

Multivariable Tangent and Secant q-derivative polynomials

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(Based on the paper by Foata-Han with the same title)

Journées Viennot, Bordeaux

June 28, 2012

The derivative polynomials

$$D \tan u = 1 + \tan^2 u$$

$$\begin{aligned} D^2 \tan u &= 2 \tan u \times D \tan u \\ &= 2 \tan u \times (1 + \tan^2 u) \\ &= 2 \tan u + 2 \tan^3 u \end{aligned}$$

$$D^3 \tan u = 2 + 8 \tan^2 u + 6 \tan^4 u$$

$$D^4 \tan u = 16 \tan u + 40 \tan^3 u + 24 \tan^5 u$$

...

Knuth-Buckholtz (1967) :

$D^n \tan u$ is a polynomial in $\tan u$

The coefficients of the polynomial can be calculated by using a recurrence relation

The derivative polynomials

$$D^n \tan u = \sum_{m \geq 0} a(n, m) \tan^m u$$

$$a(0, m) = \delta_{1, m}$$

$$a(n + 1, m) = (m - 1)a(n, m - 1) + (m + 1)a(n, m + 1)$$

$m =$	0	1	2	3	4	5	6	7
$n = 0$		1						
1	1		1					
2		2		2				
3	2		8		6			
4		16		40		24		
5	16		136		240		120	
6		272		1232		1680		720

The derivative polynomials

Again:

$$D^n \sec u := \sum_{m \geq 0} b(n, m) \tan^m u \sec u.$$

where $(b(n, m))$ satisfies the recurrence relation:

$$b_{0,m} = \delta_{0,m},$$

$$b(n+1, m) = mb(n, m-1) + (m+1)b(n, m+1).$$

The derivative polynomials

The coefficients $a(\mathbf{n}, \mathbf{m})$ and $b(n, m)$

$m =$	0	1	2	3	4	5	6		
$n = 0$	1	1						1.2^0	1
1	1	1	1					1.2^1	1
2	1	2	2	2				1.2^2	3
3	2	5	8	6	6			2.2^3	11
4	5	16	28	40	24	24		5.2^4	57
5	16	61	136	180	240	120	120	16.2^5	361

first column : tangent and secant numbers

diagonal : factorial number

right of the table : row sums

row sums of $a(\mathbf{n}, \mathbf{m})$: tangent and secant numbers, $\times 2^n$

row sums of $b(n, m)$: Springer numbers

The derivative polynomials: Generating function

Derivative polynomials:

$$A_n(x) := \sum_{m \geq 0} a(n, m) x^m$$

$$B_n(x) := \sum_{m \geq 0} b(n, m) x^m$$

The exponential generating functions for the derivative polynomials (Hoffman, 1995)

$$\sum_{n \geq 0} A_n(x) \frac{u^n}{n!} = \frac{x + \tan u}{1 - x \tan u}$$

$$\sum_{n \geq 0} B_n(x) \frac{u^n}{n!} = \frac{1}{\cos u - x \sin u}$$

The derivative poly. : Combinatorial interpretation

Hoffman (1999) has found a combinatorial interpretation of the derivative polynomials, in terms of *snakes*, a notion introduced by Arnold (1992).

snake of length n : each word $w = x_1x_2 \cdots x_n$, whose letters are integers, positive or negative, with the further property that $x_1 > x_2, x_2 < x_3, x_3 > x_4, \dots$ in an alternating way and $|x_1| |x_2| \cdots |x_n|$ is a permutation of $1\ 2 \cdots n$.

Josuat-Vergès, Novelli and Thibon (2012) have recently developed an algebraic combinatorics of snakes from the Hopf algebra point of view.

$A_n(x)$ is the generating polynomial for the set of all *snakes* of length n by the number of sign changes; they also prove an analogous result for $B_n(x)$.

This interpretation has been used again by Josuat-Vergès (2011).

q-analog ?

t-ascending factorial

$$(t; q)_n := \begin{cases} 1, & \text{if } n = 0 \\ (1 - t)(1 - tq) \cdots (1 - tq^{n-1}), & \text{if } n \geq 1 \end{cases}$$

$$(t; q)_\infty := \lim_n (t; q)_n = \prod_{n \geq 0} (1 - tq^n)$$

q-series

$$f(u) = \sum_{n \geq 0} f(n; q) \frac{u^n}{(q; q)_n}$$

q-derivative operator

$$D_q f(u) := \frac{f(u) - f(qu)}{u}$$

[Not the traditional $\frac{f(u) - f(qu)}{u(1-q)}$]

q-exponential series

$$e_q(u) = \sum_{n \geq 0} \frac{u^n}{(q; q)_n} = \frac{1}{(u; q)_\infty}$$

(a result that goes back to Euler, 1748)

q-trigonometric series (Jackson, 1904):

$$\sin_q(u) := \frac{e_q(iu) - e_q(-iu)}{2i} = \sum_{n \geq 0} (-1)^n \frac{u^{2n+1}}{(q; q)_{2n+1}} ;$$

$$\cos_q(u) := \frac{e_q(iu) + e_q(-iu)}{2} = \sum_{n \geq 0} (-1)^n \frac{u^{2n}}{(q; q)_{2n}} ;$$

q-exponential series

$$E_q(u) = \sum_{n \geq 0} q^{n(n-1)/2} \frac{u^n}{(q; q)_n} = (-u; q)_\infty$$

(a result that goes back to Euler, 1748)

q-trigonometric series (Jackson, 1904):

$$\text{Sin}_q(u) := \frac{E_q(iu) - E_q(-iu)}{2i} = \sum_{n \geq 0} (-1)^n q^{n(2n+1)} \frac{u^{2n+1}}{(q; q)_{2n+1}}$$

$$\text{Cos}_q(u) := \frac{E_q(iu) + E_q(-iu)}{2} = \sum_{n \geq 0} (-1)^n q^{n(2n-1)} \frac{u^{2n}}{(q; q)_{2n}}$$

only one q-tangent

$$\tan_q(u) := \frac{\sin_q(u)}{\cos_q(u)} = \frac{\text{Sin}_q(u)}{\text{Cos}_q(u)} = \frac{\sum_{n \geq 0} (-1)^n u^{2n+1} / (q; q)_{2n+1}}{\sum_{n \geq 0} (-1)^n u^{2n} / (q; q)_{2n}}$$

two q-secants

$$\sec_q(u) := \frac{1}{\cos_q(u)} = \frac{1}{\sum_{n \geq 0} (-1)^n u^{2n} / (q; q)_{2n}}$$

$$\text{Sec}_q(u) := \frac{1}{\text{Cos}_q(u)} = \frac{1}{\sum_{n \geq 0} (-1)^n q^{n(2n-1)} u^{2n} / (q; q)_{2n}}$$

Lemma.

$$D_q \tan_q(u) = 1 + \tan_q(u) \tan_q(qu)$$

$$D_q \sec_q(u) = \sec_q(qu) \tan_q(u)$$

$$D_q \text{Sec}_q(u) = \text{Sec}_q(u) \tan_q(qu)$$

What are

$$D_q^n \tan_q(u) = ?$$

$$D_q^n \sec_q(u) = ?$$

$$D_q^n \text{Sec}_q(u) = ?$$

q-Derivative Polynomials

q -Leibniz:

$$\begin{aligned} & D_q \prod_{1 \leq i \leq n} f_i(u) \\ &= \sum_{1 \leq i \leq n} f_1(u) \cdots f_{i-1}(u) \cdot D_q f_i(u) \cdot f_{i+1}(qu) \cdots f_n(qu) \end{aligned}$$

Remark: The LHS is symmetric in the f_i 's, so does the RHS. However we cannot see the symmetry on the present writing form of the RHS, that depends on the order of the f_i 's.

q-Derivative Polynomials

Proof.

Notation $f' := D_q f$.

$$\begin{aligned} & (f(u)g(u)h(u))' \\ &= \frac{1}{u} (f(u)g(u)h(u) - f(qu)g(qu)h(qu)) \\ &= \frac{1}{u} (f(u)g(u)h(u) - f(u)g(u)h(qu) \\ &\quad + f(u)g(u)h(qu) - f(u)g(qu)h(qu) \\ &\quad + f(u)g(qu)h(qu) - f(qu)g(qu)h(qu)) \\ &= f(u)g(u)h'(u) + f(u)g'(u)h(qu) + f'(u)g(qu)h(qu) \end{aligned}$$

q-Derivative Polynomials

We have two ways to calculate $\tan''(u) = (\tan'(u))' =$

$$=(1 + \tan(u) \tan(qu))'$$

$$= \tan(u)(\tan(qu))' + \tan(u)' \tan(q^2u)$$

$$= \tan(u)q(1 + \tan(qu) \tan(q^2u)) + (1 + \tan(u) \tan(qu)) \tan(q^2u)$$

$$= q \tan(u) + \tan(q^2u) + (1 + q) \tan(u) \tan(qu) \tan(q^2u)$$

and also

$$=(1 + \tan(qu) \tan(u))'$$

$$= \tan(qu)(\tan(u))' + \tan(qu)' \tan(qu)$$

$$= \tan(qu)(1 + \tan(u) \tan(qu)) + q(1 + \tan(qu) \tan(q^2u)) \tan(qu)$$

$$= (1 + q) \tan(qu) + \tan(u) \tan(qu)^2 + q \tan(qu)^2 \tan(q^2u)$$

Notation $\tan := \tan_q$

q-Derivative Polynomials

$$\begin{aligned} & \tan'''(u) \\ &= (q + q^2) \\ &+ (q + q^2 + q^3) \tan(u) \tan(qu) \\ &+ (1 + q + q^2) \tan(q^2u) \tan(q^3u) \\ &+ q(1 + q) \tan(u) \tan(q^3u) \\ &+ (1 + q)(1 + q + q^2) \tan(u) \tan(qu) \tan(q^2u) \tan(q^3u) \end{aligned}$$

and

q-Derivative Polynomials

$$\begin{aligned}\tan'''(u) = & (q + q^2) \\ & + (2q + 2q^2) \tan(qu) \tan(q^2u) \\ & + (1 + q) \tan(qu)^2 \\ & + (q^2 + q^3) \tan(q^2u)^2 \\ & + (q + q^2) \tan(qu)^2 \tan(q^2u)^2 \\ & + q \tan(qu)^3 \tan(q^2u) \\ & + \tan(u) \tan(qu)^3 \\ & + q^3 \tan(q^2u)^3 \tan(q^3u) \\ & + q^2 \tan(qu) \tan(q^2u)^3\end{aligned}$$

Main Theorems

Theorem. (Prototype)

Let

$$A_{n,k,a,b}(q) = \dots$$

Then

$$D_q^n \tan_q(u) = \sum_{k,a,b} A_{n,k,a,b}(q) (\tan_q(q^{k+1}u))^b (\tan_q(q^k u))^a$$

Main Theorems

Theorem. (Prototype)

For each composition $\mathbf{c} = (c_0, c_1, c_2, \dots, c_m)$ of n let

$$A_{n,\mathbf{c}}(q) = \dots$$

Then,

$$D_q^n \tan_q(u) = \sum_{\mathbf{c}} A_{n,\mathbf{c}}(q) \tan_q(q^{\mathbf{c}}u)$$

where

$$\tan_q(q^{\mathbf{c}}u) := \tan_q(q^{c_0}u) \tan_q(q^{c_0+c_1}u) \cdots \tan_q(q^{c_0+c_1+\cdots+c_{m-1}}u).$$

Main Theorems

t-composition

$$\mathbf{c} = (c_0, c_1, \dots, c_{m-1}, c_m)$$

$$c_0 \geq 0$$

$$c_1, c_2, \dots, c_{m-1} \geq 1$$

$$c_m \geq 0$$

(0) $c_0 + c_1 + \dots + c_m = n$;

(1) either $m = 0$, so that $\mathbf{c} = (n)$ and n is an *odd* integer,
or $m \geq 1$ and both c_0, c_m are *even*;

(2) if $m \geq 2$, then all the parts c_1, c_2, \dots, c_{m-1} are *odd*.

Notation $\mu \mathbf{c} := m$

Main Theorems

The set of all t -compositions of n is denoted by Θ_n .

$$\Theta_0: (0, 0);$$

$$\Theta_1: (1), (0, 1, 0);$$

$$\Theta_2: (0, 2), (2, 0), (0, 1, 1, 0);$$

$$\Theta_3: (3), (0, 1, 2), (2, 1, 0), (0, 3, 0), (0, 1, 1, 1, 0);$$

$$\Theta_4: (0, 4), (2, 2), (4, 0), (0, 3, 1, 0), (0, 1, 3, 0), (0, 1, 1, 2), \\ (2, 1, 1, 0), (0, 1, 1, 1, 1, 0).$$

(**Card** Θ_n) = (1, 2, 3, 5, 8, ...) the Fibonacci sequence

Main Theorems

A word $w = y_1 y_2 \cdots y_m$, whose letters are positive integers, is said to be *falling alternating*, or simply *alternating*, if

$$y_1 > y_2, y_2 < y_3, y_3 > y_4, \dots$$

It is said to be *rising alternating* if

$$y_1 < y_2, y_2 > y_3, y_3 < y_4, \dots$$

The length of each word w is denoted by λw

The *empty word* is denoted by ϵ

Main Theorems

tangent and *secant numbers* T_{2n+1} and E_{2n} occur as coefficients in the Taylor expansions of **tan** u and **sec** u :

$$\mathbf{tan} \, u = \sum_{n \geq 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1} = \frac{u}{1!} 1 + \frac{u^3}{3!} 2 + \frac{u^5}{5!} 16 + \frac{u^7}{7!} 272 + \dots$$

$$\mathbf{sec} \, u = \frac{1}{\cos u} = \sum_{n \geq 0} \frac{u^{2n}}{(2n)!} E_{2n} = 1 + \frac{u^2}{2!} 1 + \frac{u^4}{4!} 5 + \frac{u^6}{6!} 61 + \dots$$

Désiré André (1879):

The number of falling (resp. rising) alternating permutations $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $1\ 2 \dots n$ is equal to T_n when n is odd, and to E_n when n is even.

Main Theorems: t -permutations

Definition. A t -permutation of order n is defined to be a nonempty sequence $w = (w_0, w_1, \dots, w_m)$ of words having the properties:

- (i) the juxtaposition product $w_0 w_1 \cdots w_m$ is a permutation of $12 \cdots n$
- (ii) $(\lambda w_0, \lambda w_1, \dots, \lambda w_m)$ is a t -composition
- (iii) w_0 is rising alternating, and w_1, w_2, \dots, w_m are (falling) alternating

Notation $\mu w := m$

Main Theorems: t -permutations

The set of all t -permutations of order n will be denoted by \mathcal{T}_n .

$$\mathcal{T}_0: (\epsilon, \epsilon) ;$$

$$\mathcal{T}_1: (1); (\epsilon, 1, \epsilon);$$

$$\mathcal{T}_2: (\epsilon, 21), (12, \epsilon); (\epsilon, 2, 1, \epsilon), (\epsilon, 1, 2, \epsilon);$$

$$\begin{aligned} \mathcal{T}_3: & (132), (231); \\ & (\epsilon, 3, 21), (\epsilon, 312, \epsilon), (\epsilon, 213, \epsilon), (12, 3, \epsilon), \\ & (\epsilon, 1, 32), (23, 1, \epsilon), (13, 2, \epsilon), (\epsilon, 2, 31); \\ & (\epsilon, 1, 2, 3, \epsilon), (\epsilon, 1, 3, 2, \epsilon), (\epsilon, 2, 1, 3, \epsilon), (\epsilon, 2, 3, 1, \epsilon), \\ & (\epsilon, 3, 1, 2, \epsilon), (\epsilon, 3, 2, 1, \epsilon). \end{aligned}$$

Main Theorems: statistics

statistics on permutations:

The *descent set* of a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$, denoted by $\text{DES } \sigma$, is defined to be the set of all i such that $1 \leq i \leq n - 1$ and $\sigma(i) > \sigma(i + 1)$;

$$\text{ides } \sigma = \# \text{DES } \sigma^{-1};$$

$$\text{imaj } \sigma = \sum_i \sigma(i) \quad (\sigma(i) \in \text{DES } \sigma^{-1});$$

Let $\text{inv } \sigma$ be the *number of inversions* of σ , as being the number of pairs (i, j) such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$.

Main Theorems

statistics on t -permutations:

t -permutation $w = (w_0, w_1, \dots, w_m)$

$$\text{IDES } w := \text{IDES}(w_0 w_1 \cdots w_m)$$

$$\text{ides } w := \text{ides}(w_0 w_1 \cdots w_m)$$

$$\text{imaj } w := \text{imaj}(w_0 w_1 \cdots w_m)$$

$$\text{inv } w := \text{inv}(w_0 w_1 \cdots w_m)$$

(the juxtaposition product is a permutation)

$$\text{min } w := a \text{ if } 1 \text{ is a letter in } w_a$$

Main Theorems

Example. With the t -permutation

$$w = (4\ 5, 11\ 1\ \mathbf{3}, \mathbf{10}\ 7\ \mathbf{9}, \mathbf{6}, \mathbf{8}\ \mathbf{2}),$$

The inverse descents being reproduced in boldface, we have:

$$\text{ides } w = 6$$

$$\text{imaj } w = \mathbf{3} + \mathbf{10} + \mathbf{9} + \mathbf{6} + \mathbf{8} + \mathbf{2} = \mathbf{38}$$

$$\text{inv } w = 27$$

$$\text{min } w = 1 \quad \text{since } w_1 = (11\ 1\ \mathbf{3})$$

Main Theorems

For each triple (k, a, b) let $\mathcal{T}_{n,k,a,b}$ denote the set of all t -permutations w from \mathcal{T}_n such that $\text{ides } w = k$, $\min w = a$ and $a + b = \mu w$.

Theorem (Multivariable q -analog). *Let*

$$A_{n,k,a,b}(q) = \sum_{w \in \mathcal{T}_{n,k,a,b}} q^{\text{imaj } w}$$

Then

$$D_q^n \tan_q(u) = \sum_{k,a,b} A_{n,k,a,b}(q) (\tan_q(q^{k+1}u))^b (\tan_q(q^k u))^a,$$

where $0 \leq k \leq n - 1$, $0 \leq a + b \leq n + 1$.

Example: $n=3$

\mathcal{T}_3	k	imaj	a	weight Multi
(132)	1	2	0	$q^2 \cdot 1 \cdot 1$
(231)	1	1	0	$q \cdot 1 \cdot 1$
$(\epsilon, 3, 21)$	2	3	2	$q^3 \cdot 1 \cdot \tan(q^2 u)^2$
$(\epsilon, 312, \epsilon)$	1	2	1	$q^2 \cdot \tan(q^2 u) \cdot \tan(qu)$
$(\epsilon, 213, \epsilon)$	1	1	1	$q \cdot \tan(q^2 u) \cdot \tan(qu)$
$(12, 3, \epsilon)$	0	0	0	$1 \cdot \tan(qu)^2 \cdot 1$
$(\epsilon, 1, 32)$	1	2	1	$q^2 \cdot \tan(q^2 u) \cdot \tan(qu)$
$(23, 1, \epsilon)$	1	1	1	$q \cdot \tan(q^2 u) \cdot \tan(qu)$
$(13, 2, \epsilon)$	1	2	0	$q^2 \cdot \tan(q^2 u)^2 \cdot 1$
$(\epsilon, 2, 31)$	1	1	2	$q \cdot 1 \cdot \tan(qu)^2$
$(\epsilon, 1, 2, 3, \epsilon)$	0	0	1	$1 \cdot \tan(qu)^3 \cdot \tan(u)$
$(\epsilon, 1, 3, 2, \epsilon)$	1	2	1	$q^2 \cdot \tan(q^2 u)^3 \cdot \tan(qu)$
$(\epsilon, 2, 1, 3, \epsilon)$	1	1	2	$q \cdot \tan(q^2 u)^2 \cdot \tan(qu)^2$
$(\epsilon, 2, 3, 1, \epsilon)$	1	1	3	$q \cdot \tan(q^2 u) \cdot \tan(qu)^3$
$(\epsilon, 3, 1, 2, \epsilon)$	1	2	2	$q^2 \cdot \tan(q^2 u)^2 \cdot \tan(qu)^2$
$(\epsilon, 3, 2, 1, \epsilon)$	2	3	3	$q^3 \cdot \tan(q^3 u) \cdot \tan(q^2 u)^3$

Main Theorems

Theorem (Composition q -analog). For each t -composition \mathbf{c} of n let

$$A_{n,\mathbf{c}}(q) = \sum_{\substack{w \in \mathcal{T}_n, \\ (\lambda w_0, \lambda w_1, \dots, \lambda w_m) = \mathbf{c}}} q^{\text{inv } w}.$$

Then,

$$D_q^n \text{tan}_q(u) = \sum_{\mathbf{c} \in \Theta_n} A_{n,\mathbf{c}}(q) \text{tan}_q(q^{\mathbf{c}}u),$$

where

$$\text{tan}_q(q^{\mathbf{c}}u) := \text{tan}_q(q^{c_0}u) \text{tan}_q(q^{c_0+c_1}u) \cdots \text{tan}_q(q^{c_0+c_1+\cdots+c_{m-1}}u).$$

Example: $n=3$

\mathcal{T}_3	inv	weight	Compo
(132)	1	q	
(231)	2	q^2	
$(\epsilon, 3, 21)$	3	q^3	$\tan(u) \tan(qu)$
$(\epsilon, 312, \epsilon)$	2	q^2	$\tan(u) \tan(q^3u)$
$(\epsilon, 213, \epsilon)$	1	q	$\tan(u) \tan(q^3u)$
$(12, 3, \epsilon)$	0	1	$\tan(q^2u) \tan(q^3u)$
$(\epsilon, 1, 32)$	1	q	$\tan(u) \tan(qu)$
$(23, 1, \epsilon)$	2	q^2	$\tan(q^2u) \tan(q^3u)$
$(13, 2, \epsilon)$	1	q	$\tan(q^2u) \tan(q^3u)$
$(\epsilon, 2, 31)$	2	q^2	$\tan(u) \tan(qu)$
$(\epsilon, 1, 2, 3, \epsilon)$	0	1	$\tan(u) \tan(qu) \tan(q^2u) \tan(q^3u)$
$(\epsilon, 1, 3, 2, \epsilon)$	1	q	$\tan(u) \tan(qu) \tan(q^2u) \tan(q^3u)$
$(\epsilon, 2, 1, 3, \epsilon)$	1	q	$\tan(u) \tan(qu) \tan(q^2u) \tan(q^3u)$
$(\epsilon, 2, 3, 1, \epsilon)$	2	q^2	$\tan(u) \tan(qu) \tan(q^2u) \tan(q^3u)$
$(\epsilon, 3, 1, 2, \epsilon)$	2	q^2	$\tan(u) \tan(qu) \tan(q^2u) \tan(q^3u)$
$(\epsilon, 3, 2, 1, \epsilon)$	3	q^3	$\tan(u) \tan(qu) \tan(q^2u) \tan(q^3u)$

Proposition. *We have*

$$\sum_{k \geq 0, a+b=m} A_{n,k,a,b}(q) = \sum_{\mathbf{c} \in \Theta_{n,m}} A_{n,\mathbf{c}}(q);$$

Proof. Second fundamental transformation (Foata, Schützenberger)

q-derivative polynomials

$$A_n(x, q) := \sum_{m \geq 0} x^m \sum_{k \geq 0, a+b=m} A_{n,k,a,b}(q)$$

There is also a theory for sec_q and Sec_q . The underlying polynomials being $B_{n,k,a,b}(q)$.

Main Theorems

For each (k, a, b) let $\mathcal{T}_{n,k,a,b+1}^-$ denote the set of all t -permutations $w = (w_0, w_1, \dots, w_m, w_{m+1})$ in $\mathcal{T}_{n,k,a,b+1}$ such that

$$w_{m+1} = \epsilon.$$

Theorem (Multivariable q -analog). *Let*

$$(1.10) \quad B_{n,k,a,b}(q) = \sum_{w \in \mathcal{T}_{n,k,a,b+1}^-} q^{\text{imaj } w}$$

be the generating polynomial for the set $\mathcal{T}_{n,k,a,b+1}^-$ by “**imaj**.”
Then

$$D_q^n \text{sec}_q(u) = \sum_{k,a,b} B_{n,k,a,b}(q) \times (\text{tan}_q(q^{k+1}u))^b \text{sec}_q(q^{k+1}u) (\text{tan}_q(q^k u))^a;$$

$$D_q^n \text{Sec}_q(u) = \sum_{k,a,b} q^{n(n-1)/2} B_{n,n-1-k,b,a}(q^{-1}) \times (\text{tan}_q(q^{k+1}u))^b \text{Sec}_q(q^k u) (\text{tan}_q(q^k u))^a.$$

where $0 \leq k \leq n - 1$ and $0 \leq a + b \leq n$.

Example: n=3

\mathcal{T}_3	k	imaj	a	weight Multi
$(\epsilon, 312, \epsilon)$	1	2	1	$q^2 \cdot \tan(q^2 u) \sec(q^2 u) \tan(qu)$
$(\epsilon, 213, \epsilon)$	1	1	1	$q \cdot \tan(q^2 u) \sec(q^2 u) \tan(qu)$
$(12, 3, \epsilon)$	0	0	0	$1 \cdot \tan(qu)^2 \sec(qu) 1$
$(23, 1, \epsilon)$	1	1	1	$q \cdot \tan(q^2 u) \sec(q^2 u) \tan(qu)$
$(13, 2, \epsilon)$	1	2	0	$q^2 \cdot \tan(q^2 u)^2 \sec(q^2 u) 1$
$(\epsilon, 1, 2, 3, \epsilon)$	0	0	1	$1 \cdot \tan(qu)^3 \sec(qu) \tan(u)$
$(\epsilon, 1, 3, 2, \epsilon)$	1	2	1	$q^2 \cdot \tan(q^2 u)^3 \sec(q^2 u) \tan(qu)$
$(\epsilon, 2, 1, 3, \epsilon)$	1	1	2	$q \cdot \tan(q^2 u)^2 \sec(q^2 u) \tan(qu)^2$
$(\epsilon, 2, 3, 1, \epsilon)$	1	1	3	$q \cdot \tan(q^2 u) \sec(q^2 u) \tan(qu)^3$
$(\epsilon, 3, 1, 2, \epsilon)$	1	2	2	$q^2 \cdot \tan(q^2 u)^2 \sec(q^2 u) \tan(qu)^2$
$(\epsilon, 3, 2, 1, \epsilon)$	2	3	3	$q^3 \cdot \tan(q^3 u) \sec(q^3 u) \tan(q^2 u)^3$

q-derivative polynomials

$$B_n(x, q) := \sum_{m \geq 0} x^m \sum_{k \geq 0, a+b=m} B_{n,k,a,b}(q)$$

Hoffman (1995)

$$\sum_{n \geq 0} A_n(x) \frac{u^n}{n!} = \frac{x + \tan u}{1 - x \tan u};$$

$$\sum_{n \geq 0} B_n(x) \frac{u^n}{n!} = \frac{1}{\cos u - x \sin u}.$$

Rewritten as:

$$\sum_{n \geq 0} A_n(x) \frac{u^n}{n!} = \tan(u) + \frac{\sec(u) \times x \sec(u)}{1 - x \tan(u)},$$

$$\sum_{n \geq 0} B_n(x) \frac{u^n}{n!} = \frac{\sec(u)}{1 - x \tan(u)}.$$

Main Theorems

Theorem (q-Hoffman).

$$\sum_{n \geq 0} A_n(x, q) \frac{u^n}{(q; q)_n} = \tan_q(u) + \frac{\sec_q(u) \times x \operatorname{Sec}_q(u)}{1 - x \tan_q(u)},$$

$$\sum_{n \geq 0} B_n(x, q) \frac{u^n}{(q; q)_n} = \frac{\sec_q(u)}{1 - x \tan_q(u)}.$$

Thank you!