Staircase Macdonald polynomials and the q-Discriminant

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September 4, 2011

Abstract

We prove that a q-deformation $\mathfrak{D}_k(\mathbb{X};q)$ of the powers of the discriminant is equal, up to a normalization, to a specialization of a Macdonald polynomial indexed by a staircase partition. We investigate the expansion of $\mathfrak{D}_k(\mathbb{X};q)$ on different basis of symmetric functions. In particular, we show that its expansion on the monomial basis can be explicitly described in terms of standard tableaux and we generalize a result of King-Toumazet-Wybourne about the expansion of the q-discriminant on the Schur basis.

1 Introduction

Let $\mathbb{X} = \{x_1, \dots, x_n\}$ be an alphabet. The *q-discriminant*

$$\mathfrak{D}_1(\mathbb{X};q) := \prod_{i\neq j}(qx_i - x_j),$$

is a polynomial encountered in different fields of mathematics. In particular, its specialization at $q = 1$ is the discriminant which is an example of a symmetric function invariant under the transformation $x \to x + 1$ and which has been the subject of many works in invariant theory (by Cayley, Sylvester and MacMahon).

In condensed matter physics, it plays a crucial role in the context of the fractional quantum Hall effect. Laughlin [13] described it through a wavefunction whose expression involves an even power of the Vandermonde determinant

$$\Psi^k_{\text{Laughlin}}(\mathbb{X}) = \pm\mathfrak{D}_1(\mathbb{X};1)^k\Psi^0_{\text{Laughlin}}(\mathbb{X}).$$

In this paper, we give the links between the q-discriminant and the Macdonald polynomials. More precisely, our main result is that the "polarized powers" of the q-discriminant

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$$\mathfrak{D}_k(\mathbb{X};q) := \prod_{l=1}^{k} \mathfrak{D}_1(\mathbb{X};q^{2l-1}),$$

appear when one evaluates some specialization of "staircase" Macdonald polynomials.

The powers of the discriminant ($q = 1$) are encountered also in the context of generalizations of the Selberg integral [10, 12, 25]. These integrals are closely related to the notion of Hankel hyperdeterminant [19, 20] and Jack polynomials [8, 9]. The Selberg integral admits q-analogue involving the q-discriminant (see *e.g.* [22] ex3 p374). It is interesting to remark that such integrals are related to Macdonald polynomials [27].

More generally, the specializations $t^a q^b = 1$ rise deeper identities related to the generalization of the Izergin and Korepin determinant due to Gaudin [18].

The paper is organized as follow. In Section 2, we recall notations and properties related to symmetric functions. Section 3 is devoted to the main theorem of the paper. We prove that the polynomial $\mathfrak{D}_k(\mathbb{X};q)$ is a staircase Macdonald polynomial for a specialization of the parameters q and t. As an application, in Section 4, we give a formula for the coefficients arising in the expansion of an even power of the Vandermonde determinant in terms of monomial functions. Finally, in Section 5, we generalize a theorem of King *et al.* about the expansion of the q-discriminant in terms of Schur functions.

2 Background and notations

2.1 Symmetric functions

We consider the $\mathbb{C}[[q,t,q^{-1},t^{-1}]]$-algebra Sym of symmetric functions over an alphabet $\mathbb{X}$, *i.e.* the functions which are invariant under permutations of commuting indeterminates called letters. There exists various families of such functions. We shall need the generating series of complete function:

$$\sigma_z(\mathbb{X}) := \sum_i S^i(\mathbb{X}) z^i = \prod_{x\in\mathbb{X}} \frac{1}{1-xz}.$$

This notation is compatible with the sum $\mathbb{X}+\mathbb{Y}$ and the product $\mathbb{X}\mathbb{Y} := \sum_{x\in\mathbb{X},y\in\mathbb{Y}} xy$ in the following sense

$$\sigma_z(\mathbb{X}+\mathbb{Y}) = \sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}) = \sum_i S^i(\mathbb{X}+\mathbb{Y}) z^i$$

(see *e.g.* [17] 1.3 p 5), and

$$\sigma_t(\mathbb{X}\mathbb{Y}) = \sum_i S^i(\mathbb{X}\mathbb{Y}) t^i = \prod_{x\in\mathbb{X}} \prod_{y\in\mathbb{Y}} \frac{1}{1-xyt}$$

(see *e.g.* [17] 1.5 p13). In particular, if $\mathbb{X} = \mathbb{Y}$ one has $\sigma_z(2\mathbb{X}) = \sigma_z(\mathbb{X})^2$. This definition can be extended for any complex number α by putting $\sigma_z(\alpha\mathbb{X}) = \sigma_z(\mathbb{X})^\alpha$.

We will use the Schur basis whose elements S_λ are indexed by decreasing partitions and defined by

$$S_\lambda := \det\left(S^{\lambda_i - i + j}\right)_{1 \le i,j \le n},$$

see *e.g.* [22] I.3.4 p41 and [17] 1.4.2 p8.

2.2 Macdonald Polynomials

The Macdonald polynomials $(P_\lambda(\mathbb{X}; q, t))_\lambda$ form the unique basis of symmetric functions orthogonal for the standard q, t deformation of the usual scalar product on symmetric functions (see *e.g.* [22] VI.4 p322), verifying

$$P_\lambda(\mathbb{X}; q, t) = m_\lambda(\mathbb{X}) + \sum_{\mu \le \lambda} u_{\lambda\mu} m_\mu(\mathbb{X}). \tag{1}$$

where m_λ is a monomial function in the notation of [22] I.2.1 p8. Their generating function is (see *e.g.* [22] VI.4.13 p324)

$$K_{q,t}(\mathbb{X}, \mathbb{Y}) := \sigma_1\left(\frac{1-t}{1-q}\mathbb{X}\mathbb{Y}\right) = \sum_\lambda P_\lambda(\mathbb{X}; q, t) Q_\lambda(\mathbb{Y}; q, t),$$

where $Q_\lambda(\mathbb{X}; q, t) = b_\lambda(q, t) P_\lambda(\mathbb{Y}; q, t)$ with

$$b_\lambda(q, t) = \prod_{(i,j) \in \lambda} \frac{1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i}}{1 - q^{\lambda_i - j} t^{\lambda'_j - i + 1}},$$

see *e.g.* [22] VI.6.19 p339.

Alternatively, when $\mathbb{X} = \{x_1, \ldots, x_n\}$ is a finite alphabet, the Macdonald polynomials can be defined as the eigenfunctions of the Sekiguchi-Debiard operator $\mathfrak{M}_1$ (see *e.g* [22] VI.3 p315 and VI.4 p325). Indeed,

$$P_\lambda(\mathbb{X}; q, t)\mathfrak{M}_1 = [[\lambda]]_{q,t} P_\lambda(\mathbb{X}; q, t), \tag{2}$$

where, for any $v \in \mathbb{N}^n$, $[[v]]_{q,t}$ is defined as

$$[[v]]_{q,t} := q^{v_1} t^{n-1} + q^{v_2} t^{n-2} + \cdots + q^{v_n}. \tag{3}$$

This operator may be defined in terms of divided differences

$$f(\mathbb{X})\mathfrak{M}_1 = f(\mathbb{X} - (1-q)x_1) R(tx_1; \mathbb{X} - x_1) \partial_1 \ldots \partial_{n-1}. \tag{4}$$

where, for each $i = 1 \ldots n-1$, ∂_i, denoted on the right, is the operator (see *e.g.* [15])

$$f(x_1, \ldots, x_n)\partial_i := \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}.$$

3 Staircase Macdonald polynomials

Let us denote by $\rho := [n-1, \ldots, 1, 0]$ and set $m\rho := [m(n-1), \ldots, m, 0]$ for $m \in \mathbb{N}$. We need the following lemma.

Lemma 3.1 *Under the specialization $t \to q^{\frac{(1-2k)}{2}}$, the Macdonald polynomial $P_{2k\rho}(\mathbb{X}; q, q^{\frac{(1-2k)}{2}})$ belongs to an eigenspace of $\mathfrak{M}_1$ whose dimension is 1 and its associated eigenvalue is*

$$[[2k\rho]]_{q,q^{\frac{1-2k}{2}}} = \sum_{i=1}^{n} q^{(2k+1)(n-1)/2}. \tag{5}$$

Proof From Equation (3), the eigenvalue associated to a partition λ is

$$[[\lambda]]_{q,q^{\frac{1-2k}{2}}} = \sum_{i=1}^{n} q^{(1-2k)(n-i)/2+\lambda_i}.$$

Then, if $[[\lambda]]_{q,q^{\frac{1-2k}{2}}} = [[2k\rho]]_{q,q^{\frac{1-2k}{2}}}$, it exists a permutation $\sigma \in \mathfrak{S}_n$ such that, for each $1 \leq i \leq n$, one has $\frac{1}{2}(2k+1)(n-\sigma(i)) = \frac{1}{2}(1-2k)(n-i)+\lambda_i$. It follows that

$$\lambda_i - \lambda_{i+1} = \frac{1}{2}(2k+1)(\sigma(i+1)-\sigma(i)) - \frac{1}{2}(1-2k). \tag{6}$$

Since λ is a partition, one has necessarily $\lambda_i - \lambda_{i+1} \geq 0$ and Equality (6) implies $\sigma(i+1)-\sigma(i) \geq \frac{1-2k}{1+2k} > -1$. This implies that σ is the identity and $\lambda = 2k\rho$. □

For simplicity, we set $p := q^{-\frac{1}{2}}$ and we will consider a finite alphabet $\mathbb{X} = \{x_1, \cdots, x_n\}$. Our main result is that the polarized powers $\mathfrak{D}_k(\mathbb{X}, p)$ of the discriminant are staircase Macdonald polynomials for the specialization considered here.

Theorem 3.2 *One has*

$$\mathfrak{D}_k(\mathbb{X}; p) = (-p)^{\frac{1}{2}k^2 n(n-1)} P_{2k\rho}(\mathbb{X}; q, p^{2k-1}). \tag{7}$$

Proof Reordering factors in $\mathfrak{D}_k(qx_1, x_2, \ldots, x_n; p)R(p^{2k-1}x_1; \mathbb{X} - x_1)$, one obtains

$$\mathfrak{D}_k(qx_1, x_2, \ldots, x_n; p)R(p^{2k-1}x_1; \mathbb{X} - x_1) = \mathfrak{D}_k(\mathbb{X}; p)R(p^{-(2k+1)}x_1; \mathbb{X} - x_1). \tag{8}$$

Hence, applying Equation (8), the polynomial $\mathfrak{D}_k(\mathbb{X}; p)\mathfrak{M}_1$ can be rewritten as

$$\mathfrak{D}_k(\mathbb{X}; p)\mathfrak{M}_1 = \mathfrak{D}_k(\mathbb{X}; p)R(p^{-2k-1}x_1; \mathbb{X} - x_1)\partial_1 \cdots \partial_{n-1}.$$

Since the polynomial $\mathfrak{D}_k(\mathbb{X}; p)$ is symmetric in $\mathbb{X}$, it commutes with $\partial_1, \ldots, \partial_{n-1}$ and then

$$\mathfrak{D}_k(\mathbb{X}; p)R(p^{-2k-1}x_1; \mathbb{X}-x_1)\partial_1 \cdots \partial_{n-1} = R(p^{-2k-1}x_1; \mathbb{X}-x_1)\partial_1 \cdots \partial_{n-1}\mathfrak{D}_k(\mathbb{X}; p).$$

The remaining factor $R(p^{-2k-1}x_1; \mathbb{X} - x_1)$ is of total degree $n-1$ and therefore is sent to a constant under $\partial_1 \dots \partial_{n-1}$. We use the following lemma to compute this constant.

Lemma 3.3 *For any letters a, b,*

$$R(ax_1; bx_2, \cdots, bx_n)\partial_1 \cdots \partial_{n-1} = \sum_{i+j=n-1} a^i b^j. \tag{9}$$

Proof Rewrite $R(ax_1; bx_2, \dots, bx_n)$ as

$$S_{n-1}(ax_1 - b(\mathbb{X} - x_1)) = S_{n-1}((a+b)x_1 - b\mathbb{X}) = \sum x_1^i S_i(a+b) S_{n-1-i}(-b\mathbb{X}).$$

The image of this sum under $\partial_1 \dots \partial_{n-1}$ is $S_{n-1}(a+b)S_0(-\mathbb{X})$ as wanted. □

Applying Lemma 3.3, one obtains the value of $\Bbbk$,

$$\Bbbk = \sum_{i=1}^{n} p^{(2k+1)(i-n)} = \sum_{i=1}^{n} p^{(2k-1)(n-i)-4k(n-i)}. \tag{10}$$

From Equality (5), one recognizes that $\Bbbk = [|2k\rho|]_{q,p^{2k-1}}$. This shows that

$$\mathfrak{D}_k(\mathbb{X}; p) = \beta_{k,n}(p) P_{2k\rho}(\mathbb{X}; q, p^{2k-1}), \tag{11}$$

where $\beta_{k,n}(p)$ is a constant depending only on p, k and n. It remains to compute the coefficient $\beta_{k,n}(p)$. Since we know that the dominant coefficient in $P_{2k\rho}(\mathbb{X}; q, p^{2k-1})$ is 1 by definition, it suffices to compute the coefficient of the monomial $x_n^{2k(n-1)} \cdots x_2^{2k}$ in $\mathfrak{D}_k(\mathbb{X}, p)$. One finds

$$\beta_{k,n}(p) = (-p)^{\frac{1}{2}k^2 n(n-1)}.$$

This ends the proof.□

Example 3.4 *For $k = 2$ and $n = 4$, one obtains*

$$P_{[12\ 840]}(x_1 + x_2 + x_3 + x_4; q, q^{3/2}) = q^{12} \prod_{i \neq j} \left((qx_i - x_j)(q^3 x_i - x_j)\right)$$

4 Expansion of Macdonald polynomials in terms of monomial functions

Macdonald gives in [22] VI.7.10 p345 the following expansion of the polynomials Q_λ in terms of monomial functions:

$$Q_\lambda = \sum_\mu \left(\sum_T \phi_T(q,t) \right) m_\mu, \tag{12}$$

where the inner sum is over the tableaux of shape λ and evaluation μ and each $\phi_T(q,t)$ is an explicit rational function given in [22] VI.7.11 p346.

Theorem 3.2 and Equality (12) furnish an expansion of $\mathfrak{D}_k(\mathbb{X};p)$ according to the monomial basis,

$$\mathfrak{D}_k(\mathbb{X};p) = \frac{(-p)^{\frac{1}{2}k^2n(n-1)}}{b_{2k\rho}(q,p^{2k+1})} \sum_\lambda \left(\sum_T \phi_T(q,p^{2k-1}) \right) m_\lambda \tag{13}$$

where the inner sum is over the tableaux of shape $2k\rho$ and evaluation λ.

Recall that Jack polynomials [8, 9] $P_\lambda^{(\alpha)}(\mathbb{X})$ are obtained from $P_\lambda(\mathbb{X};q,t)$ setting $q=t^\alpha$ and taking the limit when t tends to 1 (see [22] VI 10). One has

$$P_\lambda^{(\alpha)}(\mathbb{X}) = \lim_{t\to 1} P_\lambda(\mathbb{X};t^\alpha,t), \tag{14}$$

and

$$Q_\lambda^{(\alpha)}(\mathbb{X}) = \lim_{t\to 1} Q_\lambda(\mathbb{X};t^\alpha,t) = b_\lambda^{(\alpha)} P_\lambda^{(\alpha)} \tag{15}$$

where $b_\lambda^{(\alpha)} := \lim_{t\to 1} b_\lambda(t^\alpha,t)$. Putting

$$\phi_T^{(\alpha)} := \lim_{t\to 1} \phi_T(t^\alpha,t),$$

one get from Equation (13) an expansion of integral powers of the discriminant.

Corollary 4.1 *One has*

$$\begin{aligned} \mathfrak{D}_1(\mathbb{X};1)^k = \mathfrak{D}_k(\mathbb{X};1) &= (-1)^{\frac{kn(n-1)}{2}} P_{2k\rho}^{(\alpha_k)}(\mathbb{X}) \\ &= (-1)^{\frac{kn(n-1)}{2}} \left(b_{2k\rho}^{(\alpha_k)}\right)^{-1} \sum_\lambda \left(\sum_T \phi_T^{(\alpha_k)} \right) m_\lambda, \end{aligned} \tag{16}$$

where $\alpha_k = \frac{-2}{2k-1}$ *and the inner sum is over the tableaux of shape* $2k\rho$ *and evaluation* λ.

Example 4.2 Consider an alphabet $\mathbb{X}=\{x_1,x_2,x_3\}$ of size 3. One has,

$$Q_{42}(\mathbb{X};q,t) = \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\!\!\begin{array}{|c|c|}\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array} m_{42} + \begin{array}{|c|c|}\hline 2&3\\\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\!\!\begin{array}{|c|c|}\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array} m_{411} + \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\!\!\begin{array}{|c|c|}\hline 1&2\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array} m_{33}$$
$$+ \left(\begin{array}{|c|c|}\hline 2&3\\\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\!\!\begin{array}{|c|c|}\hline 1&2\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array} + \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\!\!\begin{array}{|c|c|}\hline 1&3\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array} \right) m_{321}$$
$$+ \left(\begin{array}{|c|c|}\hline 3&3\\\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\!\!\begin{array}{|c|c|}\hline 2&2\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array} + \begin{array}{|c|c|}\hline 2&3\\\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\!\!\begin{array}{|c|c|}\hline 2&3\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array} + \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\!\!\begin{array}{|c|c|}\hline 3&3\\\hline\multicolumn{2}{c}{}\\\cline{1-2}\end{array} \right) m_{222}.$$

Each tableau T is interpreted as the function Φ_T,

$$\begin{aligned} Q_{42}(\mathbb{X};q,t) &= \left(\frac{1-t}{1-q}\right)^2 \left(\frac{1-tq}{1-q^2}\right)^2 \left(\frac{1-t^2q^2}{1-tq^3}\right) \left(\frac{1-t^2q^3}{1-tq^4}\right) m_{42} \\ &\quad + \left(\frac{1-t}{1-q}\right)^3 \left(\frac{1-tq}{1-q^2}\right) \left(\frac{1-t^2q^3}{1-tq^4}\right) \left(\frac{1-t^2q^2}{1-q^3t}\right) m_{411} + \dots \end{aligned}$$

Setting $q = t^{-2}$ and taking the limit $t \to 1$, the algorithm described here allows to compute the expansion of the Jack polynomials according to the monomial functions. After simplification, one obtains

$$Q_{42}^{(-2)}(\mathbb{X}) = \frac{1}{280}\, m_{4,2} - \frac{1}{140}\, m_{4,1,1} - \frac{1}{140}\, m_{3,3} + \frac{1}{140}\, m_{3,2,1} - \frac{3}{140}\, m_{2,2,2}.$$

And finally,

$$\mathfrak{D}_1(\mathbb{X}; 1) = -m_{4,2} + 2\, m_{4,1,1} + 2\, m_{3,3} - 2\, m_{3,2,1} + 6\, m_{2,2,2}.$$

Corollary 4.1 can be applied to expand Hankel hyperdeterminants. Hyperdeterminants are polynomials defined by Cayley in the aim of generalizing the notion of determinant to higher dimensional arrays[1] [4, 5]. Given a mth order tensor $\mathrm{M} = (M_{i_1 \dots i_m})_{1 \leq i_1, \dots, i_m \leq n}$ on a n dimensional space, its hyperdeterminant is

$$\mathrm{Det}(\mathrm{M}) = \frac{1}{n!} \sum_{\sigma_1, \dots, \sigma_m \in \mathfrak{S}_n} \mathrm{sign}(\sigma_1 \dots \sigma_m) \prod_{i=1}^{m} M_{\sigma_1(i) \dots \sigma_m(i)}.$$

Note that this polynomial vanishes when m is odd. Suppose that $m = 2k$ is an even integer. An Hankel hyperdeterminant is an hyperdeterminant whose entries depend only on the sum of the indices $M_{i_1 \dots i_{2k}} = f(i_1 + \cdots + i_{2k})$. This kind of hyperdeterminant have been already considered by the authors in collaboration with Thibon and Belbachir [19, 20, 2]. In particular, it is shown that the coefficients $C_\lambda(n, l)$ arising in the expression

$$\mathrm{Det}\left(M_{i_1 + \cdots + i_{2k}}\right) = \sum_{\lambda} C_\lambda(n, k) \prod_{i=1}^{n} f(\lambda_i),$$

are equal (up to a multiplicative term equal to the number of permutations of λ divided by $n!$) to those arising in the expansion of $\mathfrak{D}_k(\mathbb{X}; 1)$ in terms of monomial functions.

Example 4.3 From the expansion of the Jack polynomial $P_{84}^{(-2/3)}$, for an alphabet of size 3,

$$\begin{aligned} P_{84}^{(-2/3)}(x_1 + x_2 + x_3) \quad = \quad & m_{84} - 4m_{831} + 6m_{822} - 4m_{75} + 12m_{741} - 8m_{732} + 6m_{66} - 8m_{651} \\ & -22m_{642} + 48m_{633} + 48m_{552} - 36m_{543} + 90m_{44}, \end{aligned}$$

one deduces the expansion of the Hankel hyperdeterminant

[1] Note that Cayley proposed several generalizations of determinants. The polynomial considered here is the simplest one in the sense that it generalizes the expansion of determinant as an alternated sum. Reader can refer to [19, 20, 21, 23, 26] for more informations on the subject.

$$\begin{aligned}\mathrm{Det}\left(f(i_1+i_2+i_3+i_4)\right)_{0\le i_1,i_2,i_3,i_4\le 3} &= f(8)f(4)f(0)-4f(8)f(3)f(1)+3f(8)f(2)^2\\ &\quad -4f(7)f(5)f(0)+12f(7)f(4)f(1)-8f(7)f(3)f(2)\\ &\quad +3f(6)^2f(0)-8f(6)f(5)f(1)-22f(6)f(4)f(2)\\ &\quad +24f(6)f(3)^2+24f(5)^2f(2)-36f(5)f(4)f(3)\\ &\quad +15f(4)^3.\end{aligned}$$

Furthermore, in [15] Lapointe *et al.* gave a determinantal expression of Jack polynomial in terms of monomial functions. These computations leads naturally to a determinantal expression for Hankel hyperdeterminants.

Note that the formula for the Macdonald polynomials $\tilde{H}_\lambda$, given by Haglund, Haiman and Loehr [7], provides an expansion of $\mathfrak{D}_k(\mathbb{X};q)$ in terms of modified monomial functions $m_\lambda(\mathbb{X}(1-t))$ having a combinatorial interpretation.

5 Expansion of the polarized powers of the q-discriminant in terms of Schur functions

Di Francesco *et al.* [6] considered the problem of the expansion of the discriminant in terms of Schur functions. They defined the *n-admissible* partitions to be the partitions in the interval $[(n-1)^n]$, $[2(n-1),\ldots,2,0]$ (with respect to the dominance order). They conjectured that they are exactly those occurring in the expansion of the discriminant. This conjecture is false as shown by Scharf *et al.* [24]. However, Kind *et al.* [11] proved that it becomes true when replacing the discriminant by the q-discriminant.

In this section, we generalize this property to $\mathfrak{D}_k(\mathbb{X};q)$. We define (n,m)-*admissible* partitions to be the partitions which appear in the expansion

$$m_\rho(\mathbb{X})^{m-1}S_\rho(\mathbb{X})=\sum_\lambda b_\lambda^{n,m}m_\lambda(\mathbb{X}) \tag{17}$$

where $\mathbb{X}$ is an alphabet of size n. When $m=2k$ is even, the $(n,2k)$-admissible partitions are those of the interval $[(k(n-1))^n]$, $[2k(n-1),\ldots,2k,0]$. We prove that a partition appear in the expansion of $\mathfrak{D}_k(\mathbb{X};q)$ in terms of Schur functions if and only if it is a $(n,2k)$-partition.

5.1 Computing admissible partitions

Let us denote by $A_{n,m}$ the set defined recursively by

$$\begin{aligned}&A_{n,1}:=\{\lambda=[\lambda_1,\ldots,\lambda_n]|\rho\ge\lambda\}\\ &A_{n,m}:=\{((\lambda_1+\sigma(1)-1,\ldots,\lambda_n+\sigma(n)-1))|\sigma\in\mathfrak{S}_n \text{ and } \lambda\in A_{n,m-1}\}.\end{aligned} \tag{18}$$

Lemma 5.1 *Let λ be a partition. The following assertions are equivalent.*

1. *The partition λ belongs to $A_{n,m}$.*

2. *The partition λ is (n,m)-admissible.*

3. *λ is partition of length n less or equal to $m\rho$ with respect to the dominance order.*

Proof The equivalence between the assertions 1 and 2 is straightforward from Equations (17) and (18). Furthermore, from Equation (18), the maximal partition of $A_{n,m}$ is $m\rho$. It remains to prove $3 \Rightarrow 1$. We proceed by induction on m, if $m = 1$ then the result is trivial. Suppose that $m > 1$. Let λ be a partition of size n less or equal to $m\rho$ with respect to the dominance order. Then $((\lambda - \rho))$ is a partition less or equal to $(m-1)\rho$. Indeed, putting $((\lambda-\rho)) = (\mu_1, \ldots, \mu_n)$, for a permutation $\sigma \in \mathfrak{S}_n$, one has $\mu_i = \lambda_{\sigma(i)} + n - \sigma(i)$. Hence, for each i

$$\begin{array}{rcl} \mu_1 + \cdots + \mu_i & \leq & \lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(i)} + n - \sigma(1) + \cdots + n - \sigma(i) \\ & \leq & \lambda_1 + \cdots + \lambda_i + n - 1 + \cdots + n - i \\ & \leq & (m-1)(n - \frac{1}{2}i(i+1)) \end{array}$$

implies $((\lambda - \rho)) \leq (m-1)\rho$ for the dominance order.

By induction, $((\lambda - \rho))$ belongs to $A_{n,m-1}$. Furthermore, it exists a permutation σ such that $((\lambda - \rho)) + \rho^\sigma = \lambda$. Hence, from Equation (18), $\lambda \in A_{n,m}$.□

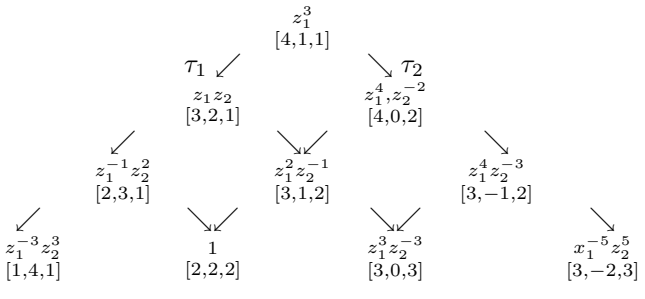
5.2 Counting admissible partitions

One considers the free commutative monoid $\mathfrak{T}$ generated by the symbols $T = \{\tau_1, \ldots, \tau_{n-1}\}$ acting on the vectors of size n by

$$\tau_i[v_1, \ldots, v_n] = [v_1, \ldots, v_{i-1}, v_i - 1, v_{i+1} + 1, v_{i+1}, \ldots, v_n].$$

For a given vector $v \in \mathbb{Z}^n$, $\mathfrak{T}.v$ is the set of the vectors $w = [w_1, \ldots, w_n] \in \mathbb{Z}^n$ of same weight (*i.e.* $v_1 + \cdots + v_n = w_1 + \ldots w_n$) lower or equal to v for the dominance order. In particular, if $v = \lambda$ is a partition then $\mathfrak{T}.\lambda$ contains all the partition of size n lower or equal to λ. To each vector $v \in \mathbb{Z}^n$, one associates the monomial $z^v = z_1^{v_1 - v_2} \ldots z_{n-1}^{v_{n-1} - v_n}$. For a given weight, the monomial z^v characterizes completely v, furthermore v is a (decreasing) partition if and only if its weight is non negative and the degree of the monomial z^v in each variable z_i is non-negative.

Example 5.2



(19)

Acting on v by τ_i is equivalent to multiply z^v by

$$t_i = \begin{cases} \frac{z_{i-1}z_{i+1}}{z_i^2} & \text{if } 1 < i < n-1 \\ \frac{z_2}{z_1^2} & \text{if } i = 1 \\ \frac{z_{n-2}}{z_{n-1}^2} & \text{if } i = n-1. \end{cases}$$

Since there is no algebraic relations between the $t_i's$, each vector appears in $\mathfrak{T}.v$ with multiplicity 0 or 1. In other words, one has

$$\sigma_q(T).z^v = \prod_i \frac{1}{1-tq}.z^v = \sum_{w \leq v} q^{\alpha_{v,w}} z^w. \tag{20}$$

where $\alpha_{v,w}$ is the degree of the monomial acting on v to obtain w. Extracting the monomial which encodes a partition is equivalent to extract the part of the series (20) constituted only with non-negative exponents. This operation is performed by the MacMahon Omega operator (see *e.g* [1])

$$\Omega_{x_1,\ldots,x_p} \sum_{n_1,\ldots,n_p \in \mathbb{Z}} \alpha_{n_1,\ldots,n_p} x_1^{n_1} \ldots . x_p^{n_p} = \sum_{n_1,\ldots,n_p \in \mathbb{N}} \alpha_{n_1,\ldots,n_p} x_1^{n_1} \ldots . x_p^{n_p}.$$

Example 5.3 One has

$$\Omega_{z_1,z_2} \frac{z_1^3}{(1-\frac{z_1}{z_2^2}q)(1-\frac{z_2}{z_1^2}q)} = z_1^3 + qz_1z_2 + q^3$$

which implies that the set of the partitions of size 3 lower or equal to [411] is $\{[411],[321],[222]\}$.

Hence,

Proposition 5.4 *The size $n \geq 2$ of the alphabet being fixed, the generating series of the (n,k)-admissible partitions is the rational function*

$$\mathfrak{A}_n(q,t;z_1,\ldots,z_{n-1}) = \Omega_{z_1,\ldots,z_{n-1}} \left((1-tz_1\ldots z_{n-1})(1-\frac{z_0z_2}{z_1^2}q)(1-\frac{z_1z_3}{z_2^2}q)\ldots(1-\frac{z_{n-2}z_n}{z_{n-1}^2}q) \right)^{-1},$$

where $z_0 = z_n = 1$.

Example 5.5 Let us give the first value of $\mathfrak{A}_n(q,t;z_1,\ldots,z_{n-1})$.

1. First, one considers the special case $n=2$,

$$\begin{aligned} \mathfrak{A}_1(q,t;z_1) &= \Omega_{z_1} \left((1-tz_1)(1-\tfrac{q}{z_1^2}) \right)^{-1} = \left((1-qt^2)\,(1-z_1\,t) \right)^{-1} \\ &= 1 + z_1t + (q+z_1^2)t^2 + (qz_1+z_1^3)t^3 + (q^2+qz_1^2+z_1^4)t^4 + \ldots. \end{aligned}$$

This means that, for $k=1$ the only admissible partition is [21], for $k=2$, there is two admissibles partitions [42] and [33], for $k=3$ the admissibles partitions are [63] and [54] etc...

2. If $n=3$, $\mathfrak{A}_3(q,t;z_1,z_2) = -\frac{1-{z_1}^3{z_2}^3q^2t^4}{(1-tz_1\,z_2)(1-q{z_2}^3t^2)(1-{z_1}^3qt^2)(1-q^2t)}$.

3. If $n=4$, $\mathfrak{A}_4(1,t;1,1) = \frac{t^4+5\,t^3+7\,t^2+2\,t+1}{(1+t)^2(1-t)^4}$.

4. If $n=5$, $\mathfrak{A}_5(1,t;1,1) = \frac{3\,t^6+21\,t^5+61\,t^4+68\,t^3+39\,t^2+7\,t+1}{(1-t)^5(t+1)^3}$

5.3 Characterization of the partitions arising in the expansion of $\mathfrak{D}_k(\mathbb{X};q)$

In this paragraph, one extends the result of King-Toumazet-Wybourne to the polynomials $\mathfrak{D}_k(\mathbb{X};q)$.

Theorem 5.6 *Expand $\mathfrak{D}_k(\mathbb{X};q)$ in terms of Schur functions,*

$$\mathfrak{D}_k(\mathbb{X};q) = \sum_\lambda c_\lambda(q) S_\lambda(\mathbb{X}).$$

Then, $c_\lambda(q) \neq 0$ if and only if λ is a $(n,2k)$-admissible partition.

Proof Let us prove first the only if part. From Theorem 3.2, the polynomial $\mathfrak{D}_k(\mathbb{X};q)$ equals (up to a multiplicative coefficient) a specialization of the Macdonald $P_{2k\rho}(\mathbb{X};q,t)$. But it is well known that the partitions arising in the expansion of $P_{2k\rho}(\mathbb{X};q,t)$ in terms of Schur functions belong to the interval $[(k(n-1))^n]$, $2k\rho$ (see *e.g.* the determinantal expression of Macdonald polynomials given in [15]). From Lemma 5.1, this is equivalent to the fact that λ is $(n,2k)$-admissible.

Conversely, to prove that the admissibility of λ implies the non nullity of $c_\lambda(q)$, it suffices to prove it for a specialization. We will set $q=-1$. In this case,

$$\mathfrak{D}_k(\mathbb{X};q) = \prod_{i\neq j}(x_i+x_j)^k = S_\rho(\mathbb{X})^{2k}.$$

We will prove a stronger result showing that the coefficient $c_\lambda^{n,m}$ in the expansion

$$S_\rho(\mathbb{X})^m = \sum_\lambda c_\lambda^{n,m} S_\lambda(\mathbb{X})$$

is non-zero if and only if λ is (n,m)-admissible. We proceed by induction on m. Note that the initial case ($m=2$) have been proved by King-Toumazet-Wybourne in [11] Corollary 3.2 as a consequence of an important result of Bereinstein-Zelevinsky [3].

One needs the two following lemmas

Lemma 5.7 *If λ is a (n,m)-admissible partition ($m>1$), then $((\lambda-\rho))$ is a $(n,m-1)$-admissible partition.*

Proof From Equality (17), each (n,m)-admissible partition can be obtained by adding a permutation of ρ to a $(n,m-1)$-admissible partition. This is equivalent to our statement. □

Lemma 5.8 *Let $\mu \subset \lambda$ be a partition and $\nu := ((\lambda_1 - \mu_1, \ldots, \lambda_{n-1} - \mu_{n-1}, \lambda_n - \mu_n))$. Then, the Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu} = \langle S_\lambda, S_\mu S_\nu \rangle$ equals* 1.

Proof The Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of tableaux of shape ν and evaluation $\lambda - \mu$. But $\lambda - \mu$ is a permutation if ν and Theorem 11.4.3 of [17] implies that such a tableau exists and is unique. This ends the proof. □

End of the proof of Theorem 5.6 Let λ be a (n, m)-admissible partition. Since $\rho \subset \lambda$, Lemma 5.7 implies that the partition $\mu = ((\lambda - \rho))$ is $(n, m-1)$-admissible. And by induction, S_μ appears with a non-zero coefficient in S^{m-1}_ρ. The positivity of the Littlewood Richardson coefficients implies that each partition ν such that $c^{\nu}_{\mu,\rho} \neq 0$ appears with a non-zero coefficient in the expansion of S^m_ρ. In particular, from Lemma 5.8, it is the case of λ. This shows that $c^{n,m}_{\lambda} \neq 0$ if and only if λ is (n, m)-admissible and proves the Theorem.□.
Note that other expansion of Macdonald functions can be found in literature (for example Hall-Littlewood polynomials can be expanded in terms of plane partitions [16]), it should be interesting to investigate the properties $\mathfrak{D}_k(\mathbb{X}; q)$ which can be deduced from these expansions.

Acknowledgments

We are grateful to Alain Lascoux for fruitful discussions.

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