Baxter permutations and meanders

Éric Fusy (LIX, École Polytechnique)
Meanders on two lines

- A 2-line meander
Meanders on two lines

• A 2-line meander

encoded by a permutation
Meanders on two lines

- A 2-line meander encoded by a permutation

- Monotone 2-line meander: can be obtained from two monotone lines (one in $x$, the other in $y$)
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associated permutation
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Which permutations can be obtained this way?
Meanders on two lines

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- A 2-line meander encoded by a permutation

- Monotone 2-line meander:
  can be obtained from two monotone lines (one in $x$, the other in $y$)

Which permutations can be obtained this way?
Maps odd numbers to odd numbers, even numbers to even numbers
Permutations for monotone 2-line meanders
[Baxter’64, Boyce’68&’81]
Permutations for monotone 2-line meanders

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then has to go left
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Permutations for monotone 2-line meanders

[Baxter’64, Boyce’68&’81]

white points are either:

rising or descending
Permutations for monotone 2-line meanders

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white points are either:

- rising
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white points are either:

rising

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or
Permutations for monotone 2-line meanders

[Baxter’64, Boyce’68&’81]

White points are either:
- Rising
- Descending
Permutations mapping even to even, odd to odd, and satisfying condition shown on the right are called **complete Baxter permutations**
Permutations for monotone 2-line meanders
[Baxter’64, Boyce’68&’81]

Permutations mapping even to even, odd to odd, and satisfying condition shown on the right are called complete Baxter permutations.

**Theorem ([Boyce’81] reformulated bijectively):**

Monotone 2-line meanders with $2n - 1$ crossings are in bijection with complete Baxter permutations on $2n - 1$ elements.
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:

- 
- or
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:

1) Draw the blue curve

or
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:

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or
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:

1) Draw the blue curve
2) Draw the red curve
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:

1) Draw the blue curve
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Inverse construction

From a complete Baxter permutation to a monotone 2-line meander

white points are either:

1) Draw the blue curve
2) Draw the red curve

The two curves meet only at the permutation points (because of the empty area-property at white points)
Complete and reduced Baxter permutations
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- complete one can be recovered from reduced one
Complete and reduced Baxter permutations

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Case of a descent

- Complete one can be recovered from reduced one
Complete and reduced Baxter permutations

- complete one can be recovered from reduced one

Case of a rise
Complete and reduced Baxter permutations

• complete one can be recovered from reduced one

case of a rise
Complete and reduced Baxter permutations

• complete one can be recovered from reduced one

complete

reduced

case of a rise
Complete and reduced Baxter permutations

- Complete one can be recovered from reduced one
- Reduced one is characterized by forbidden patterns $2 - 41 - 3$ and $3 - 14 - 2$
Complete and reduced Baxter permutations

- complete one can be recovered from reduced one
- reduced one is characterized by forbidden patterns $2 - 41 - 3$ and $3 - 14 - 2$
- permutation on white points (called anti-Baxter) is characterized by forbidden patterns $2 - 14 - 3$ and $3 - 41 - 2$
Counting results

- Baxter permutations
  - Number of reduced Baxter permutations with $n$ elements
    \[ b_n = \sum_{r=0}^{n-1} \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2} \]

[Chung et al’78] [Mallows’79]
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    \]
    [Chung et al’78] [Mallows’79]
  - Bijective proof: [Viennot’81], [Dulucq-Guibert’98]

5 7 6 2 1 4 3  \Leftrightarrow  

Diagram: 

5 7 6 2 1 4 3
Counting results

- Baxter permutations
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- Subfamilies
  - alternating [Cori-Dulucq-Viennot’86], [Dulucq-Guibert’98]
    \[
    \text{Cat}_k \text{Cat}_k \text{ if } n = 2k \quad \text{Cat}_k \text{Cat}_{k+1} \text{ if } n = 2k + 1
    \]
  - doubly alternating [Guibert-Linusson’00]
    \[
    \text{Cat}_k \text{ where } k = \lfloor n/2 \rfloor
    \]
Counting results

- Baxter permutations
  - Number of reduced Baxter permutations with $n$ elements
    \[ b_n = \sum_{r=0}^{n-1} \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2} \]
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    \[ \text{Cat}_k \text{Cat}_k \text{ if } n = 2k \]
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  - doubly alternating [Guibert-Linusson’00]
    \[ \text{Cat}_k \text{ where } k = \lfloor n/2 \rfloor \]

- anti-Baxter permutations
  \[ a_n = \sum_{i=0}^{[(n+1)/2]} (-1)^i \binom{n+1-i}{i} b_{n+1-i} \]
  [Asinowski et al’10]
Local conditions for monotone 2-line meanders
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Conditions
- two bipartite matchings missing a black point (one above, one below the blue line)
- white points are either rising or descending
Local conditions for monotone 2-line meanders

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Proof of $\Leftarrow$
Assume there is a red loop (say, clockwise):
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Proof of $\Leftarrow$
Assume there is a red loop (say, clockwise):
then the leftmost and the rightmost point on the loop are of different colors
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Assume there is a red loop (say, clockwise):

then the leftmost and the rightmost point on the loop are of different colors

$\Rightarrow$ we have a 2-line meander
Local conditions for monotone 2-line meanders

Conditions
- two bipartite matchings missing a black node (one above, one below the blue line)
- white nodes are either or

Proof of $\Leftarrow$: construct permutation step by step
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Conditions
- two bipartite matchings missing a black node (one above, one below the blue line)
- white nodes are either \[ \begin{array}{c}
\text{or}
\end{array} \]

Proof of \( \Leftarrow \): construct permutation step by step

Important observation:

\[ \begin{array}{c}
i
\end{array} \]

already labelled

By similar argument as to show there is no red loop
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important observation:
\[
\begin{array}{c}
\text{i} & \text{i+1} \\
\text{already labelled} & \text{already labelled}
\end{array}
\]
Local conditions for monotone 2-line meanders

\[ \iff \]

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\end{array}
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\end{array}
\end{array}
\end{array} \)

Proof of \( \iff \): construct permutation step by step

<table>
<thead>
<tr>
<th>3</th>
<th>2</th>
<th>1</th>
<th>4</th>
<th>6</th>
<th>5</th>
</tr>
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- $i+1$ already labelled
- $i+1$ already labelled
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<th>i</th>
<th>$i+1$</th>
</tr>
</thead>
<tbody>
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<td>already labelled</td>
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Proof of $\Leftarrow$: construct permutation step by step

Similarly:
Encoding a monotone 2-line meander
Encoding a monotone 2-line meander
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Encoding a monotone 2-line meander
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Encoding a monotone 2-line meander

Each path has length $n - 1$
Encoding a monotone 2-line meander
close to encoding in [Viennot’81, Dulucq-Guibert’98]
Encoding a monotone 2-line meander

close to encoding in [Viennot’81, Dulucq-Guibert’98]
exactly coincides with encoding in [Felsner-F-Noy-Orden’11]
(uses “equatorial line” in separating decompositions of quadrangulations)
Enumeration using the LGV lemma

Each path has length $n - 1$
Enumeration using the LGV lemma

Let \( a_{i,j} = \# \) (upright lattice paths from \( A_i \) to \( B_j \)) = \( \left( x(B_j) - x(A_i) \right)^{n-1} \)

By the Lindstroem-Gessel-Viennot Lemma (used in [Viennot’81])

the number \( b_{n,r} \) of such nonintersecting triples of paths is

\[
b_{n,r} = \text{Det}(a_{i,j}) = \begin{vmatrix}
(n-1) & (n-1) & (n-1) \\
\begin{array}{c}
\binom{n-1}{r} \\
\binom{n-1}{r+1} \\
\binom{n-1}{r+2}
\end{array} & \begin{array}{c}
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\end{array} \\
(n-1) & (n-1) & (n-1) \\
\begin{array}{c}
\binom{n-1}{r-2} \\
\binom{n-1}{r-1} \\
\binom{n-1}{r}
\end{array} & \begin{array}{c}
\binom{n-1}{r-2} \\
\binom{n-1}{r-1} \\
\binom{n-1}{r}
\end{array} & \begin{array}{c}
\binom{n-1}{r-2} \\
\binom{n-1}{r-1} \\
\binom{n-1}{r}
\end{array}
\end{vmatrix} = \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2} \]

each path has length \( n - 1 \)
Enumeration using the LGV lemma

Let $a_{i,j} = \#$ (upright lattice paths from $A_i$ to $B_j) = \left( x(B_j) - x(A_i) \right)^{n-1}$

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$$b_{n,r} = \text{Det}(a_{i,j}) = \left| \begin{array}{ccc} \binom{n-1}{r} & \binom{n-1}{r+1} & \binom{n-1}{r+2} \\ \binom{n-1}{r-1} & \binom{n-1}{r} & \binom{n-1}{r+1} \\ \binom{n-1}{r-2} & \binom{n-1}{r-1} & \binom{n-1}{r} \end{array} \right| = \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}$$

this is also the number of Baxter permutations of size $n$ with $r$ rises
Alternating (reduced) Baxter permutations

[Cori-Dulucq-Viennot’86], [Dulucq-Guibert’98]

1) Case $n$ even, $n = 2k$

\[ \pi = 8 \quad 9 \quad 7 \quad 10 \quad 1 \quad 4 \quad 3 \quad 5 \quad 2 \quad 6 \]
Alternating (reduced) Baxter permutations

[Cori-Dulucq-Viennot’86], [Dulucq-Guibert’98]

1) Case $n$ even, $n = 2k$

$\pi = 8 \ 9 \ 7 \ 10 \ 1 \ 4 \ 3 \ 5 \ 2 \ 6$

alternation $\Rightarrow$ middle word is 010101…0
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middle path is

\[ \text{Diagram with path and steps} \]
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1) Case $n$ even, $n = 2k$

$\pi = 8 \ 9 \ 7 \ 10 \ 1 \ 4 \ 3 \ 5 \ 2 \ 6$

There are $\text{Cat}_k \text{Cat}_k$ alternating (reduced) Baxter permutations of size $n$.
Alternating (reduced) Baxter permutations

[Cori-Dulucq-Viennot’86], [Dulucq-Guibert’98]

2) Case $n$ odd, $n = 2k + 1$

There are $\text{Cat}_k \text{Cat}_{k+1}$ alternating (reduced) Baxter permutations of size $n$
Doubly alternating (reduced) Baxter permutations
[Guibert-Linusson’00]

1) Case $n$ even, $n = 2k$

$\pi = 5 \ 7 \ 6 \ 8 \ 3 \ 4 \ 1 \ 2$

Alternation of $\pi$: middle word is $010101\ldots 0$

Middle path is

Alternation of $\pi^{-1}$: black points are $\blacklozenge$ or $\blackdiamondsuit$
Doubly alternating (reduced) Baxter permutations
[Guibert-Linusson’00]

1) Case $n$ even, $n = 2k$

$\pi = 5 \ 7 \ 6 \ 8 \ 3 \ 4 \ 1 \ 2$

$\Rightarrow$

mirror of each other

There are $\text{Cat}_k$ doubly alternating (reduced) Baxter permutations of size $n$
Doubly alternating (reduced) Baxter permutations

[Guibert-Linusson’00]

2) Case \( n \) odd, \( n = 2k + 1 \)

\( \pi = 1 \ 3 \ 2 \ 5 \ 4 \ 9 \ 7 \ 8 \ 6 \)

\( \pi^{-1} \) has black points:

\( \Rightarrow \) mirror of each other

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\( \Rightarrow \) mirror of each other

There are \( \text{Cat}_k \) doubly alternating (reduced) Baxter permutations of size \( n \).