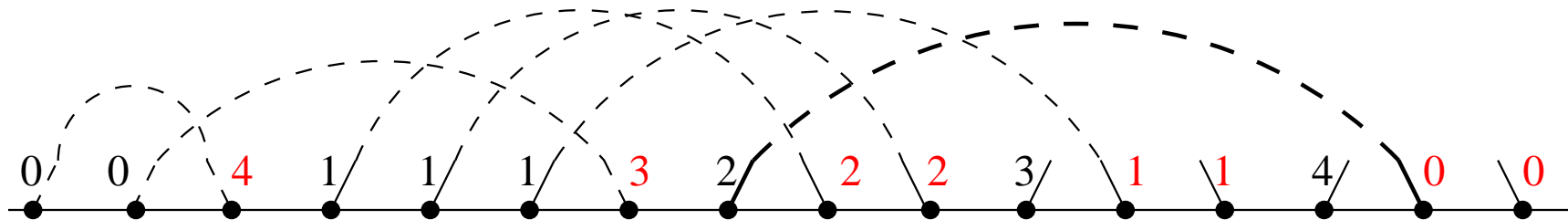
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# From involutions with no neighbour nesting to interval orders

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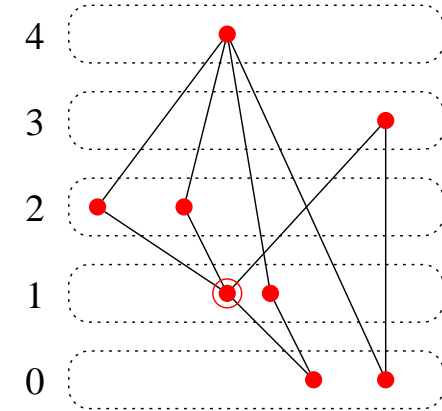
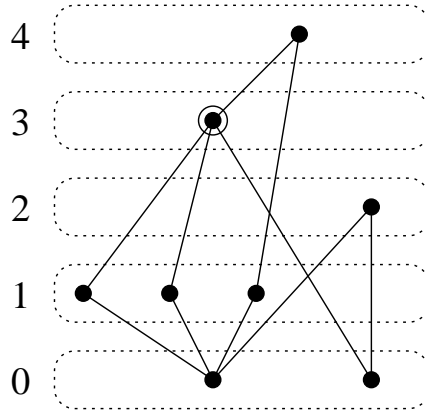
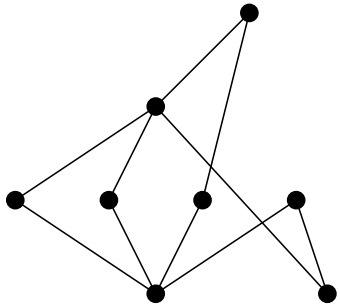
- Display  $P$  in **canonical form**: if  $\{D(x), x \in P\} = \{D_0, D_1, \dots, D_k\}$  with  $\emptyset = D_0 \subset D_1 \subset \dots \subset D_k$ , put  $x$  at level  $i$  if  $D(x) = D_i$ .
- Display the dual poset  $P^*$  in canonical form
- Form runs of opening/**closing** arches by reading the levels in  $P$  (resp.  $P^*$ )



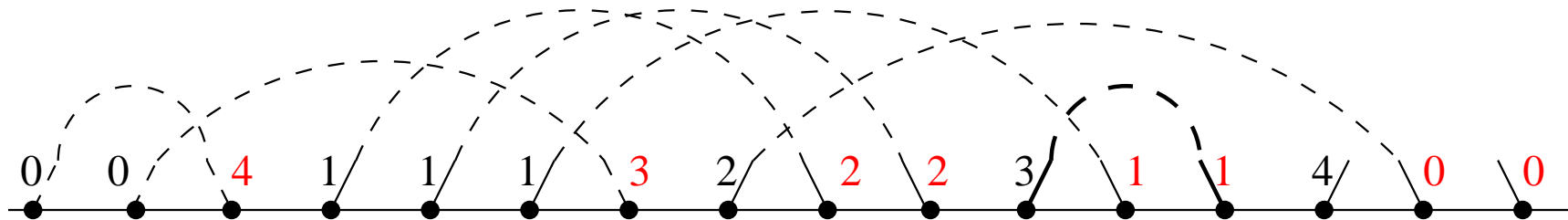
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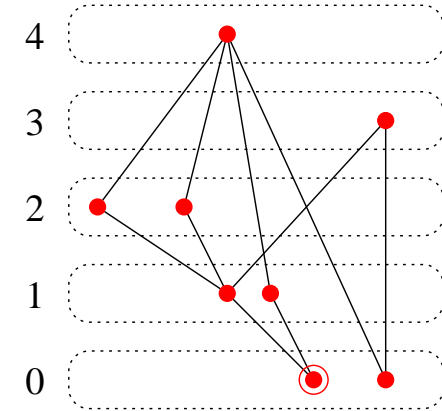
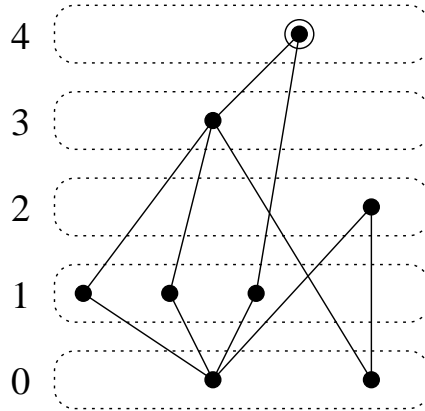
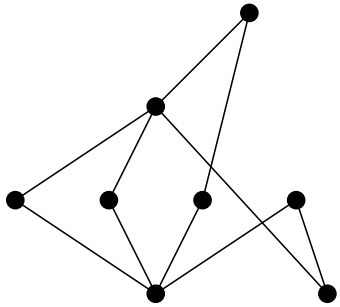
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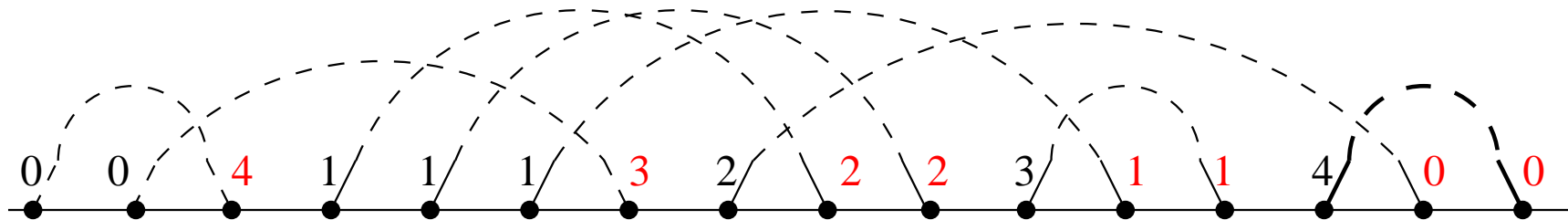
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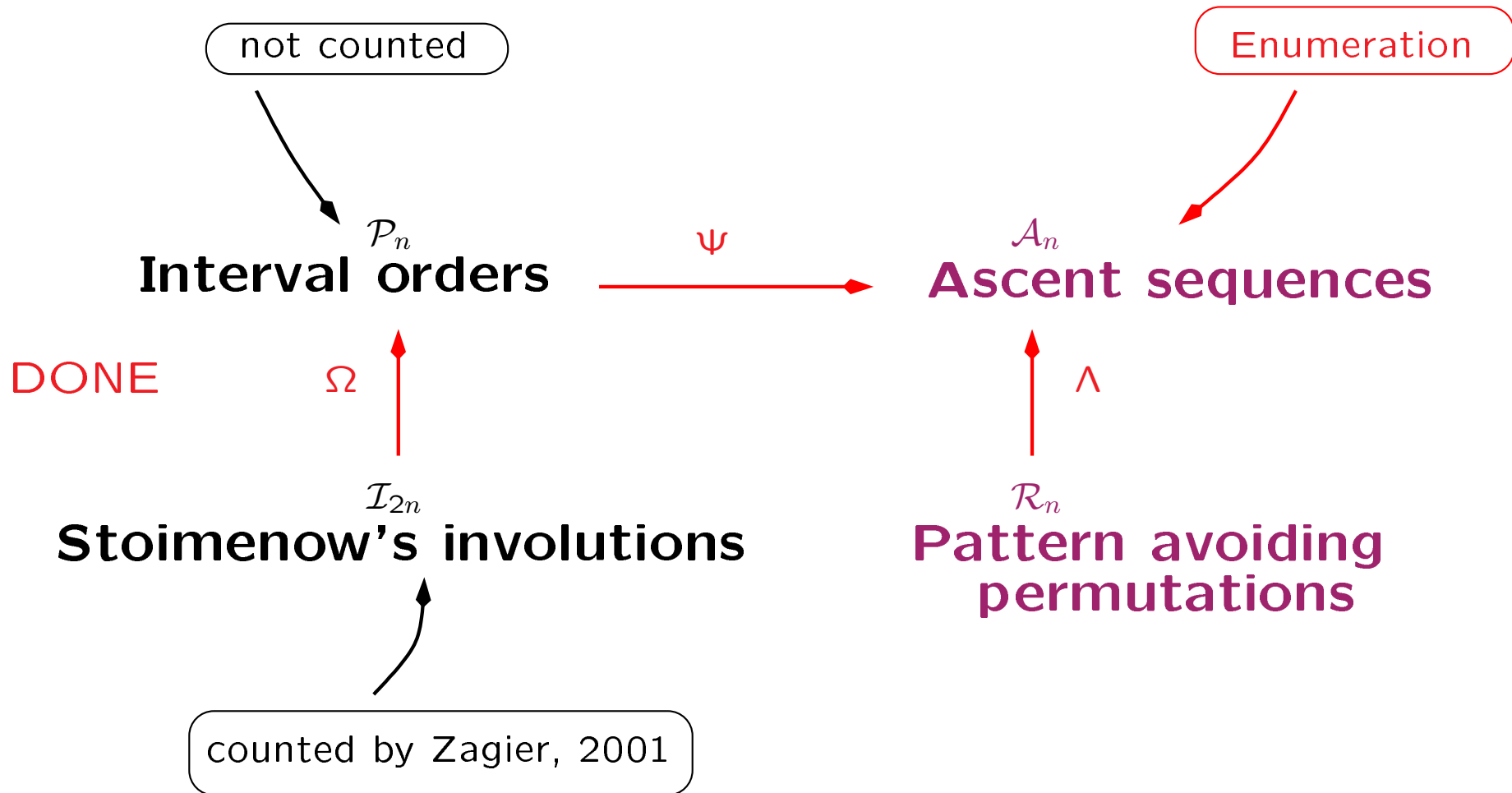


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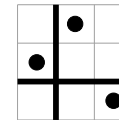
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# Where we stand



### III. From permutations avoiding

to ascent sequences



## Decomposition of permutations of $\mathcal{R}_n$

- Consider  $\pi = 61832547 \in \mathcal{R}_8$  and remove the largest entry:

$$\tau = 6132547 \in \mathcal{R}_7$$

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$$\tau = \star 6 1 \star 3 2 \star 5 4 \star 7 \star$$

- **Active sites:**
  - to the left of  $\tau$ ,
  - just after the entry  $i$ , provided  $i - 1$  does not appear to the right of  $i$ .

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- Labelling of the active sites:

$$0 6 1 1 3 2 2 5 4 3 7 4$$

The 8 appears in the active site with label 1

$\Rightarrow \pi \in \mathcal{R}_8$  may be recovered from the pair  $(\tau, 1)$  where  $\tau \in \mathcal{R}_7$ .

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$\Rightarrow \pi \in \mathcal{R}_8$  may be recovered from the pair  $(\tau, 1)$  where  $\tau \in \mathcal{R}_7$ .

- Repeating this until ending up with the empty permutation, we have the sequence

$$\Lambda(\pi) = (0, 1, 1, 2, 2, 0, 3, 1)$$

from which  $\pi$  can be reconstructed.

## Ascent sequences

**Definition.** A sequence  $(x_1, \dots, x_n)$  of non-negative integers is an **ascent sequence** if

- $x_1 = 0$ , and
- $x_i \in [0, 1 + \text{asc}(x_1, \dots, x_{i-1})]$  for all  $1 < i \leq n$ .

**Ascent number:**  $\text{asc}(0, 0, 1, 0, 1, 2, 0) = 3$

**Examples:**

All ascent sequences of length 3:

$(0, 0, 0)$   $(0, 0, 1)$   $(0, 1, 0)$   $(0, 1, 1)$   $(0, 1, 2)$

All ascent sequences of length 4:

$(0, 0, 0, 0)$   $(0, 0, 0, 1)$   $(0, 0, 1, 0)$   $(0, 0, 1, 1)$   $(0, 0, 1, 2)$   $(0, 1, 0, 0)$   $(0, 1, 0, 1)$   
 $(0, 1, 0, 2)$   $(0, 1, 1, 0)$   $(0, 1, 1, 1)$   $(0, 1, 1, 2)$   $(0, 1, 2, 0)$   $(0, 1, 2, 1)$   $(0, 1, 2, 2)$   
 $(0, 1, 2, 3)$

## Second bijection:

from permutations avoiding  to ascent sequences

- Definition of  $\Lambda$ : if  $\pi \in \mathcal{R}_n$  is obtained by inserting  $n$  in the active site labelled  $i$  of  $\tau \in \mathcal{R}_{n-1}$  then

$$\Lambda(\pi) = (x_1, x_2, \dots, x_{n-1}, i) \quad \text{where} \quad \Lambda(\tau) = (x_1, x_2, \dots, x_{n-1}).$$

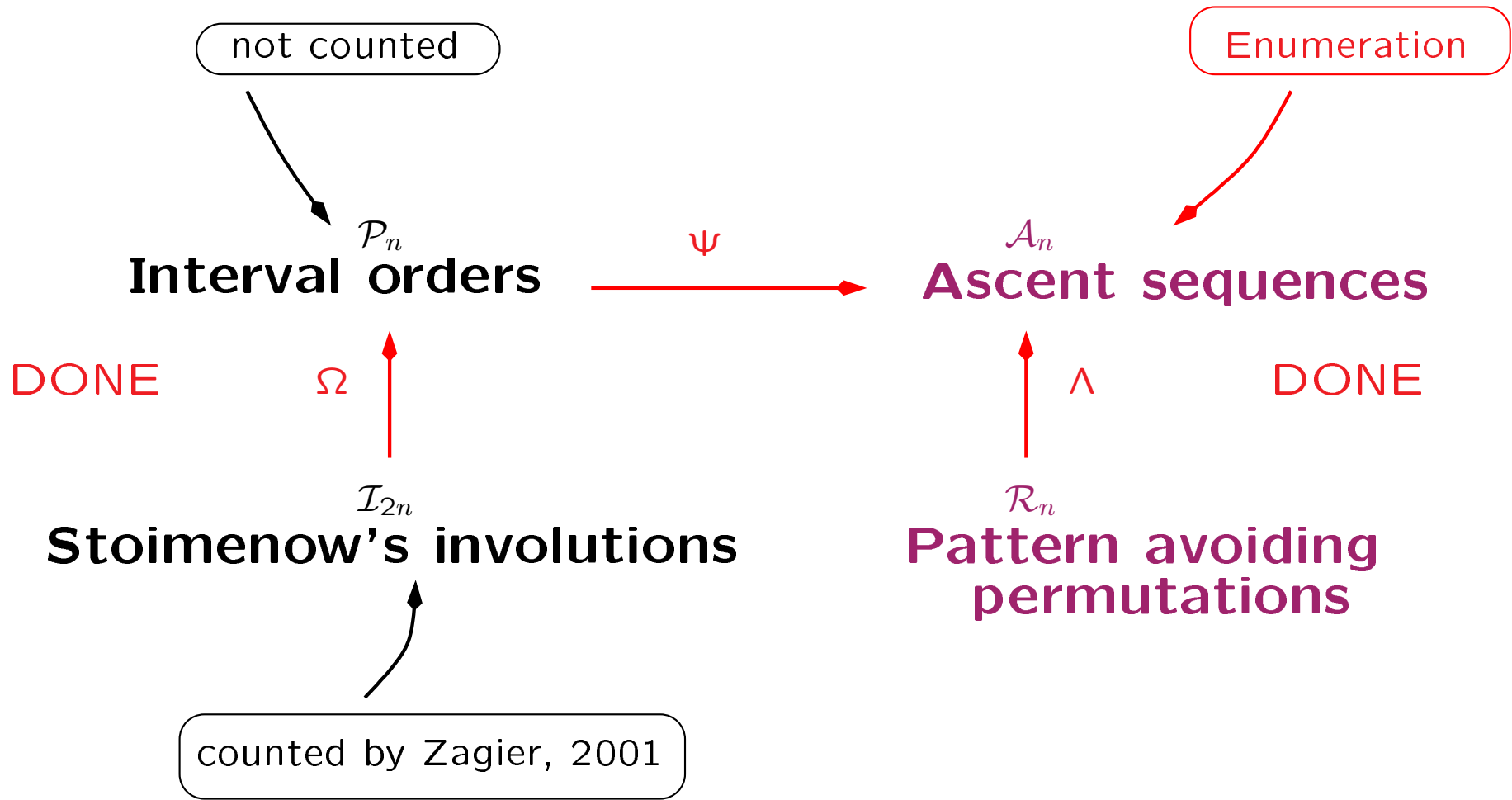
**Theorem:** The map  $\Lambda$  is a bijection between permutations of  $\mathcal{R}_n$  and ascent sequences of length  $n$ .

- Proof: by induction on  $n$ , using

$$s(\pi) = 2 + \text{asc}(x) \quad \text{and} \quad b(\pi) = x_n$$

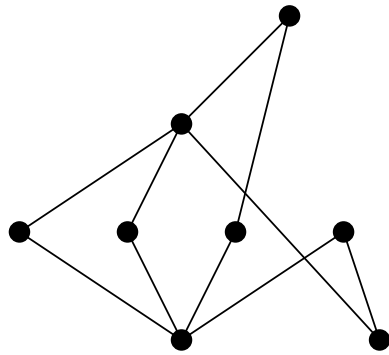
where  $s(\pi)$  is the number of active sites, and  $b(\pi)$  the label of the active site located just before the largest entry of  $\pi$ .

# Where we stand

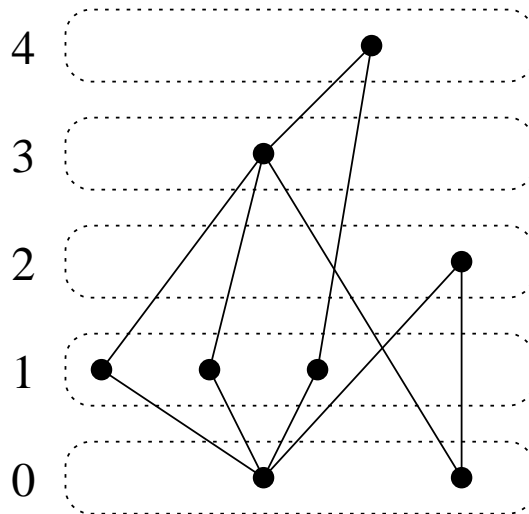


## **IV. From interval orders to ascent sequences**

## Third bijection: from interval orders to ascent sequences



Display  $P$  in **canonical form**: if  $\{D(x), x \in P\} = \{D_0, D_1, \dots, D_k\}$  with  $\emptyset = D_0 \subset D_1 \subset \dots \subset D_k$ , put  $x$  at level  $i$  if  $D(x) = D_i$ . We denote  $\ell(P) = k$ .



## Decomposition of $(2 + 2)$ -free posets

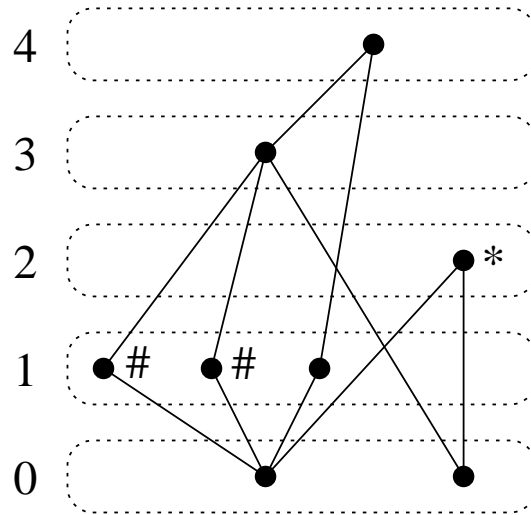
Let  $\ell^*(P)$  be the minimum level containing a maximal element. Let  $L_i(P)$  be the set of elements of  $P$  at level  $i$ . All the maximal elements located at level  $i$  are order-equivalent in  $P$ . We will remove one of them. Let  $Q$  be the poset that results from applying:

- (i) If  $|L_i(P)| > 1$ , or if  $|L_i(P)| = 1$  and  $i = \ell(P)$ , then remove one of the maximal elements at level  $i$ .
- (ii) If  $|L_i(P)| = 1$  and  $i < \ell(P)$  then set  $\mathcal{N} = D_{i+1}(P) \setminus D_i(P)$ . Make each element in  $\mathcal{N}$  a maximal element of the poset by deleting from the order all relations  $x < y$  where  $x \in \mathcal{N}$ . Finally, remove the unique element lying at level  $i$ .

- Definition of the map  $\Psi$  between posets and ascent sequences:

$$\Psi(P) = (x_1, \dots, x_{n-1}, i) \quad \text{where} \quad \Psi(Q) = (x_1, \dots, x_{n-1})$$

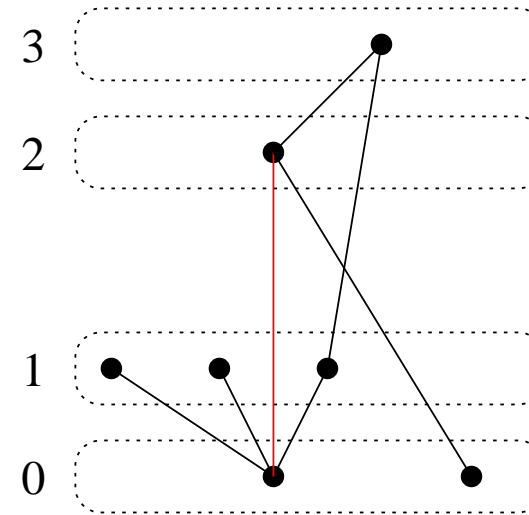
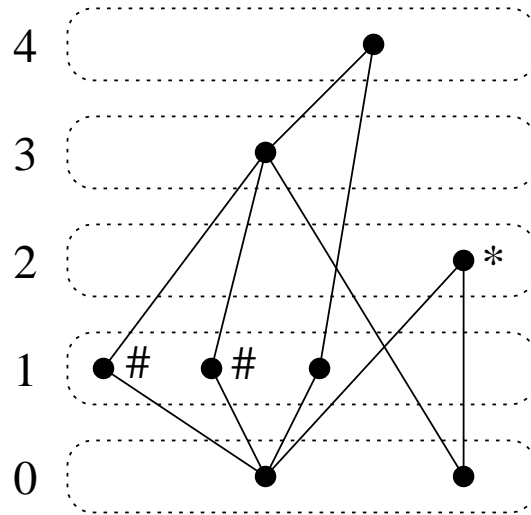
## Decomposition of $(2 + 2)$ -free posets



$$\ell(P) = 4, \quad i = 2 < \ell(P) \quad \Rightarrow x_8 = 2$$

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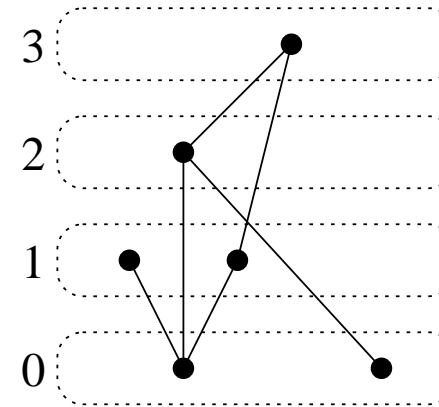
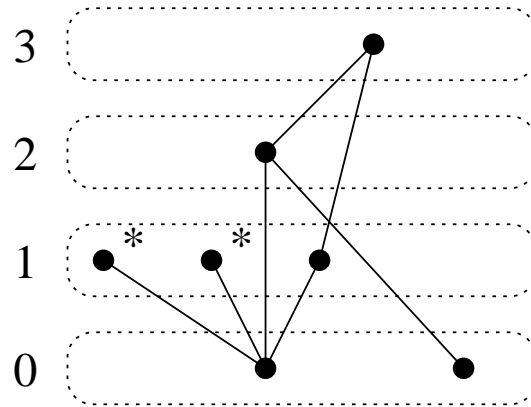
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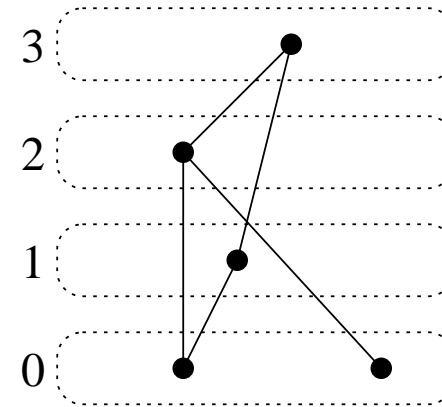
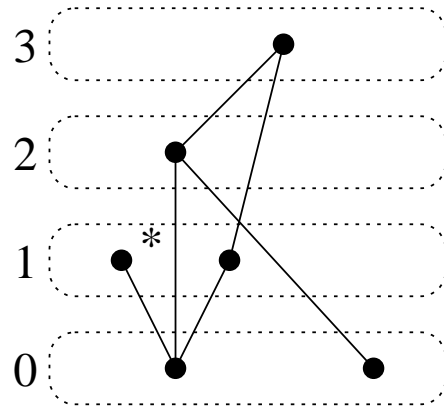
## Decomposition of $(2 + 2)$ -free posets



$$\ell(P) = 3, \quad i = 1 < \ell(P) \quad \Rightarrow \quad x_7 = 1$$

- (i) If  $|L_i(P)| > 1$ , or if  $|L_i(P)| = 1$  and  $i = \ell(P)$ , then remove one of the maximal elements at level  $i$ .

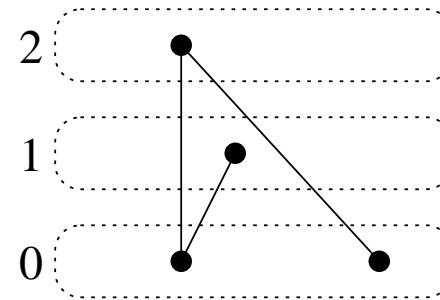
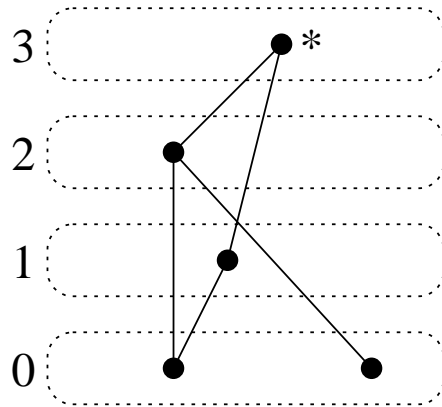
## Decomposition of $(2 + 2)$ -free posets



$$\ell(P) = 3, \quad i = 1 < \ell(P) \quad \Rightarrow x_6 = 1$$

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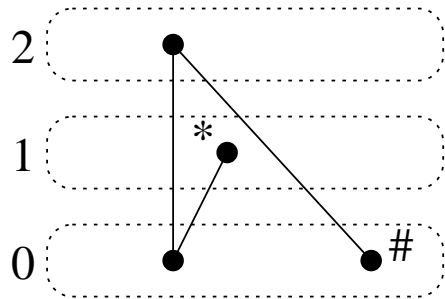
## Decomposition of $(2 + 2)$ -free posets



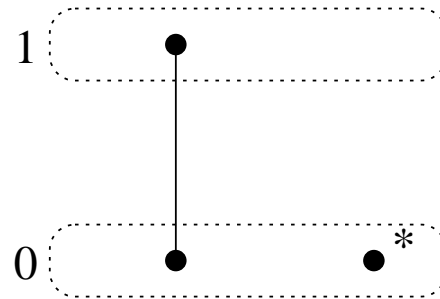
$$\ell(P) = 3, \quad i = 3 = \ell(P) \quad \Rightarrow \quad x_5 = 3$$

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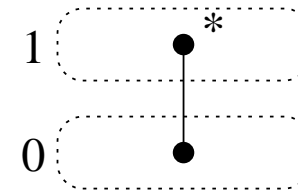
## Third bijection: from interval orders to ascent sequences



$$x_4 = 1$$



$$x_3 = 0$$



$$x_2 = 1$$



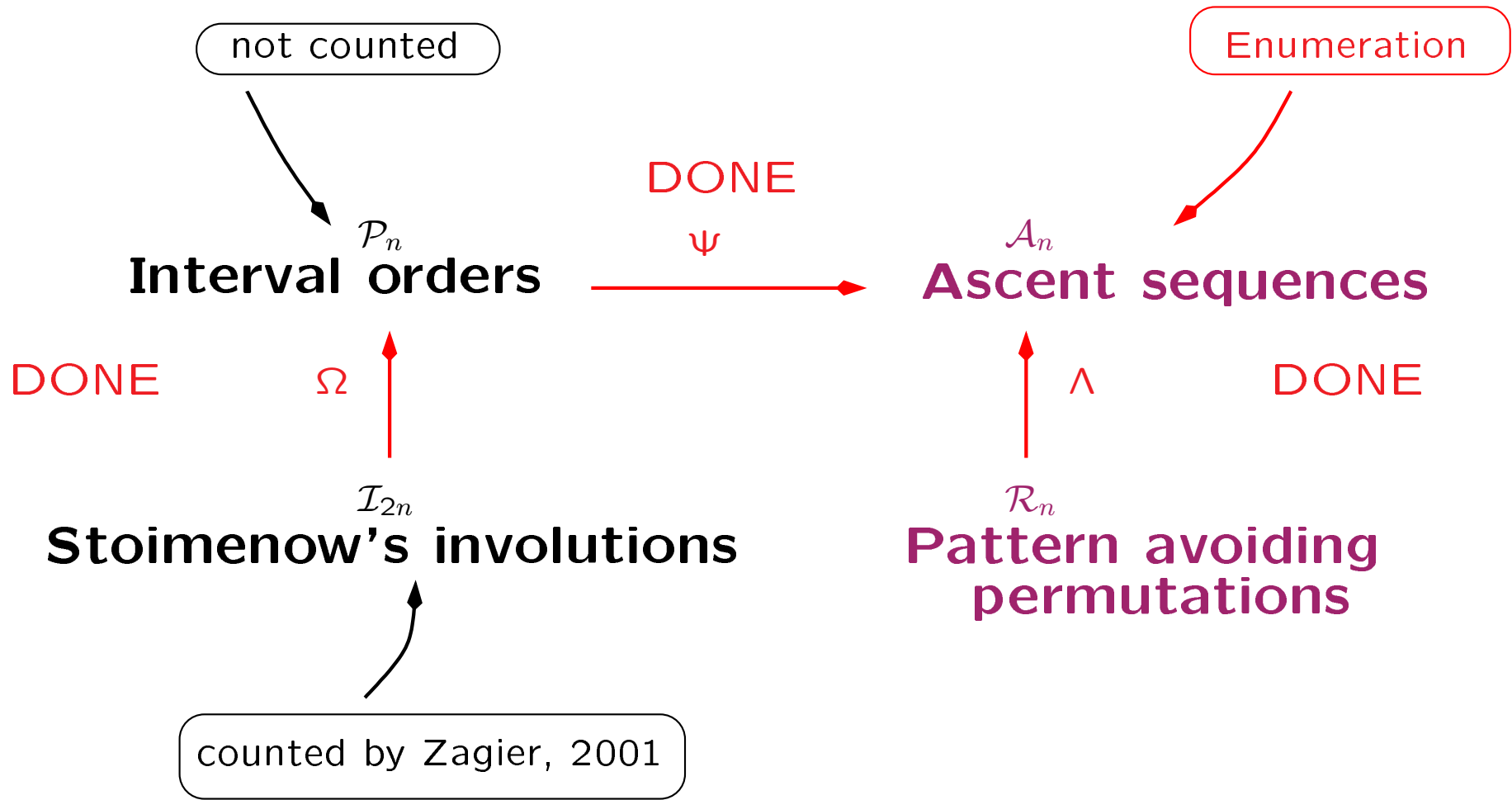
$$x_1 = 0$$

$$\Psi(P) = (0, 1, 0, 1, 3, 1, 1, 2)$$

**Theorem:** The map  $\Psi$  is a bijection between  $(2 + 2)$ -free posets on  $n$  elements and ascent sequences of length  $n$ .

**Remark:** other decompositions given by [El-Zahar 89], [Khamis 04], [Haxel, McDonald, Thomasson 87]

# Where we stand



## Some properties of our bijections

Given an ascent sequence  $x$ , let  $P$  and  $\pi$  be the poset and permutation corresponding to  $x$ , and let  $\sigma$  be the involution corresponding to  $P$ . Then

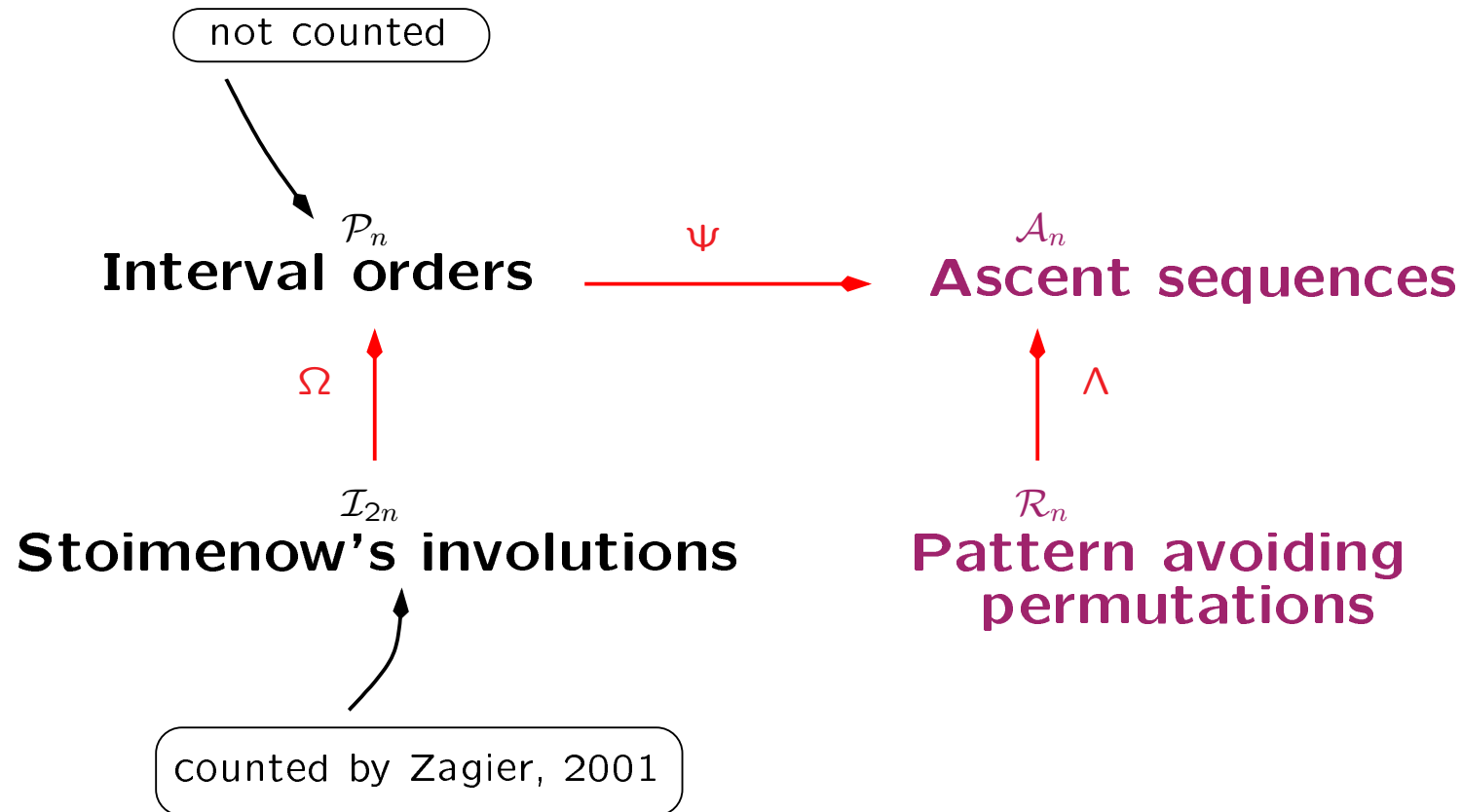
$$\begin{aligned} \text{lir}(\sigma) &= \min(P) = \text{zeros}(x) = \text{ldr}(\pi) \\ \ell^*(P) &= \text{last}(x) = b(\pi) \\ \text{or}(\sigma) - 1 &= \ell(P) = \text{asc}(x) = \text{asc}(\pi^{-1}) = s(\pi) - 2 \end{aligned}$$

with

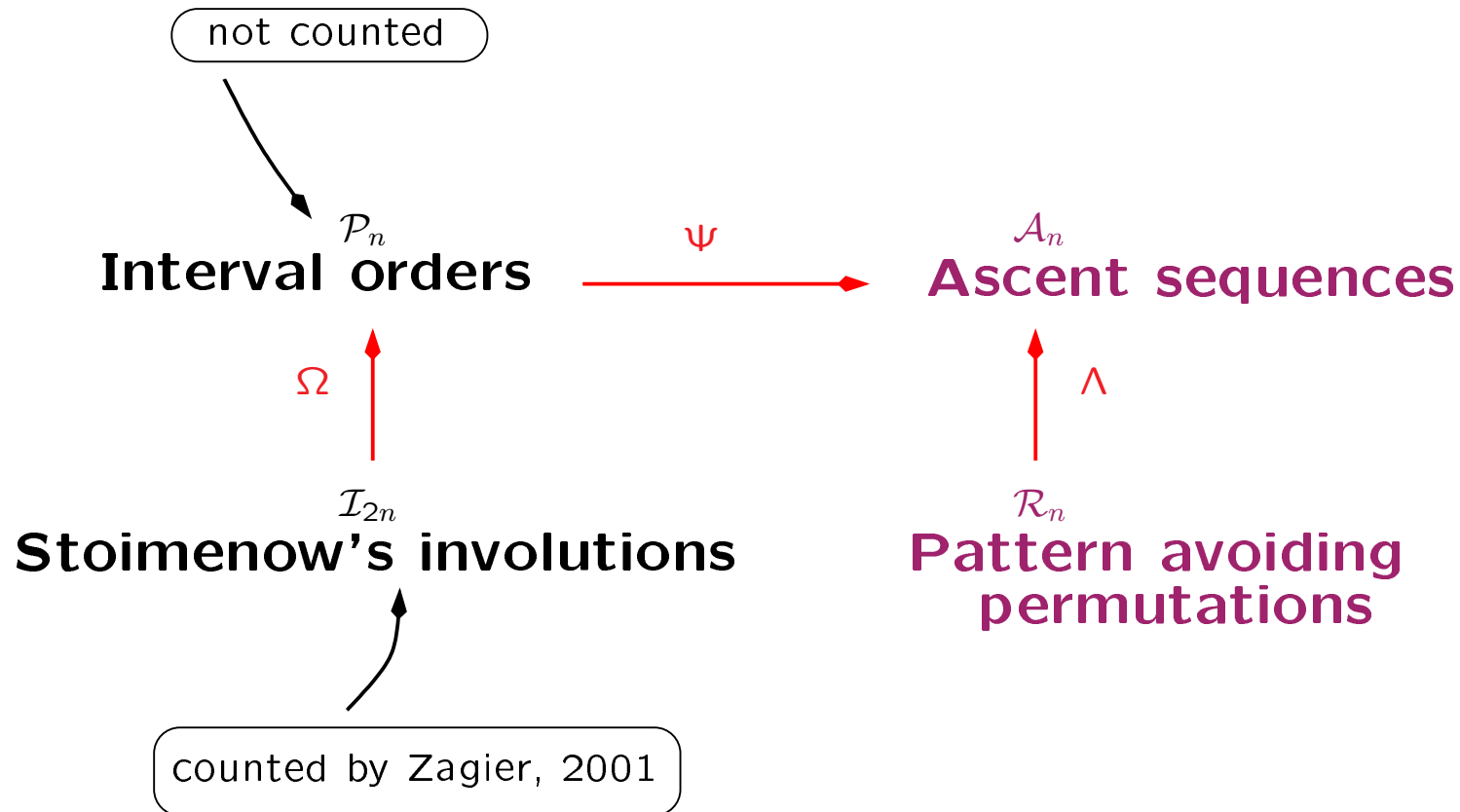
- $\text{lir}(\pi)$ : leftmost increasing run (the largest integer  $i$  such that  $\pi_1 < \pi_2 < \dots < \pi_i$ )
- $\text{ldr}(\pi)$ : leftmost decreasing run
- $\text{or}(\sigma)$ : number of opening runs of  $\sigma$
- $s(\pi)$ : number of active sites
- $b(\pi)$ : label of the active site located just before the largest entry of  $\pi$ .

## V. Enumeration

# Which is the most manageable class of objects?



# Which is the most manageable class of objects?



Ascent sequences

## Recursive description of ascent sequences

The only ascent sequence of length 1 is (0).

For  $n \geq 1$ , the sequence  $(x_1, \dots, x_n)$  is an ascent sequence iff

- $(x_1, \dots, x_{n-1})$  is an ascent sequence, and
- $x_n \in [0, 1 + \text{asc}(x_1, \dots, x_{n-1})]$ .

$\Rightarrow$  Obvious recursive description, keeping track of

- the number of ascents,
- the value of the last entry.

## A functional equation

Let  $F(t; u, v) \equiv F(u, v)$  be the generating function of ascent sequences, counted by length ( $t$ ), number of ascents ( $u$ ) and last entry ( $v$ ). This is a formal power series in  $t$  with coefficients in  $\mathbb{Q}[u, v]$ . The first few terms of  $F(t; u, v)$  are

$$F(t; u, v) = 1 + t + (1 + uv)t^2 + (1 + 2uv + u + u^2v^2)t^3 + O(t^4).$$

**Proposition:** The generating function  $F(t; u, v)$  satisfies

$$(v - 1 - tv(1 - u)) F(u, v) = (v - 1)(1 - tuv) - tF(u, 1) + tuv^2F(uv, 1)$$

**Remark:** one unknown series only on the right-hand side

## First expression of the solution

$$(v - 1 - tv(1 - u)) F(u, v) = (v - 1)(1 - tuv) - tF(u, 1) + tuv^2 F(uv, 1).$$

The **kernel** vanishes when  $v = V(u)$ , with  $V(u) = 1/(1 - t + tu)$ . Replacing  $v$  by  $V(u)$  in the equation results in:

$$F(u, 1) = \frac{(1 - u)(1 - t)}{(1 - t + tu)^2} + \frac{u}{(1 - t + tu)^2} F\left(\frac{u}{1 - t + tu}, 1\right).$$

Iterating this equation gives:

**Proposition:** The series  $F(t; u, 1)$  counting ascent sequences by their length and ascent number, seen as a series in  $u$ , has rational coefficients in  $t$ , and satisfies

$$F(t; u, 1) = \sum_{k \geq 1} \frac{(1 - u) u^{k-1} (1 - t)^k}{(u - (u - 1)(1 - t)^k) \prod_{i=1}^k (u - (u - 1)(1 - t)^i)}.$$

A formal series in  $u$  with rational coefficients in  $t$ !

## Massaging the solution

$$F(t; u, 1) = \sum_{k \geq 1} \frac{(1-u) u^{k-1} (1-t)^k}{(u - (u-1)(1-t)^k) \prod_{i=1}^k (u - (u-1)(1-t)^i)}.$$

**Lemma.** For  $m \geq 1$ , let

$$S(t; u) = \sum_{k \geq 1} \frac{(u-1)^m u^{k-1} (1-t)^{mk}}{\prod_{i=1}^k (u - (u-1)(1-t)^i)}.$$

Then  $S(t; u)$  is actually a **polynomial** in  $u$  and  $t$ :

$$S(t; u) = - \sum_{j=0}^{m-1} (u-1)^j u^{m-1-j} (1-t)^j \prod_{i=j+1}^{m-1} (1 - (1-t)^i).$$

## Second expression of the solution

- **Theorem.** Let  $n \geq 0$ , and consider the following polynomial in  $t$  and  $u$ :

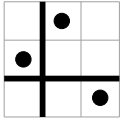
$$F_n(t; u) = \sum_{\ell=0}^n (u-1)^{n-\ell} u^\ell \sum_{m=\ell}^n (-1)^{n-m} \binom{n}{m} (1-t)^{m-\ell} \prod_{i=m-\ell+1}^m (1 - (1-t)^i).$$

Then  $F_n(t; u)$  is a **multiple of  $t^n$** .

The generating function of ascent sequences, counted by the length and the ascent number, is

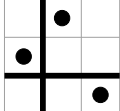
$$F(t; u, 1) = \sum_{n \geq 0} F_n(t; u).$$

- When  $u = 1$ ,  $F_n(t; 1) = \prod_{i=1}^n (1 - (1-t)^i)$ , so that the generating function of

ascent sequences,  $(2+2)$ -free posets, permutations avoiding , involutions with no neighbour nesting is:

$$F(t; 1, 1) = \sum_{n \geq 0} \prod_{i=1}^n (1 - (1-t)^i).$$

## What a beautiful series!

The generating function of ascent sequences,  $(2 + 2)$ -free posets, permutations avoiding , involutions with no neighbour nesting is:

$$P(t) = \sum_{n \geq 0} \prod_{i=1}^n (1 - (1 - t)^i).$$

Obtained by Zagier for involutions with no neighbour nesting, using a decomposition of involutions found in [Stoimenow 98].

- Asymptotic behaviour:

$$\frac{p_n}{n!} \sim \kappa \left( \frac{6}{\pi^2} \right)^n \sqrt{n}, \quad \text{where} \quad \kappa = \frac{12\sqrt{3}}{\pi^{5/2}} e^{\pi^2/12}.$$

- A remarkable formula:

$$P(1 - e^{-24x}) = e^x \sum_{n \geq 0} \frac{T_n}{n!} x^n,$$

where

$$\sum_{n \geq 0} \frac{T_n}{(2n + 1)!} x^{2n+1} = \frac{\sin 2x}{2 \cos 3x}.$$

[Zagier 01]