Covariance Trees

Faster *Learning* & *Querying* for Multi-Scale Local Gaussian Models

Thierry Guillemot Andrés Almansa Tamy Boubekeur



Journées Bordelaises d'Analyse Mathématique des Images

Surface reconstruction







3D point cloud

Point Set Surfaces RIMLS [Öztireli et al. 2009]

Non-local Point Set Surface

Reconstruction from random samples



20% of original pixels



Cubic





PSNR : 24.9 dB PSNR : 27.2 dB

Outline

- Motivation
 - Example-based image inpainting
 - From example-based to model-based
 - Local Gaussian Models vs GMM
- Covariance Tree
- Applications
 - Learning-based Image denoising
 - Random Sampling
 - Surface reconstruction from 3D point clouds

Example-based image inpainting [Efros-Leung 1999, Wexler 2005]

Input: *Visible part* of the image $u|_{O^c}$ Output: reconstruction of the *occluded part* $u|_O$ via

$$\min_{u|_O}\sum_{m\in O}\|p_m(u)-p_{\varphi(m)}(u)\|^2$$

where

$$\varphi(m) = \underset{n \in O^c}{\arg\min} \|p_m(u) - p_n(u)\|^2$$

is the *nearest neighbour* of $p_m(u)$: patches in Images and videos:





Example-based image inpainting [Arias-Caselles-Facciolo 2012]

Input: *Visible part* of the image $u|_{O^c}$ Output: reconstruction of the *occluded part* $u|_O$ via

$$\min_{w,u|_{O}} \sum_{m \in O, n \in O^{c}} w(m,n) \|p_{m}(u) - p_{n}(u)\|^{2} - T \sum_{m} H(w(m,\cdot))$$

under the constraint $\sum_{n} w(m, n) = 1, \forall m \in O$

where $H(f) = -\sum_{n} f(n) \log(f(n))$ is the entropy of the probability density distribution *f*.



Example-based image inpainting [Arias-Caselles-Facciolo 2012]

Input: *Visible part* of the image $u|_{O^c}$ Output: reconstruction of the *occluded part* $u|_O$ via

$$\min_{w,u|_{O}} \sum_{m \in O, n \in O^{c}} w(m,n) \|p_{m}(u) - p_{n}(u)\|^{2} - T \sum_{m} H(w(m,\cdot))$$

under the constraint $\sum_{n} w(m, n) = 1, \forall m \in O$

where $H(f) = -\sum_{n} f(n) \log(f(n))$ is the entropy of the probability density distribution *f*.

Non-convex problem

Alternated minimisation of convex problems

• *w*-minimization (Learn local distribution)

$$w(m,n) = \frac{1}{Z}e^{-\frac{1}{T}\|p_m(u)-p_n(u)\|^2}$$

• *u*-min: (a posteriori expectation)

$$\hat{p}_m = \mathsf{E}\left[p \mid p_m(u)\right] = \sum_n w(m, n)p_n$$

• Aggregation: $u(m) = \sum_n \hat{p}_n [n - m]$



Example-based image inpainting [Arias-Caselles-Facciolo 2012]

Input: Visible part of the image $u|_{O^c}$ Output: reconstruction of the occluded part $u|_O$ via

$$\min_{w,u|_{O}} \sum_{m \in O, n \in O^{c}} w(m,n) \|p_{m}(u) - p_{n}(u)\|^{2} - T \sum_{m} H(w(m,\cdot))$$

under the constraint $\sum_{n} w(m, n) = 1, \forall m \in O$

where $H(f) = -\sum_{n} f(n) \log(f(n))$ is the entropy of the probability density distribution *f*.

Non-convex problem

Alternated minimisation of convex problems

• *w*-minimization (Learn local distribution)

$$w(m,n) = \frac{1}{Z}e^{-\frac{1}{T}\|p_m(u)-p_n(u)\|^2}$$

• *u*-min: (a posteriori expectation)

$$\hat{p}_m = \mathsf{E}\left[p \mid p_m(u)\right] = \sum_n w(m, n)p_n$$

• Aggregation:
$$u(m) = \sum_n \hat{p}_n [n - m]$$

Challenges

- Computation of *w* truncated and approximated by Patch Match [Barnes 2009] Other alternatives? non-structured data?
- Non-convexity: Multi-scale
- Patch similarity: l² is ambiguous for fine textures



Example-based vs. model-based image inpainting

Example based

• *w*-minimization (Learn local distribution)

$$w(m,n) = \frac{1}{Z}e^{-\frac{1}{T}\|p_m(u)-p_n(u)\|^2}$$

• *u*-min: (a posteriori expectation)

$$\hat{p}_m = \mathsf{E}\left[p \mid p_m(u)\right] = \sum_n w(m, n)p_n$$

• Aggregation:
$$u(m) = \sum_n \hat{p}_n [n - m]$$

Model based

• *w*-minimization (Learn local model)

 $w(m, \cdot) \sim N(\mu_m, \Sigma_m)$ that fits

 $\{p_n(u) : \|p_m(u) - p_n(u)\|^2 < T\}$

- *u*-minimization: estimate \hat{p}_n by:
 - EAP (blurry), or...
 - MAP, or...
 - Random synthesis near $p_m(u)$ based on $N(\mu_m, \Sigma_m)$
- Aggregation: $u(m) = \sum_n \hat{p}_n[n-m]$
- fast algorithms on unstructured data (CovTree)
- synthesize vs. copy

Non-Local Means denoising [Buades-Coll-Morel 2005]

Input: Noisy image $\tilde{u} = u + n$ where $n \sim N(0, \sigma^2 Id)$. Output: Estimated clean image \hat{u} via

$$\max_{u}\sum_{m,n}w(m,n)\|p_m(u)-p_n(\tilde{u})\|^2-T\sum_{m}H(w(m,\cdot))$$

under the constraint $\sum_{n} w(m, n) = 1, \forall m \in O$

Example based

• w-minimization (Learn local distribution)

$$w(m,n) = \frac{1}{Z}e^{-\frac{\|p_m(u)-p_n(\tilde{u})\|^2-T}{T}}$$

• *u*-minimization: (*a posteriori expectation*)

$$\hat{p}_m = \sum_n w(m, n) p_n$$

• Aggregation: $\hat{u}(m) = \sum_n \hat{p}_n[n-m]$

Non-Local Bayes denoising [Lebrun-Buades-Morel 2013]

Input: Noisy image $\tilde{u} = u + n$ where $n \sim N(0, \sigma^2 Id)$. Output: Estimated clean image \hat{u} via

$$\max_{u(m), N(\mu_m, \Sigma_m)} \Pr\left[p_m(u) \mid p_n(\tilde{u}), N(\mu_m, \Sigma_m)\right]$$

Model based

• *w*-minimization (Learn Local Gaussian Model)

$$\mu_m = \frac{1}{Z} \sum_n e^{-\frac{\|p_m(u) - p_n(\tilde{u})\|^2}{\sigma^2}} p_n(\tilde{u})$$
$$\boldsymbol{\Sigma}_m = \frac{1}{Z} \sum_n e^{-\frac{\|p_m(u) - p_n(\tilde{u})\|^2}{\sigma^2}} \bar{p}_n(\tilde{u}) \bar{p}_n(\tilde{u})^T - \sigma^2 locometrics$$

• *u*-minimization: (MAP)

$$\hat{p}_m = \operatorname*{arg\,min}_q \frac{1}{\sigma^2} \|q - p_m(\tilde{u})\|^2 + (q - \mu_m)^T \boldsymbol{\Sigma}_m^{-1} (q - \mu_m)$$

• Aggregation: $\hat{u}(m) = \sum_{n} \hat{p}_{n}[n-m]$

Piecewise Linear Estimators [Yu-Mallat-Sapiro 2012]

Input: Perturbed image $\tilde{u} = Au + n$ where $n \sim N(0, \sigma^2 Id)$. Output: Restored image \hat{u} via

$$\max_{u(m), N(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k), k(m)} \Pr\left[p_m(u) \mid p_n(\tilde{u}), N(\boldsymbol{\mu}_{k(m)}, \boldsymbol{\Sigma}_{k(m)}) \right]$$

with k = 1, ..., 20

Model based

- initialization: \hat{u}^0 , (μ_k^0, Σ_k^0) , k(m)
- Relearn Gaussian Models (μ_k^i, Σ_k^i) to fit $\{p_m(u) : k(m) = k\}$
- Signal estimation (\hat{p}_m) and model selection (k(m))

$$(\hat{p}_m, k(m)) = \operatorname*{arg\,max}_{q,k} \Pr\left[q \mid p_m(\tilde{u}), N(\mu_k, \mathbf{\Sigma}_k)\right]$$

• Aggregation: $\hat{u}^i(m) = \sum_n \hat{p}_n[n-m]$

Learning-based restoration [Zoran-Weiss 2011]

Offline learning

Input: a huge database of *natural image patches* $\mathbf{p}_i \in \mathcal{P}$. Output: Gaussian Mixture Model { $N(\mu_k, \Sigma_k) : k = 1, ..., 250$ } fitting the data (several days worth of computation)

Restoration

Input: Perturbed image $\tilde{u} = Au + n$ where $n \sim N(0, \sigma^2 Id)$. Gaussian Mixture Model { $N(\mu_k, \Sigma_k) : k = 1, ..., 250$ } (representing the manifold of natural image patches) Output: Restored image \hat{u} via

$$\max_{u(m), k(m)} \Pr\left[p_m(u) \mid p_n(\tilde{u}), N(\mu_{k(m)}, \Sigma_{k(m)})\right]$$

Covariance Tree [Guillemot-Almansa-Boubekeur 2014]

Learning

Input: a huge database of *data points* $\mathbf{p}_i \in \mathcal{P}$. Output: pre-computed Local Gaussian Models at several *scales* and *locations*

Query

Input: a query point **q** and a scale σ Output: accurate approximation of $N(\mu_{\mathbf{q}}, \Sigma_{\mathbf{q}})$ fitting $\mathcal{P}|_{B(\mathbf{q},\sigma)}$

Bayesian Restoration





Covariance Tree

Outline of the algorithm



Outline of the algorithm



 σ_b Rayon minimal de construction

 $\sigma_{\mathbf{q}}$ Echelle de requête

Partitioning into representative cells (Gaussian kd-trees [Adams et. al 2009])







Echantillons

$$\begin{split} \boldsymbol{\eta}_{c} &= \operatorname{moyenne}(\mathbf{p}_{i}) \\ \boldsymbol{\eta}_{r} &= \max_{i} \|\mathbf{p}_{i} - \boldsymbol{\eta}_{c}\| \\ \boldsymbol{\eta}_{d} &= \operatorname{dir.} \text{ plus gde variation des } \mathbf{p}_{i} \end{split}$$







$$\begin{aligned} \boldsymbol{\eta}_{c} &= \operatorname{moyenne}(\mathbf{p}_{i}) \\ \boldsymbol{\eta}_{r} &= \max_{i} \|\mathbf{p}_{i} - \boldsymbol{\eta}_{c}\| \\ \boldsymbol{\eta}_{d} &= \operatorname{dir. \ plus \ gde \ variation \ des \ \mathbf{p}_{i}} \end{aligned}$$

























Echantillons

Outline of the algorithm



 σ_b Rayon minimal de construction

 $\sigma_{\mathbf{q}}$ Echelle de requête





$$\boldsymbol{\mu}_{\eta} := \boldsymbol{\mu}_{\eta} + w_{i} \mathbf{p}_{i}$$
$$\boldsymbol{\Sigma}_{\eta} := \boldsymbol{\Sigma}_{\eta} + w_{i} \mathbf{p}_{i}^{t} \mathbf{p}_{i}$$





$$\boldsymbol{\mu}_{\eta} := \boldsymbol{\mu}_{\eta} + w_{i} \mathbf{p}_{i}$$
$$\boldsymbol{\Sigma}_{\eta} := \boldsymbol{\Sigma}_{\eta} + w_{i} \mathbf{p}_{i}^{t} \mathbf{p}_{i}$$





$$\boldsymbol{\mu}_{\eta} := \boldsymbol{\mu}_{\eta} + w_{i} \mathbf{p}_{i}$$
$$\boldsymbol{\Sigma}_{\eta} := \boldsymbol{\Sigma}_{\eta} + w_{i} \mathbf{p}_{i}^{t} \mathbf{p}_{i}$$





Apprentissage

 $\boldsymbol{\mu}_{\eta} := \boldsymbol{\mu}_{\eta} + w_{i} \mathbf{p}_{i}$ $\boldsymbol{\Sigma}_{\eta} := \boldsymbol{\Sigma}_{\eta} + w_{i} \mathbf{p}_{i}^{t} \mathbf{p}_{i}$





Outline of the algorithm



 $\sigma_{\mathbf{q}}$ Echelle de requête






















Computational Performance



Computational Performance



Generalisation

Context domain



Value domain





Image Restoration

Applications





Approche par dictionnaire



Applications





Approche par dictionnaire



Non Local Bayes [Lebrun et al. 12]

$$\mathbb{P}(\mathbf{q}|\hat{\mathbf{q}}) = \mathcal{N}(\mathbf{0}, \sigma_d)$$
$$\mathbb{P}(\hat{\mathbf{q}}) = \mathcal{N}(\boldsymbol{\mu}', \boldsymbol{\Sigma}')$$
$$\mathbb{P}(\mathbf{q}) = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$
$$\boldsymbol{\Sigma}' = \boldsymbol{\Sigma} - \sigma_d^2 I d$$

$$argmax_{\hat{\mathbf{q}}} \mathbb{P}(\hat{\mathbf{q}}|\mathbf{q}) = argmax_{\hat{\mathbf{q}}} \mathbb{P}(\mathbf{q}|\hat{\mathbf{q}}) \mathbb{P}(\hat{\mathbf{q}})$$
$$\hat{\mathbf{q}} = \boldsymbol{\mu} + [\boldsymbol{\Sigma} - \sigma_d^2 Id] \boldsymbol{\Sigma}^{-1} (\mathbf{q} - \boldsymbol{\mu})$$

Non Local Bayes [Lebrun et al. 12]



Different grouping criteria



Results: Different grouping criteria



Compressed grouping criterion



Results: Non Local Bayes



Applications





Approche par dictionnaire



Learning-based denoising

- Data-base
 - 120 images
 - 100 000 000 patches
- Learning complexity
 - Building: 5 minutes
 - Learning: 5 hours
- Query:
 - 1 million pixels
 - Query: 30 seconds













Learning-based denoising

$$\mathbb{P}(\mathbf{q}|\hat{\mathbf{q}}) = \mathcal{N}(\mathbf{0}, \sigma_d)$$
$$\mathbb{P}(\hat{\mathbf{q}}) = \mathcal{N}(\boldsymbol{\mu}', \boldsymbol{\Sigma}')$$
$$\mathbb{P}(\mathbf{q}) = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$
$$\boldsymbol{\Sigma}' = \boldsymbol{\Sigma} - \sigma_d^2 I d$$

$$\begin{aligned} & \operatorname{argmax}_{\hat{\mathbf{q}}} \ \mathbb{P}(\hat{\mathbf{q}}|\mathbf{q}) = \operatorname{argmax}_{\hat{\mathbf{q}}} \ \mathbb{P}(\mathbf{q}|\hat{\mathbf{q}}) \mathbb{P}(\hat{\mathbf{q}}) \\ & \hat{\mathbf{q}} = \boldsymbol{\mu} + \boldsymbol{\Sigma} [\boldsymbol{\Sigma} + [\sigma_d^2 Id]^{-1} (\mathbf{q} - \boldsymbol{\mu}) \end{aligned}$$

Learning-based denoising



Reconstruction from Random Samples

$$\begin{split} \tilde{\mathbf{q}} &= S\mathbf{q} \\ \hat{\mathbf{q}} &= (\boldsymbol{\Sigma}_{\mathbf{q}}S^{H}S + \frac{\sigma^{2}}{2}Id)^{-1}(\boldsymbol{\Sigma}_{\mathbf{q}}S^{H}\tilde{\mathbf{q}} + \frac{\sigma^{2}}{2}\boldsymbol{\mu}_{\mathbf{q}}) \\ \hat{\mathbf{q}}_{t} &= (\boldsymbol{\Sigma}_{\hat{\mathbf{q}}_{t-1}}S^{H}S + \frac{\sigma^{2}}{2}Id)^{-1}(\boldsymbol{\Sigma}_{\hat{\mathbf{q}}_{t-1}}S^{H}\tilde{\mathbf{q}} + \frac{\sigma^{2}}{2}\boldsymbol{\mu}_{\hat{\mathbf{q}}_{t-1}}) \end{split}$$

Reconstruction d'image échantillonnée aléatoirement Original 20% pixel originaux





Cubique



CovTree - Dictionnaire











PSNR : 24.9 dB PSNR: 27.2 dB

Applications





Approche par dictionnaire



Surface reconstruction







3D point cloud

Point Set Surfaces RIMLS [Öztireli et al. 2009]

Non-local Point Set Surface

Surface reconstruction



Point Set Surfaces – PSS



Non Local vs Local denoising



PSS decomposition



PSS decomposition + interpolation



Non-local interpolation



Local surface descriptor: 2D patch of scalar displacements



L2 similarity measure

$$w_{NL}(\mathbf{y}, \mathbf{z}) = \frac{1}{Z_{NL}} e^{\left(-\frac{d^2}{h^2}\right)}$$

Non local vs Local



Results: Ramsès Statue



Results: 3D star



Results: Ramses Statue

Original





NLPSS



Conclusions

Contributions



Multiscale Tree of Local Gaussian Models



Denoising, Random Sampling



Perspectives

- Multi-resolution (patch size)
- Optimize building step
 - Hierarchical EM
 - On line learning, data stream
- Robust distances, geodesic distances

Thank you for your attention!
Références

Non Local Point Set Surfaces

Thierry Guillemot, Andrés Almansa et Tamy Boubekeur 3DIMPVT – 3D Imaging Modeling Processing Visualization Transmission – 2012

Covariance Trees for 2D and 3D Processing

Thierry Guillemot, Andrés Almansa et Tamy Boubekeur CVPR – Computer Vision and Pattern Recognition – 2014