

Multiscale analysis of similarities between images on Riemannian manifolds

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Coauthors:

- C. B., F. Calderero, V. Caselles, and G. Facciolo, *Multiscale Analysis of Similarities between Images on Riemannian Manifolds*. Multiscale Modeling and Simulation, 2013 (in press).
- V. Fedorov, P. Arias, R. Sadek, and C. B. On affine invariant image comparison and applications. Preprint 2014.
- V. Fedorov, PhD thesis (2nd year).
- G. Facciolo, F. Calderero, V. Fedorov, C. B., and V. Caselles. Multiscale analysis of similarities between images on Riemannian manifolds. Part II. Preprint, 2013.



Multiscale analysis of similarities between images on Riemannian manifolds

Purpose: to compare two images or two videos



Image comparison

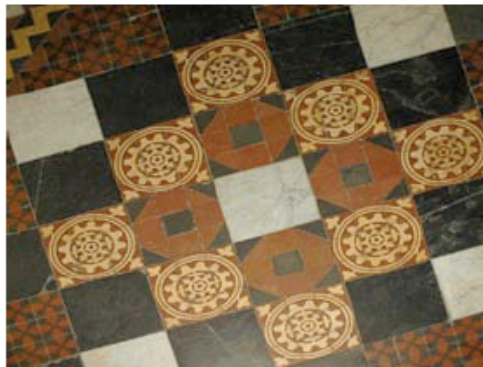
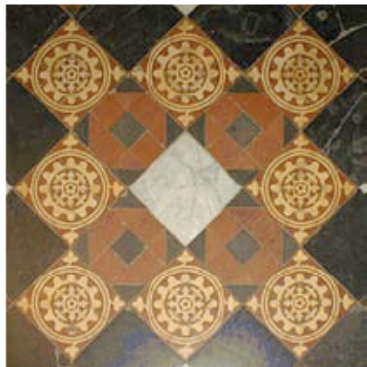


Image comparison



General problem with different objectives:



General problem with different objectives:

Stereo



General problem with different objectives:

Stereo



General problem with different objectives:

Stereo



Left stereo image



Right stereo image

General problem with different objectives



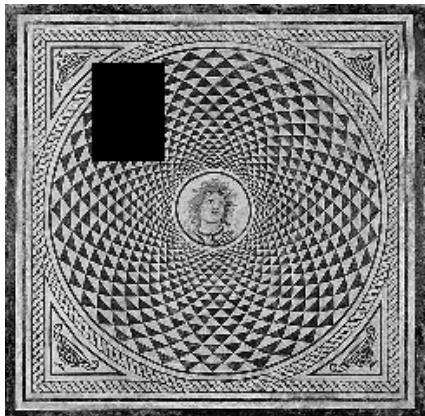
General problem with different objectives

Inpainting



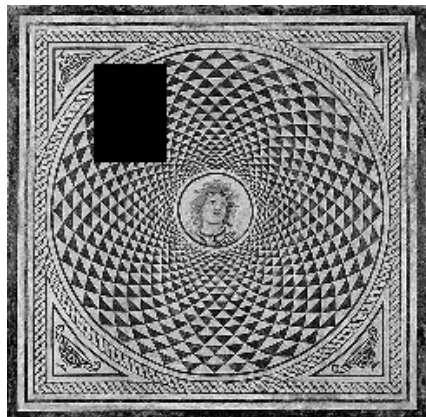
General problem with different objectives

Inpainting



General problem with different objectives

Inpainting



Exemplar-Based Methods, [P. Arias, ...]

- Underlying assumption: image **self-similarity**.
- **Image Patches** as basic units of information



Inpainting

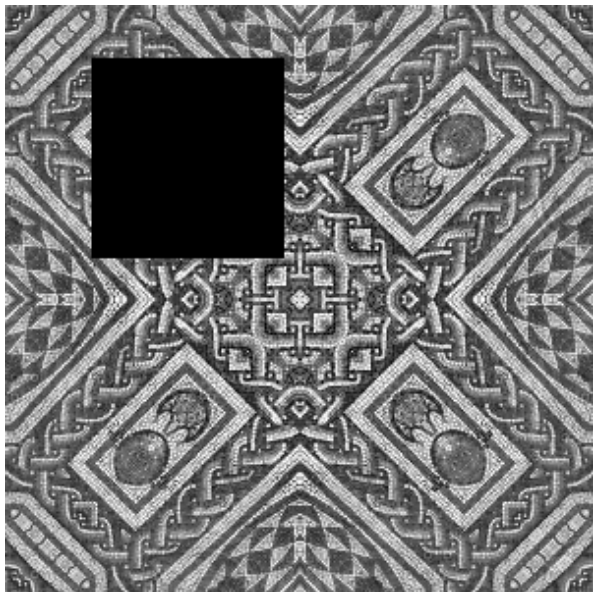


Image source: Vadim Fedorov



Inpainting

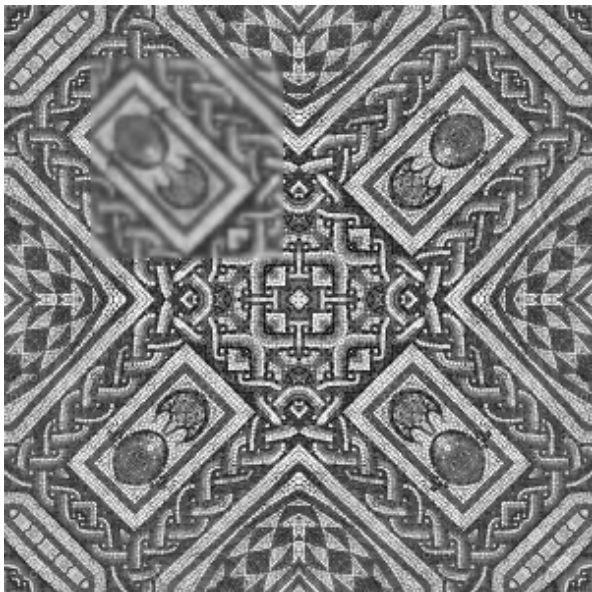


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Inpainting

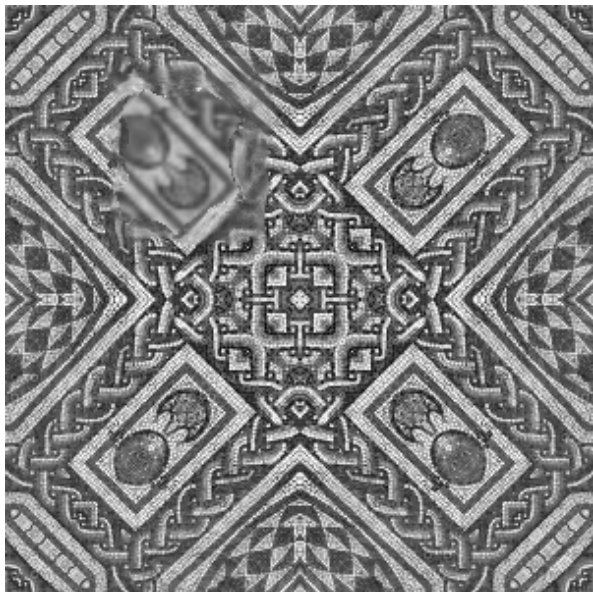


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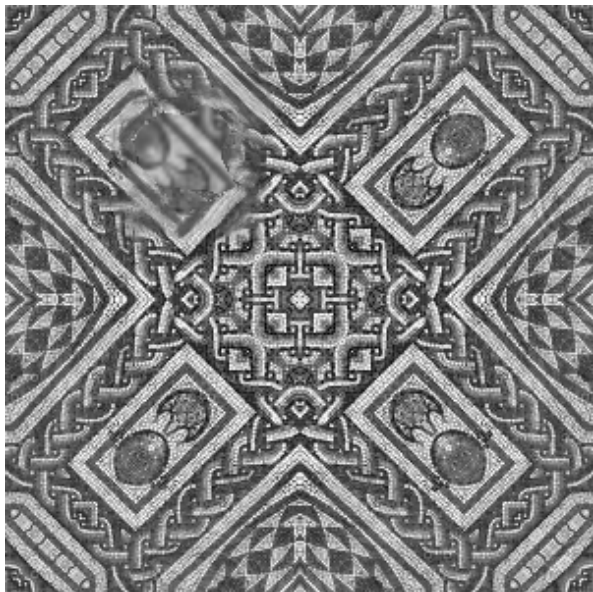


Image source: Vadim Fedorov



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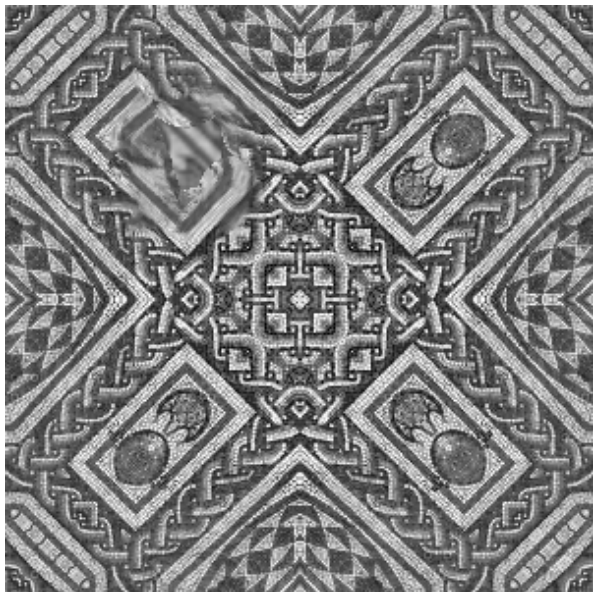


Image source: Vadim Fedorov



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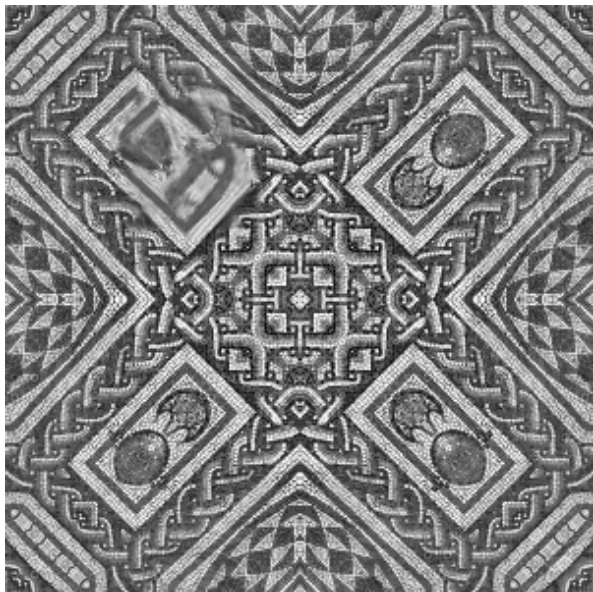


Image source: Vadim Fedorov



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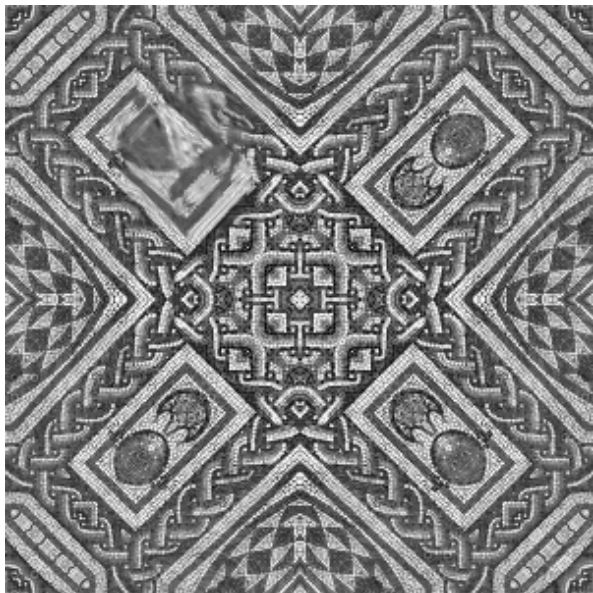


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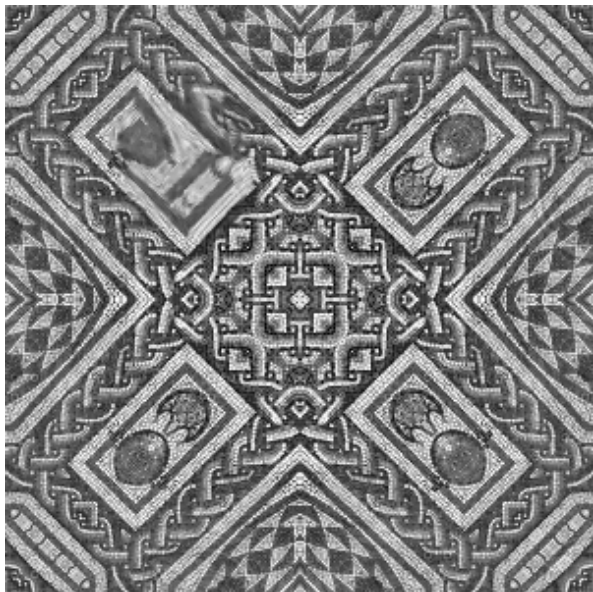


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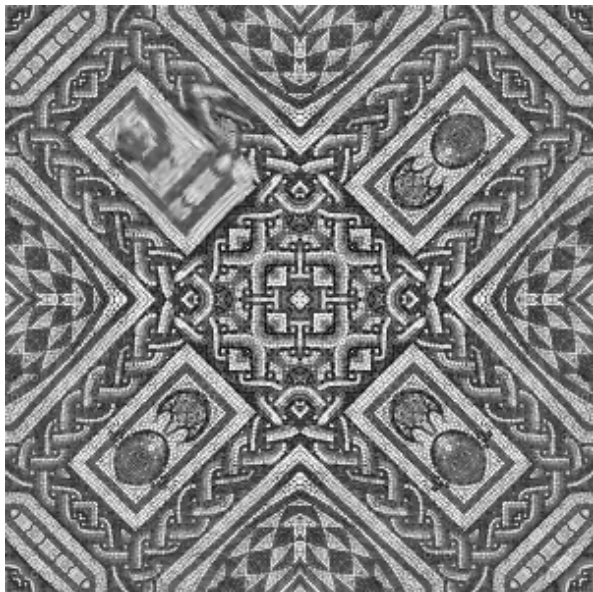


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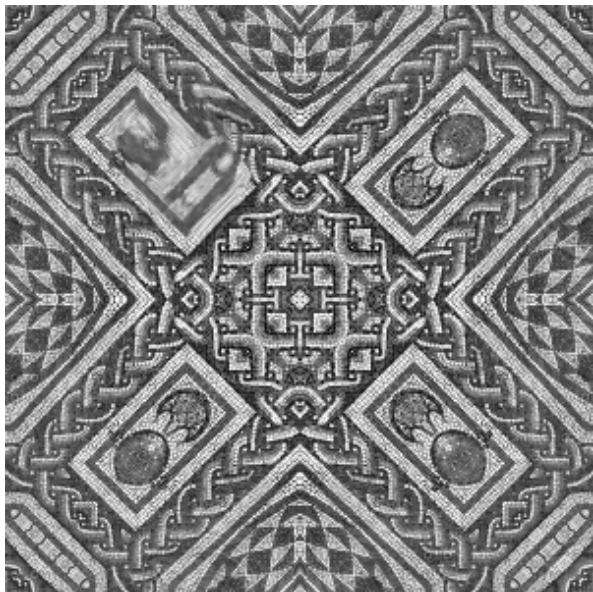


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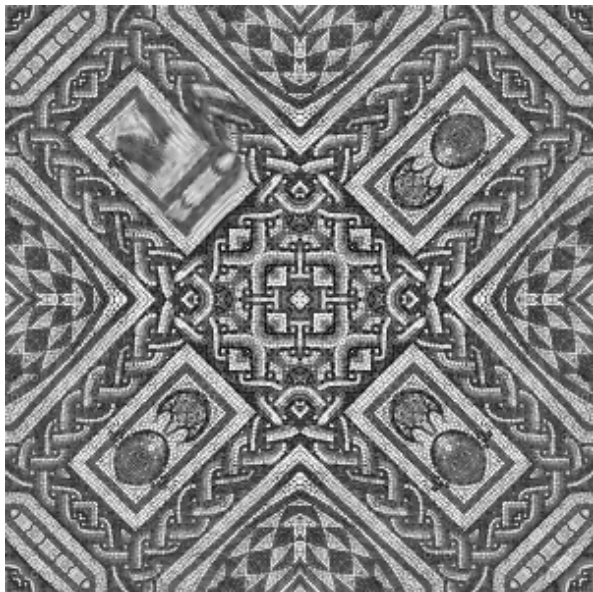


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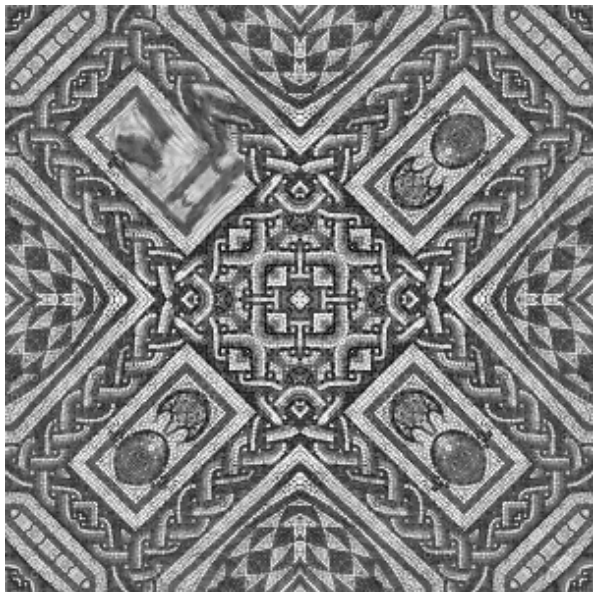


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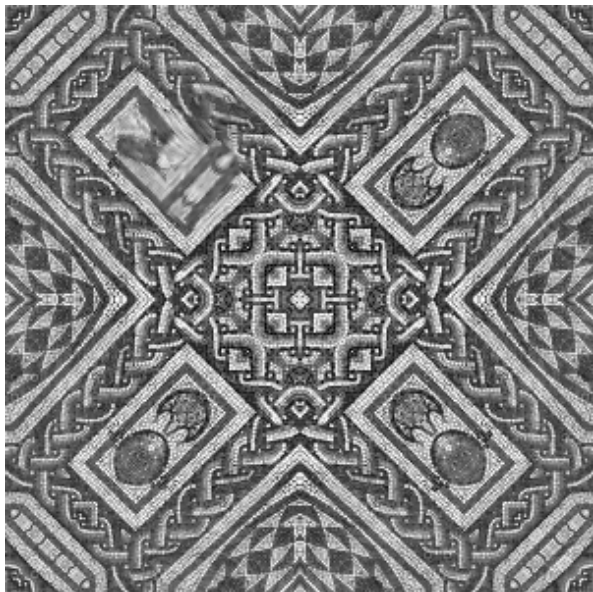


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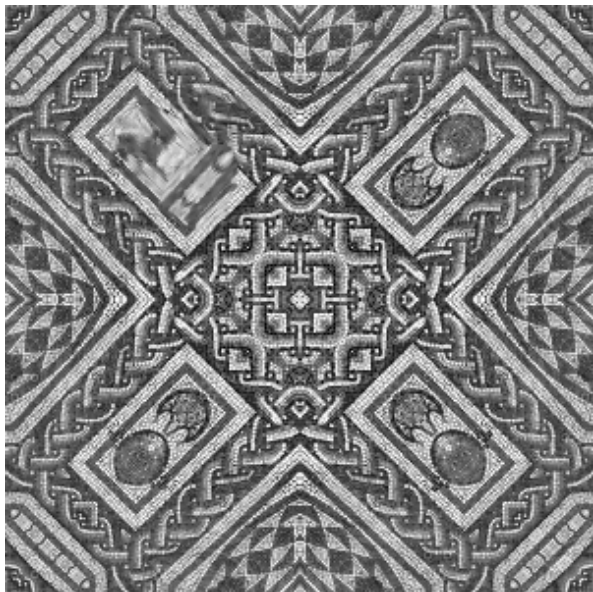


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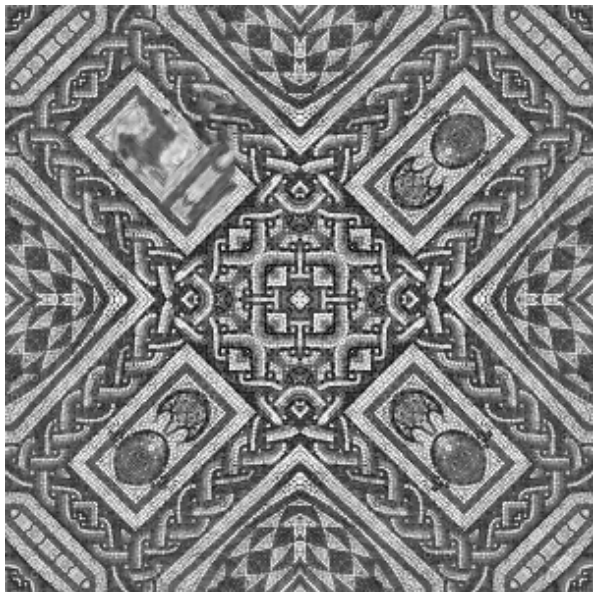


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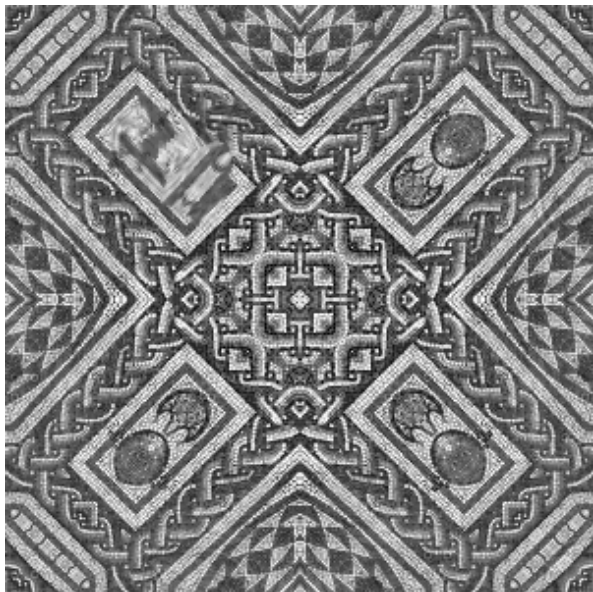


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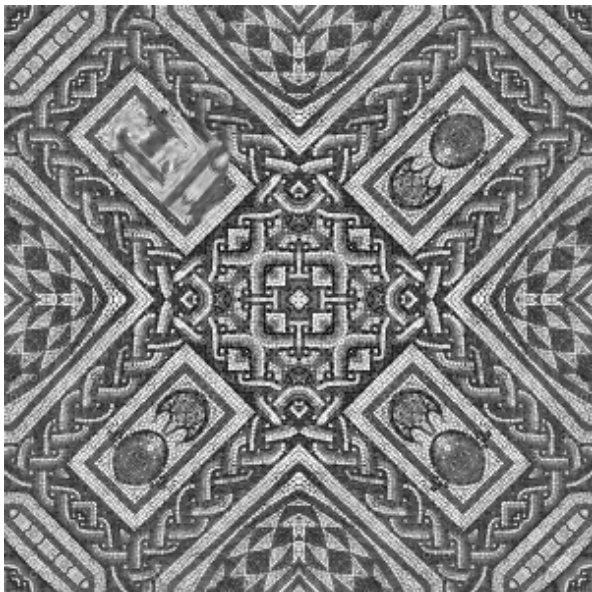


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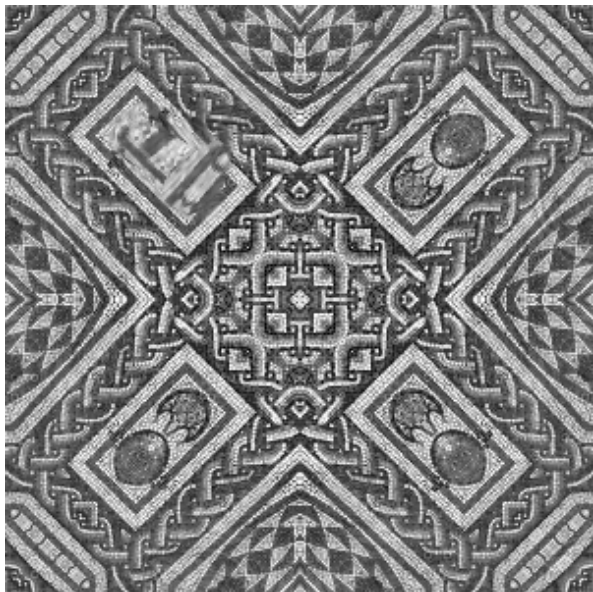


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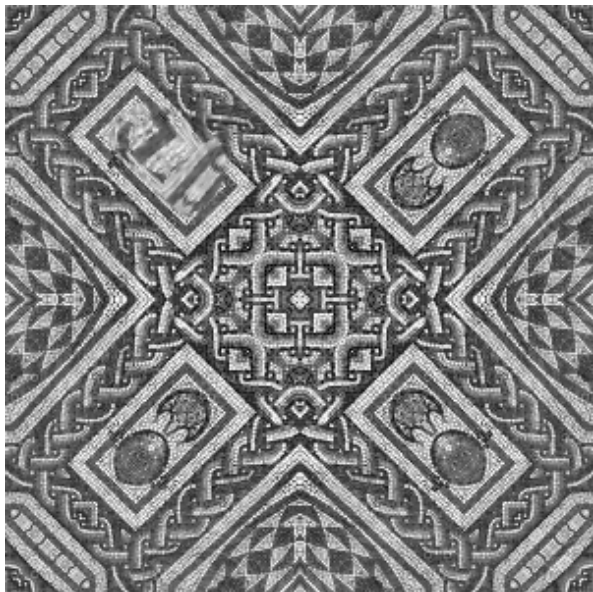


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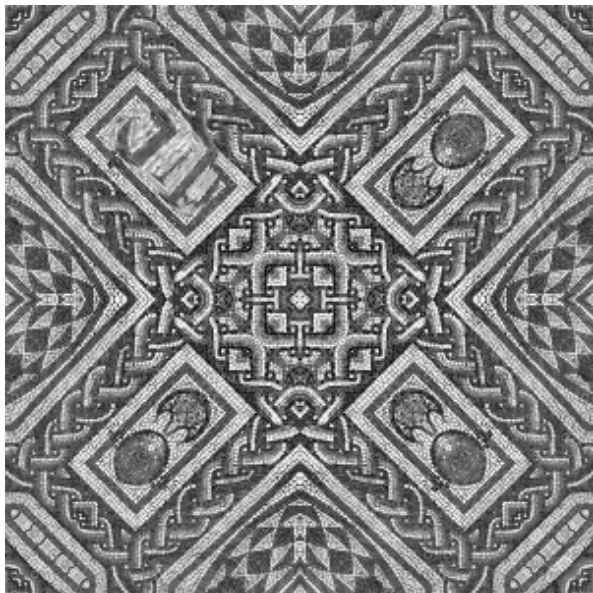


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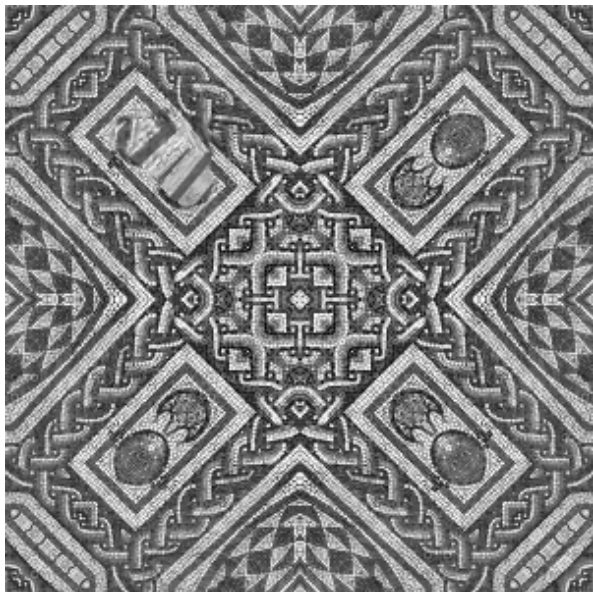


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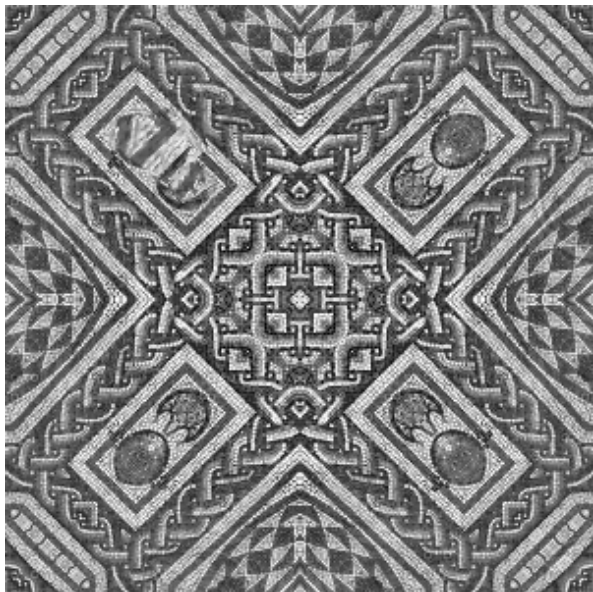


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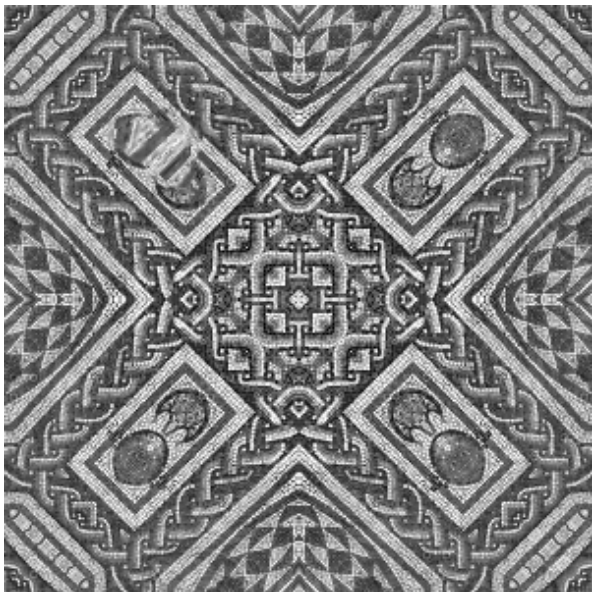


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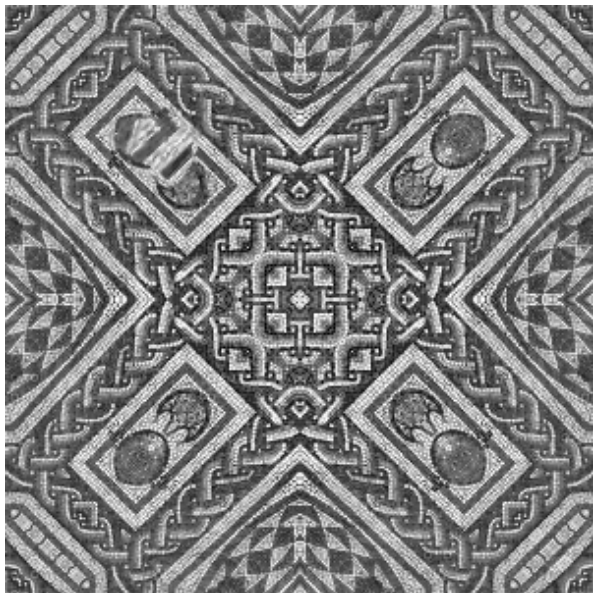


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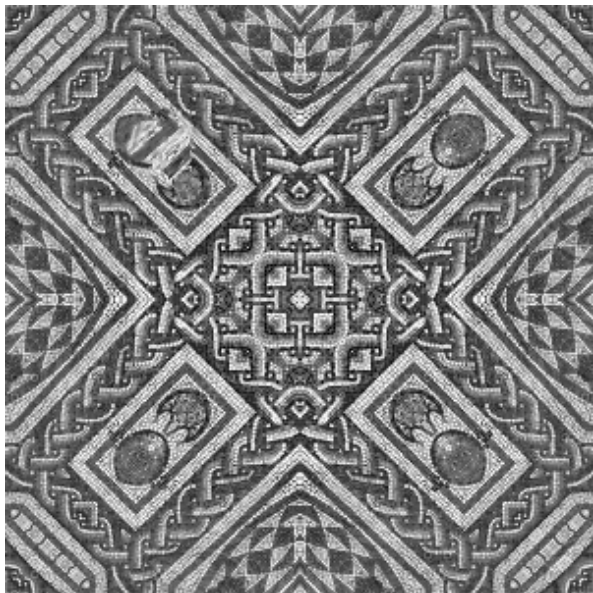


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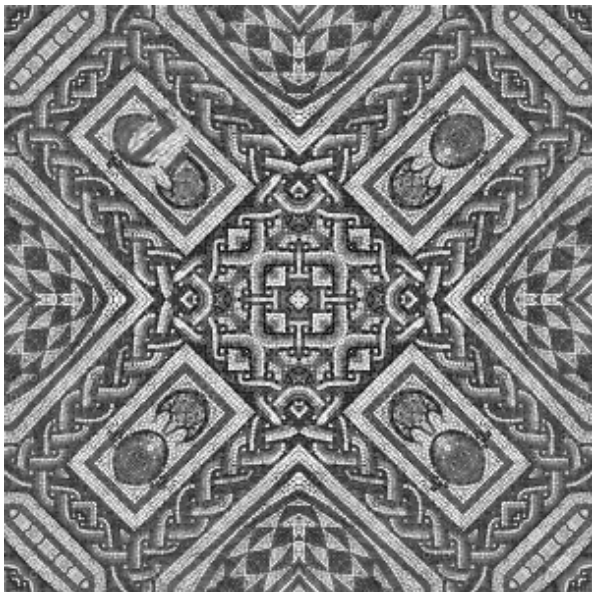


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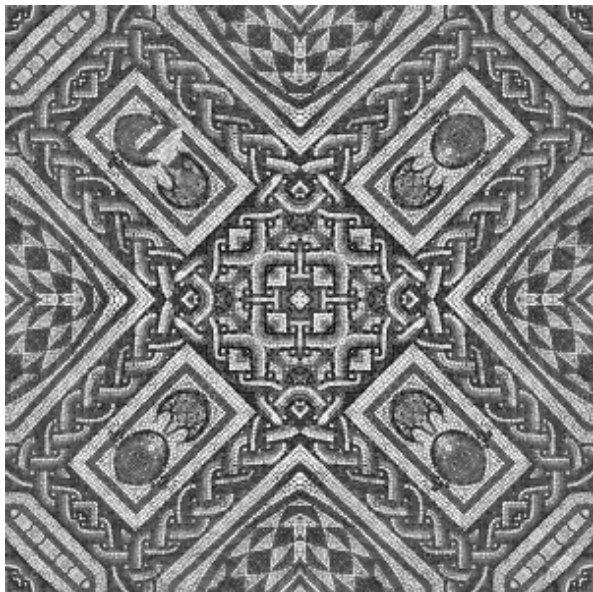


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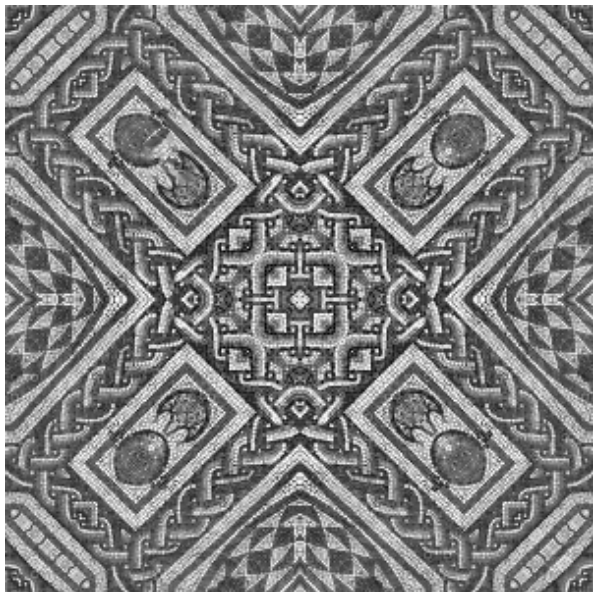


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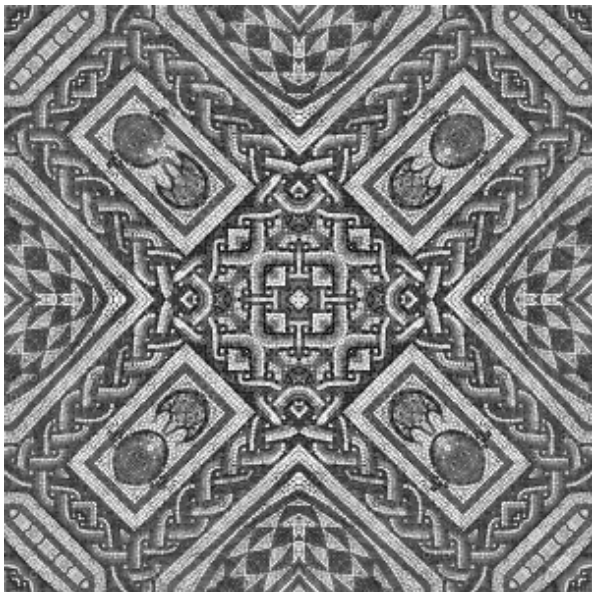


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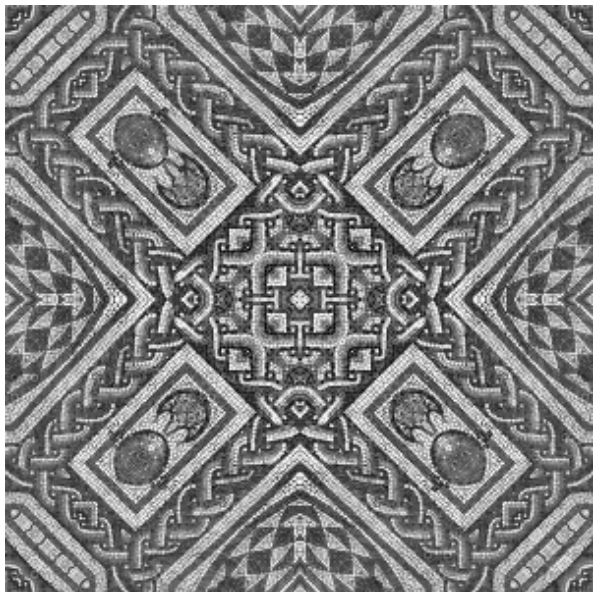


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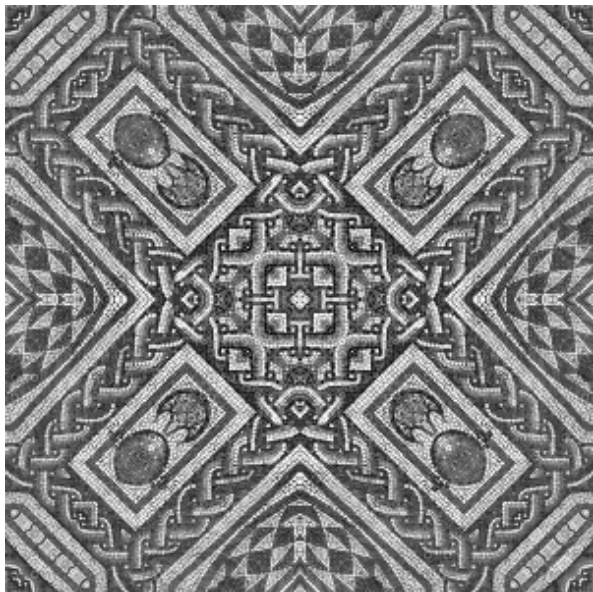


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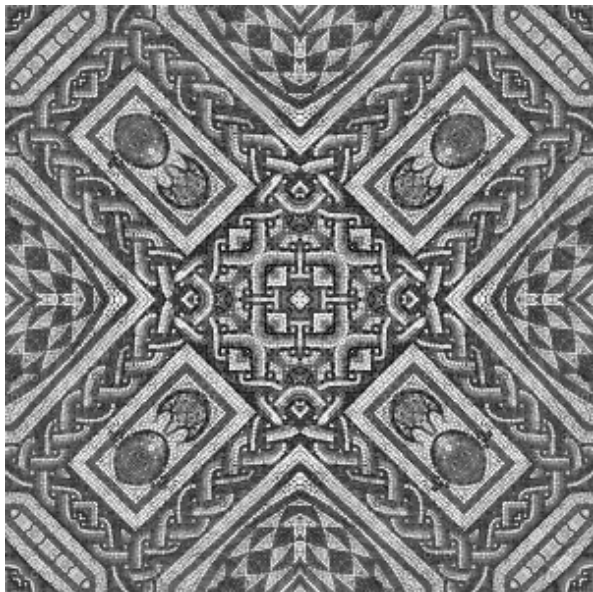


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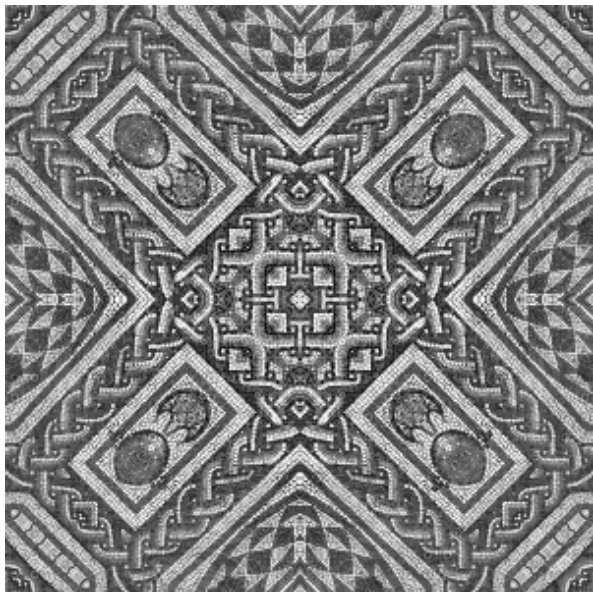


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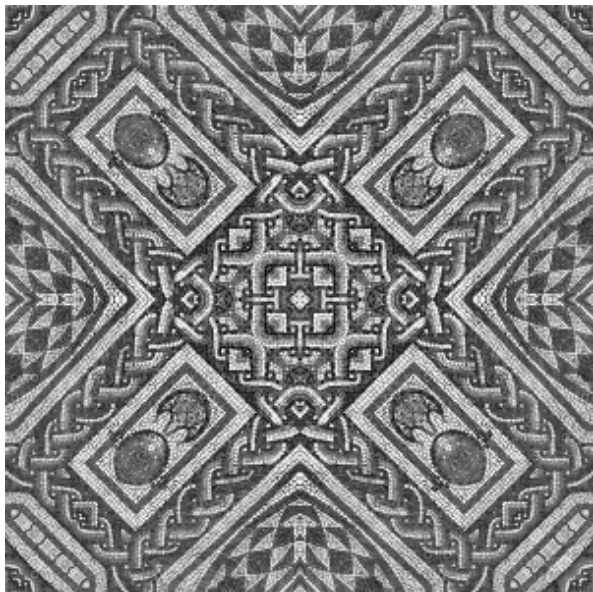


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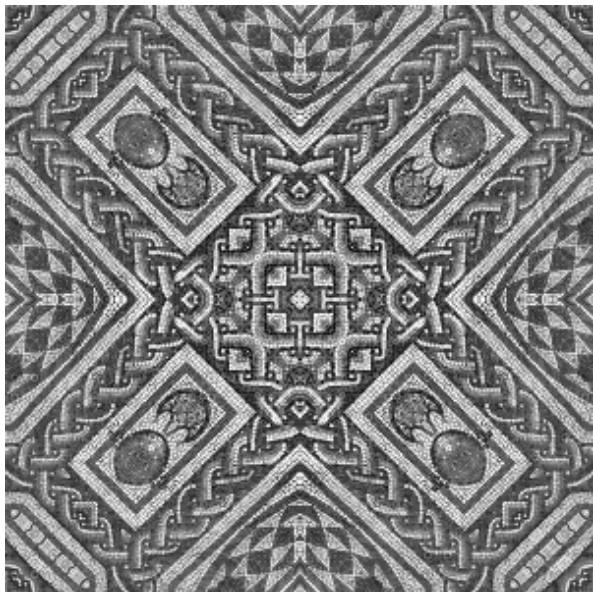


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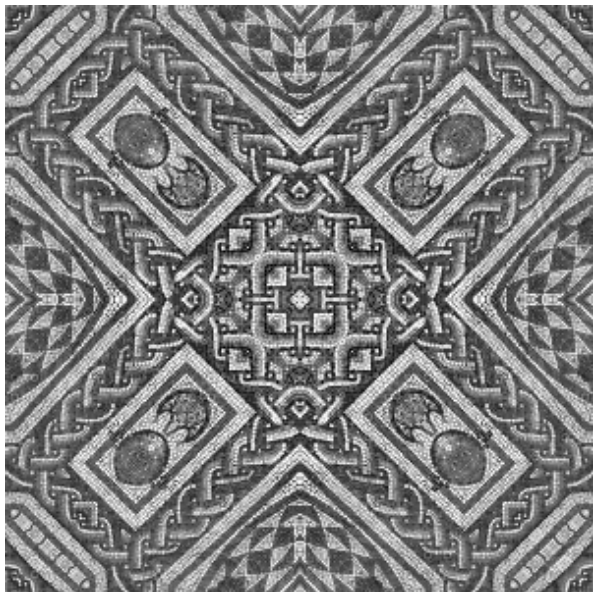


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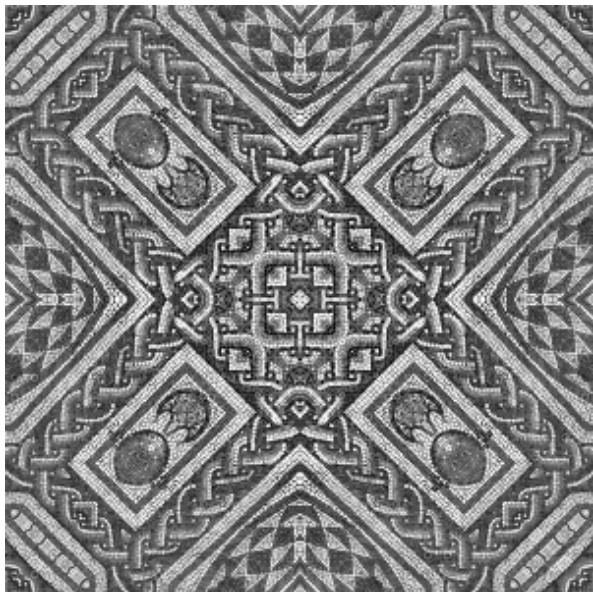


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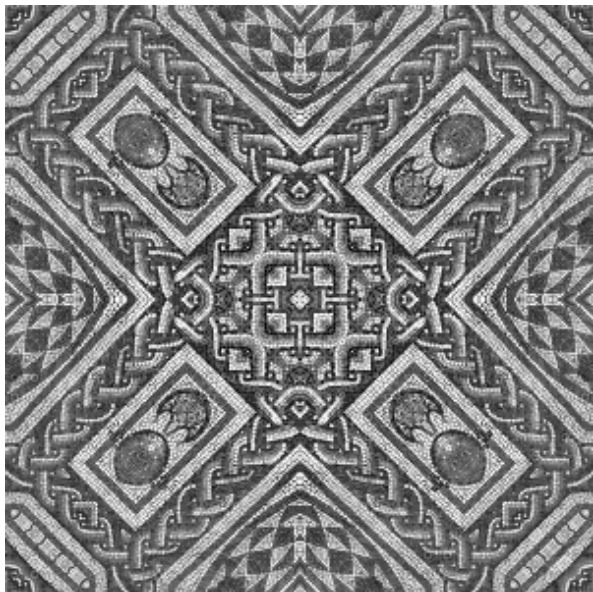


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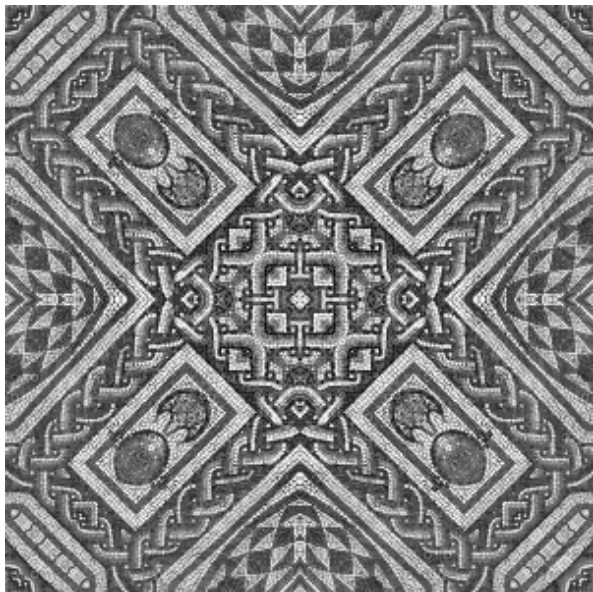


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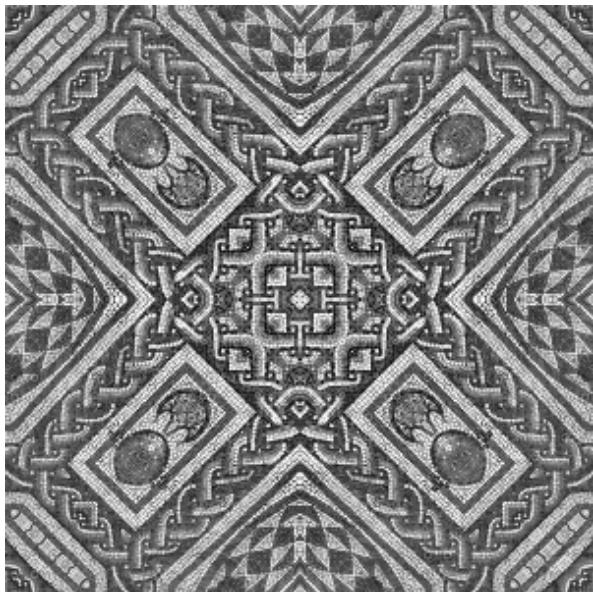


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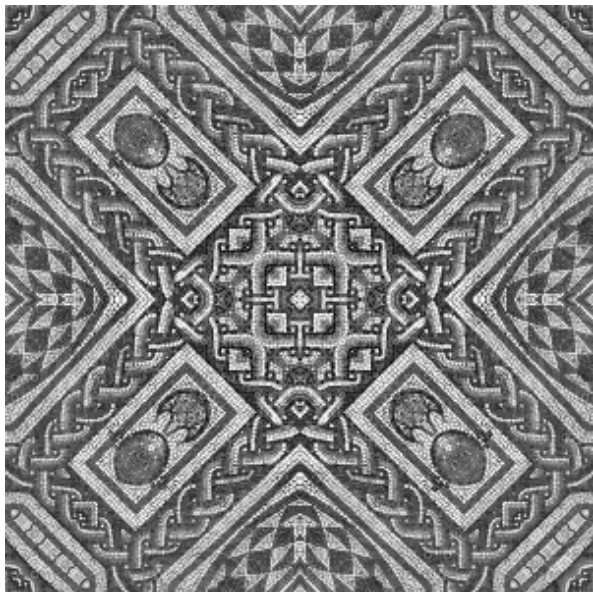


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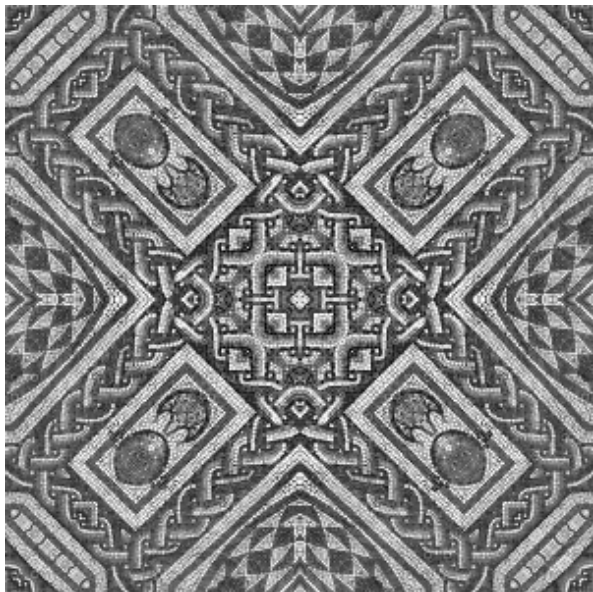


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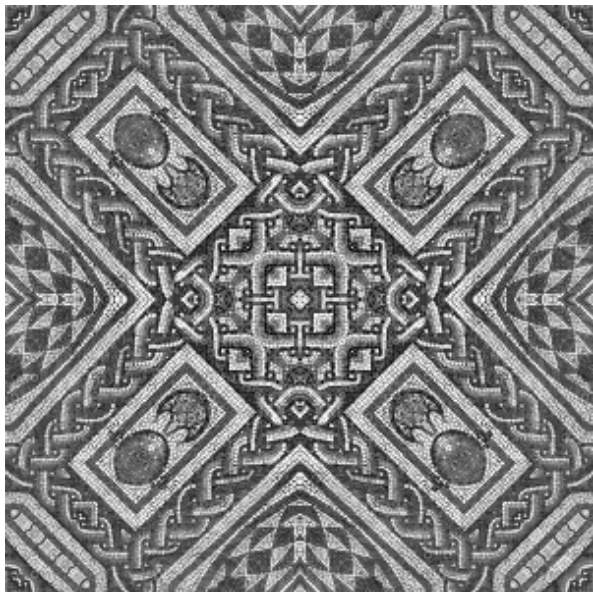


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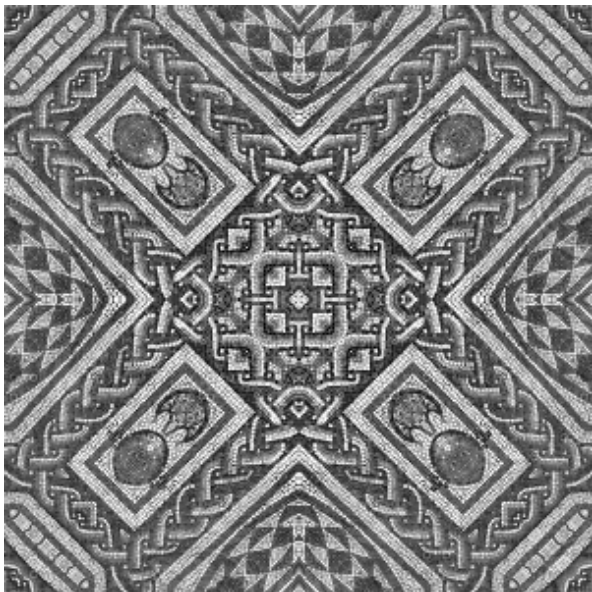


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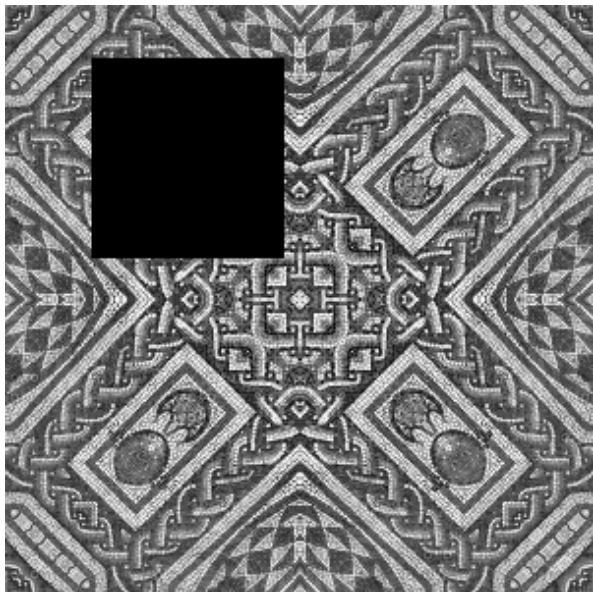


Image source: Vadim Fedorov



Plan of the talk

Plan

- Problem statement and axiomatic approach.
- Basic set of axioms.
- General classification.
- Linear case: the scale spaces are given by multiples of the Laplace-Beltrami flow.
- Case of morphological scale spaces: they are given in terms of geometric equations.
- Some experiments.



Multiscale analysis of similarities between images on Riemannian manifolds

- **Purpose: to compare images (or videos) by computing the similarity of patches defined on the images (or on the videos).**



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- We will consider images defined on Riemannian manifolds, and we will define **a multiscale comparison of the images defined on Riemannian manifolds.**



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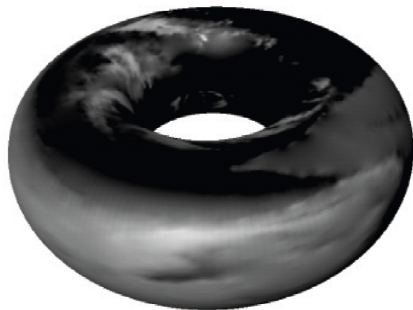
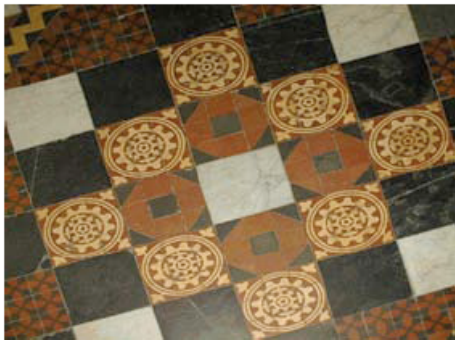
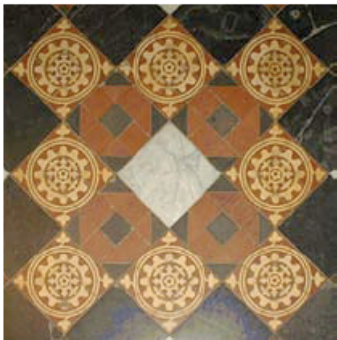


Image source: [V. Caselles & L. Izual & O. Sander. An Axiomatic Approach to Scalar Data Interpolation on Surfaces, 2006]



Multiscale analysis of similarities between images on Riemannian manifolds

- **Purpose:** to compare images (or videos) by computing the similarity of patches defined on the images (or on the videos).
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Multiscale analysis of similarities between images on Riemannian manifolds

- We will define **multiscale comparison measures** in the case of images defined on Riemannian manifolds.
- We will follow an **axiomatic approach**.
- Our approach can be considered as **a nonlocal extension (comparing two points) of the multiscale analyses** defined using the axiomatic approach in
 - Alvarez, L. and Guichard, F. and Lions, P.L. and Morel, J.M., *Axioms and fundamental equations of image processing*, 1993.



Multiscale analysis

- A multiscale analysis represents a given image at different scales of smoothing, the scale being related to the size of the neighborhood which is used to give an estimate of the brightness of the picture at a given point. It is a basic preprocessing step for shape recognition [Lowe, 2004], [Guichard-Morel, 2001], [Cao-Lisani-Morel-Musé-Sur, 2008],...
- The systematic study of multiscale analyses for images was the purpose of the axiomatic approach proposed in [Alvarez-Guichard-Lions-Morel, 1993].
- Based on a series of axioms which define the structure of the multiscale space and a set of geometric and photometric invariants, multiscale analyses were defined in terms of (viscosity) solutions of a parabolic equation.
- Linear multiscale analysis: Gaussian scale space. Classification covers: Perona-Malik equation, the Rudin-Osher-Fatemi's model, the mean curvature motion. Morphological scale spaces: given in terms of geometric equations that diffuse the level sets of the image with functions of their principal curvatures.



Multiscale analysis for image comparison

- Our purpose is extend the axiomatic approach to define **multiscale comparison measures** in the case of images defined on Riemannian manifolds, which can be defined by the image domain with a suitable metric depending on the image (e.g., the image plane endowed with an anisotropic metric, like the structure tensor [Weickert,1998], [Brox et al,2006], [Peyré,2009]).
- There will appear a large family of possibilities, derived from the axiomatic approach.
- The set of axioms will include **architectural axioms** and **comparison principle** that permit to define multiscale analyses as solutions of a degenerate parabolic PDE.
- The inclusion of geometric invariances will be subsumed under the requirement of intrinsic definition of the multiscale analysis, independent of the parameterization of the manifold. This essentially restricts the invariances to rotation invariance in the tangent plane.



Multiscale analysis of similarities between images on Riemannian manifolds

- From the mathematical point of view, the basic ingredients are the papers
 1. Alvarez, L. and Guichard, F. and Lions, P.L. and Morel, J.M., *Axioms and fundamental equations of image processing*, 1993.
 2. Caselles, V. and Morel, J.M. and Sbert, C., *An axiomatic approach to image interpolation*, 1998.
 3. Caselles, V. and Igual, L. and Sander, O., *An axiomatic approach to scalar data interpolation on surfaces*, 2006.
 4. Calderero, F. and Caselles, V., *Multiscale Analysis for Images on Riemannian Manifolds*, SIAM J. Imaging Sciences 2013 (in press).
- Actually, some of the results were proved in [1] for **multiscale analysis on images** and **extended to images on Riemannian manifolds** in [4].



Multiscale analysis for image comparison

Basic idea: Given two images u and v defined in their respective image domains (assume \mathbb{R}^2 or \mathbb{R}^N), we want to compare their neighborhoods at the points $x, y \in \mathbb{R}^2$, respectively.

The most simple way to compare them would be to compare the two neighborhoods of x, y using the Euclidean distance. That is

$$\mathcal{C}(t, x, y) = \int_{\mathbb{R}^N} g_t(h) (u(x+h) - v(y+h))^2 dh, \quad (1)$$

where g_t is a given window that we assume to be Gaussian of variance t .

This formula gives a simple explicit comparison between the neighborhoods of x, y and assumes that the image domain is the Euclidean plane.

Let us note at this point that we could have also used the integral of $u(x+h)v(y+h)$ as a comparison measure.



Multiscale analysis for image comparison

- In the case of multiscale measures between images defined on Riemannian manifolds, there will appear a large family of possibilities, derived from the axiomatic approach (expressed in terms of solutions of a degenerate parabolic PDE). An (early) **example** of **linear multiscale analysis of similarity measure** is the model

$$\mathcal{C}_t = \text{Tr}(G_1(x)^{-1}D_x^2\mathcal{C}) + 2\text{Tr}(G_1(x)^{-1/2}G_2(x)^{-1/2}D_{xy}^2\mathcal{C}) + \text{Tr}(G_2(y)^{-1}D_y^2\mathcal{C}), \quad (2)$$

where $\mathcal{M}_i = (\mathbb{R}^N, G_i(x))$, $i = 1, 2$ are two Riemannian manifolds associated to two images u, v , and $x \in \mathcal{M}_1$, $y \in \mathcal{M}_2$.

- In particular, if we assume that the metrics are constant in both images, then the model becomes

$$\mathcal{C}_t = \text{Trace}(A^t A D_x^2 \mathcal{C}) + 2\text{Trace}(A B^t D_{xy} \mathcal{C}) + \text{Trace}(B^t B D_y^2 \mathcal{C}). \quad (3)$$

where A, B are two square matrices. The **multiscale similarity measure**

$$\mathcal{C}(t, x, y) = \int_{\mathbb{R}^N} g_t(h) \mathcal{C}(0, x + Ah, y + Bh) dh,$$

between the patches centered at x and y , where g_t is the Gaussian of scale t , and $\mathcal{C}(0, x, y) = (u(x) - v(y))^2$, satisfies equation (3).



Multiscale analysis for image comparison

Remark The Euclidean comparison measure

$$\mathcal{C}(t, x, y) = \int_{\mathbb{R}^N} g_t(h) (u(x+h) - v(y+h))^2 dh, \quad (4)$$

is not an exception: it solves the equation

$$\mathcal{C}_t = \Delta_x \mathcal{C} + 2\text{Trace}(D_{xy}^2 \mathcal{C}) + \Delta_y \mathcal{C}, \quad (5)$$

which is possibly the simplest case of linear PDE expressing the multiscale comparison of two image patches.



Some notation and definitions

- Let (\mathcal{N}, h) be a **smooth Riemannian manifold** in \mathbb{R}^{N+1} .
- Particular case: $\mathcal{N} = \mathbb{R}^N$ (or a domain in \mathbb{R}^N) endowed with a general metric h_{ij} .
- Given a point $\eta \in \mathcal{N}$, we denote by $T_\eta \mathcal{N}$ the tangent space to \mathcal{N} at the point η . By $T_\eta^* \mathcal{N}$ we denote its dual space.
- **Rotations in the tangent space** Let us define a rotation $R : T_\eta \mathcal{N} \rightarrow T_\eta \mathcal{N}$ as a linear map that satisfies

$$\langle Rv, Rw \rangle = \langle v, w \rangle \quad \forall v, w \in T_\eta \mathcal{N}.$$

Notice that rotations satisfy $R^t H R = H$.

- We will consider the **manifold** $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$.



Some notation and definitions

- The **manifold** $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$.

Let (\mathcal{M}^i, g^i) be a smooth Riemannian manifold with metric g^i , $i = 1, 2$. Let $\Gamma^{(i)}$ be the connection (or Christoffel symbols) on \mathcal{M}^i . We shall work here with a manifold $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$ with the metric $h = g^1 \times g^2$, so that $T_\xi \mathcal{N} = T_{\xi_1} \mathcal{M}^1 \times T_{\xi_2} \mathcal{M}^2$, $\xi = (\xi_1, \xi_2) \in \mathcal{M}^1 \times \mathcal{M}^2$. If $(v_i, w_i) \in T_{\xi_1} \mathcal{M}^1 \times T_{\xi_2} \mathcal{M}^2$, $\xi = (\xi_1, \xi_2) \in \mathcal{M}^1 \times \mathcal{M}^2$, then we consider the metric

$$\langle (v_1, w_1), (v_2, w_2) \rangle_\xi = \langle v_1, v_2 \rangle_{\xi_1} + \langle w_1, w_2 \rangle_{\xi_2} = (G^1(\xi_1)v_1, v_2) + (G^2(\xi_2)w_1, w_2).$$

With a slight abuse of notation, let us write $G(\xi) = \text{diag}(G^1(\xi_1), G^2(\xi_2))$.



A priori connections on $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$

Key concept in our results.

Idea: Suppose that both manifolds \mathcal{M}^1 and \mathcal{M}^2 coincide with \mathbb{R}^N endowed with the Euclidean metric. Let u, v be two given images in \mathbb{R}^N . Then it would be standard to use the L^2 distance to compare the patches centered at x and y ,

$$\mathcal{C}(t, x, y) = \int_{\mathbb{R}^N} g_t(h) (u(x+h) - v(y+h))^2 dh, \quad (6)$$

where g_t is a given window that we assume to be Gaussian of variance t . But **if the image v is rotated** we could also use the L^2 distance between u and a rotated version of v (around y), namely

$$\mathcal{C}(t, x, y) = \int_{\mathbb{R}^N} \mathbf{g}_t(\mathbf{h}) (\mathbf{u}(\mathbf{x} + \mathbf{h}) - \mathbf{v}(\mathbf{y} + \mathbf{R}\mathbf{h}))^2 d\mathbf{h}. \quad (7)$$

We admit that this decision is taken a priori and is done **thanks to an operator that connects the tangent plane at both points.**



A priori connections on $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$

Definition. We say that $P(\xi)$, $\xi = (\xi_1, \xi_2) \in \mathcal{N}$, is an **a priori connection map in \mathcal{N}** if $P(\xi) : (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2))$ is an isometry, i.e.

$$\langle P(\xi)v, P(\xi)w \rangle_{G^2(\xi_2)} = \langle v, w \rangle_{G^1(\xi_1)} \quad \forall v, w \in T_{\xi_1}\mathcal{M}^1,$$

and we assume also that the map is differentiable in ξ .

Given an a priori connection $P(\xi) : (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2))$, we can also define its inverse $P(\xi)^{-1} : (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2)) \rightarrow (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1))$.



A priori connections on $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$

- **Case** $\mathcal{M}^1 = \mathcal{M}^2 = \mathcal{M}$ (orientable). We can consider $P(\xi)$ as an internal a priori connection (into itself) given from parallel transport between ξ_1 and ξ_2 , which is an isometry.
- In coordinates, $P(\xi)$ expresses the a priori connection in the coordinate system $\psi_1 \rightarrow \psi_2$. The **isometry property** can be written as

$$(P(\xi)^t G^2(\xi_2) P(\xi) v, w) = (G^1(\xi_1) v, w),$$

where $P(\xi)$ is expressed in the basis of $T_{\xi_1} \mathcal{M}^1$ associated to the metric $G^1(\xi_1)$ and the basis of $T_{\xi_2} \mathcal{M}^2$ associated to the metric $G^2(\xi_2)$. Then

$$P(\xi)^t G^2(\xi_2) P(\xi) = G^1(\xi_1). \quad (8)$$



A priori connections on $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$

In another coordinate system:

If $P(\xi)$ is an a priori connection in the coordinate system $\psi_1 \rightarrow \psi_2$ then

$$\bar{P}(\xi) := B_{G^2, \bar{G}^2}(\xi_2)^{-1} P(\xi) B_{G^1, \bar{G}^1}(\xi_1), \quad (9)$$

is an a priori connection in the coordinate system $\bar{\psi}_1 \rightarrow \bar{\psi}_2$.

where $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$ is another coordinate system around ξ ,

$G^i(\xi_i), \bar{G}^i(\xi_i), i = 1, 2$, be the metric matrices represented in the coordinate system $\psi_i, \bar{\psi}_i$, respectively, $B_{G^i, \bar{G}^i}(\xi_i) = D(\psi_i^{-1} \circ \bar{\psi}_i)(0)$, $i = 1, 2$, and $B_{G, \bar{G}}(\xi) = (B_{G^1, \bar{G}^1}(\xi_1), B_{G^2, \bar{G}^2}(\xi_2))$.

Definition

We say that the coordinate systems $\psi, \bar{\psi}$ are **P(ξ)-related** if $\bar{P}(\xi)$ is defined by (9).



(Last) notations

- **Related germs of functions on $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$.** Let $C_b(\mathcal{N})$ denote the space of bounded continuous functions in \mathcal{N} with the maximum norm. We think of $C_b(\mathcal{N})$ as the space of similarity functions on $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$. We denote by $C_b^\infty(\mathcal{N})$ the space of infinitely differentiable functions on \mathcal{N} . Let $C \in C_b(\mathcal{N})$. Let us denote $(C, \psi)(x, y) = C(\psi_1(x), \psi_2(y)), \quad \forall (x, y) \in U_1 \times U_2$.

- Gradient and Hessian.

We denote by $SM_\xi(\mathcal{N})$ the set of symmetric matrices of size $2N \times 2N$ in $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$. In coordinates, we denote $D_{\mathcal{N}}C = (D_x C, D_y C)$,

$$D_{\mathcal{N}}^2 C = \begin{pmatrix} D_{\mathcal{N},xx} C & D_{\mathcal{N},xy} C \\ D_{\mathcal{N},xy} C & D_{\mathcal{N},yy} C \end{pmatrix}.$$

In coordinates, with $i, j, k \in \{1, \dots, N\}$,

$$D_{\mathcal{N}}^2 C = \begin{pmatrix} \frac{\partial^2 C}{\partial x^i \partial x^j} & \frac{\partial^2 C}{\partial x^i \partial y^j} \\ \frac{\partial^2 C}{\partial y^j \partial x^i} & \frac{\partial^2 C}{\partial y^j \partial y^k} \end{pmatrix} = \begin{pmatrix} \Gamma^{(1)k}(x) \frac{\partial C}{\partial x^k} & 0 \\ 0 & \Gamma^{(2)k}(y) \frac{\partial C}{\partial y^k} \end{pmatrix}.$$

$\mathcal{Q}((\kappa)) := \{C \in C_b^\infty(\mathcal{N}) : \|D^\alpha C\|_\infty \leq \kappa_n \quad \forall n \geq 0 \quad \forall |\alpha| \leq n\}$, where $(\kappa) := \kappa_n$ be an increasing sequence of non-negative constants.



Axioms. Architectural axioms

[Recursivity] $T_0(C) = C$, $T_s(T_t C) = T_{s+t} C$, $\forall s, t \geq 0$, $\forall C \in C_b(\mathcal{N})$.

The recursivity axiom is a strong version of causality which implies that the similarity measure at a coarser scale can be deduced from a finer one, which is a natural property in image analysis and a sound hypothesis in human vision [Alvarez-Guichard-Lions-Morel], [...].



Architectural axioms

[Infinitesimal generator] $\frac{T_h(C, \psi)(\xi) - (C, \psi)(\xi)}{h} \rightarrow (\mathcal{A}(C), \psi)$ as $h \rightarrow 0+$ for any $C \in C_b^\infty(\mathcal{N})$ and any coordinate system $\psi = (\psi_1, \psi_2)$ around ξ . We assume that

$$T_t(R(C, \psi))(\xi) = R(T_t(C), \psi)(\xi) + o(t) = T_t(C)(\xi) + o(t) \quad \text{as } t \rightarrow 0+$$

for any $C \in C_b(\mathcal{N})$, any coordinate system $\psi = (\psi_1, \psi_2)$, and any R which are P -related rotations (i.e.,). We have denoted by $R(C, \psi)$ the function in the coordinate system ψ which is $P(\xi)$ -related (or R -related) to ψ .

It can be shown $\mathcal{A}(R(C, \psi))(0) = R\mathcal{A}(C, \psi)(0) = \mathcal{A}(C, \psi)(0)$ for any $C \in C_b(\mathcal{N})$, any coordinate system $\psi = (\psi_1, \psi_2)$, and any R P -related rotations.

Remark. It can be shown that **the Infinitesimal generator axiom contains the invariance with respect to diagonal rotations in the tangent plane of $\mathcal{M}^1 \times \mathcal{M}^2$.**

When $(\mathcal{M}^i, G^i) = (\mathcal{M}, G) = (\mathbb{R}^N, I)$ it amounts to invariance with respect to Euclidean diagonal rotations in \mathbb{R}^{2N} . That is, $T_t(RC) = RT_t(C) \forall t \geq 0$,

$\forall C \in C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $\forall R \in O(N)$ (Euclidean rotations in \mathbb{R}^N) where $RC(x, y) = C(Rx, Ry)$.



Architectural axioms

[Regularity axiom] $\|T_t(C + h\tilde{C}) - (T_t(C) + h\tilde{C})\|_\infty \leq Mht$
 $\forall h, t \in [0, 1], \forall C, \tilde{C} \in \mathcal{Q}(\kappa)$ where the constant M depends on $\mathcal{Q}(\kappa)$.

[Locality] $T_t(C)(x) - T_t(\tilde{C})(x) = o(t)$ as $t \rightarrow 0+$, $x \in \mathbb{R}^N$,
 $\forall C, \tilde{C} \in C_b(\mathcal{N})$ such that $D^\alpha C(x) = D^\alpha \tilde{C}(x)$ for all multiindices α .



Comparison principle

[Comparison principle] $T_t C \leq T_t \tilde{C} \quad \forall t \geq 0$ and all $C, \tilde{C} \in C_b^\infty(\mathcal{N})$ such that $C \leq \tilde{C}$.

The comparison principle is an order-preserving property. It means that if a similarity measure is always smaller than another, then applying a multiscale analysis does not invert this relation. Intuitively, the multiscale analysis produces low resolution versions of the similarity measures, which should be consistent with the initial ones [Alvarez-Guichard-Lions-Morel], [...].



Morphological axioms

[Gray level shift invariance] $T_t(0) = 0$, $T_t(C + \kappa) = T_t(C) + \kappa$
 $\forall t \geq 0$, $\forall C \in C_b^\infty(\mathcal{N})$, $\forall \kappa \in \mathbb{R}$.

[Gray scale invariance] $T_t(f(C)) = f(T_t(C)) \forall t \geq 0$, $\forall C \in C_b^\infty(\mathcal{N})$,
and for any strictly increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$.

- The inclusion of geometric invariances will be subsumed under the requirement of intrinsic definition of the multiscale analysis, independent of the parameterization of the manifold. This essentially restricts the invariances to rotation invariance in the tangent plane. The consideration of other geometric invariance (translation or rotation) will be discussed as a separate chapter for images defined in \mathbb{R}^N , out of the general classification.



Multiscale analysis of image similarity measures

Theorem

Let T_t be a multiscale analysis satisfying the **Recursivity, Infinitesimal generator and Regularity axioms**. Then $\mathcal{A}(C_r) \rightarrow \mathcal{A}(C)$ in $C_b(\mathcal{N})$ if $C_r, C \in C_b^\infty(\mathcal{N})$ and $D^\alpha C_r \rightarrow D^\alpha C$ in $C_b(\mathcal{N})$ for all α with $|\alpha| \geq 0$.

The proof follows the same lines of the corresponding result in [Alvarez-Guichard-Lions-Morel,1993], and also the same result for manifolds in [Calderero-Caselles, 2013].



Multiscale analysis of image similarity measures

Theorem

Let T_t be a multiscale analysis satisfying all **Architectural axioms**, and the **Comparison principle**. Then there exists a function

$F : SM_\xi(\mathcal{N}) \times T_\xi^* \mathcal{N} \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}$ increasing with respect to its first argument such that

$$\frac{T_t(C, \psi) - (C, \psi)}{t} \rightarrow F(D^2(C \circ \psi)(0), D(C \circ \psi)(0), C(\xi), \xi, G, \Gamma^k) \text{ in } C_b(\mathcal{N}) \text{ as } t \rightarrow 0$$

for all $C \in C_b^\infty(\mathcal{N})$, ψ being a coordinate system around $\xi \in \mathcal{N}$. The function F is continuous in its first three arguments.

If we assume that T_t is **gray level shift invariant**, then the function F does not depend on C .

(We have denoted $G = (G^1, G^2)$ and $\Gamma = \Gamma^{(1)} \otimes \Gamma^{(2)}$. We did not denote explicitly the arguments for G, Γ^k . The first argument in F is a symmetric map from $T_\xi \mathcal{N}$ to $T_\xi^* \mathcal{N}$.)



Multiscale analysis of image similarity measures

Lemma

Let $\xi \in \mathcal{N}$, and $\psi : U \rightarrow \mathcal{N}$ be a coordinate system around ξ . Let G, Γ^k be the metric coefficients and the Christoffel symbols of \mathcal{N} in the coordinate system ψ at the point ξ . Let $A_1, A_2 : T_\xi \mathcal{N} \rightarrow T_\xi^* \mathcal{N}$ be two matrices such that A_1, A_2 are symmetric, $p \in T_\xi^* \mathcal{N}$, $c \in \mathbb{R}$. If $A_1 \leq A_2$, then

$$F(A_1, p, c, \xi, G, \Gamma^k) \leq F(A_2, p, c, \xi, G, \Gamma^k).$$

Thus F is **elliptic**.

Theorem

Let T_t be a multiscale analysis satisfying the all **Architectural axioms**, the **Comparison principle**, and **Gray level shift invariance**. If $C(t, \xi) = T_t C(\xi)$, then C is a viscosity solution of

$$\mathbf{C}_t = \mathbf{F}(\mathbf{D}_{\mathcal{N}}^2 \mathbf{C}, \mathbf{DC}, \xi, \mathbf{G}, \Gamma^k), \quad (10)$$

with $\mathbf{C}(\mathbf{0}, \xi) = \mathbf{C}(\xi)$.



Theorem

Let T_t be a multiscale analysis on \mathcal{N} satisfying the **Architectural axioms** and the **Comparison principle**. Let $\psi = (\psi_1, \psi_2) : U = U_1 \times U_2 \rightarrow \mathcal{N}$ be a coordinate systems around $\xi \in \mathcal{N}$. Let G, Γ be the metric coefficients and the Christoffel symbols of \mathcal{N} in the coordinate system ψ at the point ξ . For any symmetric matrix $X = (X_{ij}) : (T_\xi \mathcal{N}, I) \rightarrow (T_\xi^* \mathcal{N}, I)$ in $SM_\xi(\mathcal{N}, I)$, $q \in (T_\xi^* \mathcal{N}, I)$, and $a \in \mathbb{R}$, let us define the function

$$\mathcal{H}(X, q, a, \xi) = F(X, q, a, \xi, I, 0), \quad (11)$$

that is, \mathcal{H} is the function **F** obtained when using a geodesic coordinate system. Then

$$F(A, p, a, \xi, G, \Gamma^k) = \mathcal{H}(B^t(A - \Gamma(p))B, B^t p, c, \xi) \quad (12)$$

for any matrix $A \in SM_\xi(\mathcal{N})$, and any covector p , where $BB^t = G^{-1}$. Moreover the function H satisfies

$$\mathcal{H}(A', p', c, \xi) = \mathcal{H}(R^t A' R, R^t p', c, \xi). \quad (13)$$

where $A' : (T_\xi \mathcal{N}, I) \rightarrow (T_\xi^* \mathcal{N}, I)$ is any matrix in $SM_\xi(\mathcal{N}, I)$, $p' \in (T_\xi^* \mathcal{N}, I)$, and R is any Euclidean rotation in $(T_\xi \mathcal{N}, I)$ of the form $R = \text{diag}(R_0, R_0)$ where R_0 is an Euclidean rotation in $(T_\xi \mathcal{M}, I)$.

(It makes explicit the dependence of F in the metric G and the connection Γ^k).



Multiscale analysis of image similarity measures

From previous results we obtain the **rotation invariance** of F (and \mathcal{H}) in the **tangent plane**:

$$F(X, p, \xi, G) = F(R^t X R, R^t p, \xi, G), \forall \text{diagonal rotations } R \text{ in } (T_\xi \mathcal{N}, G(\xi)),$$

Diagonal means

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

where R_1, R_2 are **related by an a priori connection P** .

$$R_1 : (T_{\xi_1} \mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_1} \mathcal{M}^1, G^1(\xi_1)), \text{ and}$$

$$R_2 : (T_{\xi_2} \mathcal{M}^2, G^2(\xi_2)) \rightarrow (T_{\xi_1} \mathcal{M}^2, G^2(\xi_2)), \text{ with } R_2 \bar{P}(\xi) = P(\xi) R_1.$$



Multiscale analysis of image similarity measures

Let us assume that $\mathcal{M}^1 = \mathcal{M}^2 = \mathcal{M}$ (with different metrics). Let us consider the axiom:

[Axiom of symmetry of the two coordinates] If $SC(\xi_1, \xi_2) = C(\xi_2, \xi_1)$, then $T_t(SC) = ST_t(C) \forall t \geq 0, \forall C \in C_b^\infty(\mathcal{M} \times \mathcal{M})$.

Lemma

Let T_t be a multiscale analysis satisfying the axioms: **all Architectural, Comparison principle, Gray level Shift invariance, and Symmetry of the two coordinates invariance**. Then F satisfies

$$F(SAS, Sp, \xi, SG(\xi), \Gamma^k(S\xi)) = F(A, p, S\xi, SG(\xi), \Gamma^k(S\xi))$$

$$\forall A \in \mathcal{S}(\mathcal{N}), \forall p \in \mathcal{N} \setminus \{0\}, \forall \xi \in \mathcal{N}.$$



The linear case

Lemma

Let \mathcal{M} be a Riemannian manifold. Let D be a matrix such that

$$RDR^t = D$$

for all rotations R in $(T_\eta\mathcal{M}, G(\eta))$. Then $D = \lambda G(\eta)^{-1}$ for some $\lambda \in \mathbb{R}$.



The linear case

Theorem

Let T_t be a multiscale analysis on similarity functions satisfying the axioms: **all Architectural, Comparison principle, and Gray level Shift invariance**. Assume that T_t is **linear**. Then

$$C_t = F(D_N^2 C, \xi, G),$$

where

$$F(X, \xi, G) = c_{11}(\xi) \text{Tr}((G^1)^{-1}(\xi_1) X_{11}) + 2c_{12}(\xi, G) \text{Tr}(\bar{D}_{12} I^1(\xi_1)^{-1} X_{12}) \\ + c_{22}(\xi) \text{Tr}((G^2)^{-1}(\xi_2) X_{22}),$$

where \bar{D}_{12} is an isometry from $(T_{\xi_1} \mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2} \mathcal{M}^2, G^2(\xi_2))$.
The ellipticity of F implies that $c_{11}, c_{22} \geq 0$.

- the operators $c_{ii}(\xi) \text{Tr}((G^i)^{-1}(\xi_i) X_{ii})$ are **multiples of the Laplace-Beltrami operator**.
- This result is equivalent to the one in [Calderero-Caselles] where they prove that **linear scales spaces on \mathcal{M} are given by the Laplace-Beltrami operator**.
- There are no first order terms in these operators. They cannot couple with vectors so that we have the invariance induced by the rotations of tangent planes.



The linear case: (1) The case of $(\mathcal{M}^r, \mathbf{g}^r(\mathbf{x})) = (\mathbb{R}^N, \mathbf{g}^r(\mathbf{x}))$

To fix ideas consider $\mathcal{M}^1 = \mathcal{M}^2 = \mathcal{M} = \mathbb{R}^N$ and $g_{ij}^r(x)$ be general metrics in \mathbb{R}^N , $r = 1, 2$.

We can define $P(x, y)(v) = G^2(y)^{-1/2} G^1(x)^{1/2} v$, $v \in \mathbb{R}^N$, as the a priori connection of x and y .

The PDE obtained is

$$C_t = a(x, y) \Delta_{\mathcal{M}_x} C + 2c_{12}(x, y) \text{Tr}(G^2(y)^{-1/2} G^1(x)^{-1/2} D_{xy} C) + c(x, y) \Delta_{\mathcal{M}_y} C,$$

where

$$\Delta_{\mathcal{M}_x} C = \text{Tr}(G^1(x)^{-1} (D_{xx} u(x) - \Gamma^{(1)}(Du)(x))).$$

Similarly for the operator $\Delta_{\mathcal{M}_y}$.

This will permit to construct also an operator in the case of video.



The linear case: (2) The case of conformal metrics in \mathbb{R}^N

Let's take $\mathcal{M}^1 = \mathcal{M}^2 = \mathcal{M} = \mathbb{R}^N$ and $g_{ij}^r(x) = \lambda^r(x)^2 \delta_{ij}$, $\lambda^r(x) > 0$ for $x \in \mathbb{R}$.

We define $P(x, y)(v) = \frac{\lambda^1(x)}{\lambda^2(y)} v$, $v \in \mathbb{R}^N$, as the **a priori connection** of x and y . Then $|P(x, y)v|_{g^2} = |v|_{g^1}$ for all $(x, y) \in \mathbb{R}^{2N}$, and

$$\begin{aligned}\text{Tr}_{g^1}(D_{\mathcal{M}^{xx}}^2 u(x)) &= \text{Tr}((G^1)(x)^{-1} D_{\mathcal{M}^{xx}}^2 u(x)) \\ &= \frac{1}{\sqrt{\det(g^1(x))}} \text{div} \left(\sqrt{\det(g^1(x))} (g^1)(x)^{-1} Du(x) \right) = \Delta_{\mathcal{M}^x} u(x)\end{aligned}$$

which is the Laplace-Beltrami operator.

The computations give that the linear operator on $C(t, x, y)$ can be written as

$$C_t = a(x, y) \Delta_{\mathcal{M}^x} C + 2 \frac{c_{12}(x, y)}{\lambda^1(x) \lambda^2(y)} \text{Tr}(D_{xy} C) + c(x, y) \Delta_{\mathcal{M}^y} C,$$

for functions $a(x, y)$, $c_{12}(x, y)$, $c(x, y)$ so that the operator is elliptic (that is, if and only if $a|v_1|^2 + 2c_{12} \langle v_1, v_2 \rangle + c|v_2|^2 \geq 0$ for all v_1, v_2).



The linear case: (3) The case where $(\mathcal{M}^r, g) = (\mathbb{R}^N, I)$

Let us consider the case $(\mathcal{M}^r, g) = (\mathbb{R}^N, I)$. We denote $\xi = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. We do not subsume this under the general manifold case, because we can assume a different set of invariances that permits that the operator has first order terms. Let us consider translation and rotation invariance:

[Translation invariance:] $T_t(\tau_{a,b}C) = \tau_{a,b}T_tC \quad \forall t \geq 0,$
 $\forall C \in C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N), \forall a, b \in \mathbb{R}^N$ where $\tau_{a,b}C(x, y) = C(x + a, y + b)$.

[Rotation invariance] $T_t(RC) = RT_tC \quad \forall t \geq 0, \forall C \in C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N),$
 $\forall R \in O(N)$ where $RC(x, y) = C(Rx, Ry)$. We have denoted by $O(N)$ the rotations in \mathbb{R}^N .



The linear case: (3) The case where $(\mathcal{M}^r, g) = (\mathbb{R}^N, I)$

Theorem

Let T_t be a multiscale analysis in (\mathbb{R}^N, I) satisfying the axioms: all Architectural, Comparison principle, Gray level Shift invariance, and Rotation invariance. Assume that T_t is linear. Then

$$C_t = F(D^2 C, DC, z)$$

where

$$F(A, v, z) = \sum_{i,j=1}^2 c_{ij}(z) \text{Tr} A_{ij} + \langle b(z), v \rangle,$$

for some functions $c_{ij}(z) \in \mathbb{R}$, $b(z) \in \mathbb{R}^{2N}$, $i, j = 1, 2$, such that $c_{ij}(R \circ z) = c_{ij}(z)$, $b(R \circ z) = R \circ b(z)$ for all $R \in O(N)$ and all $z \in \mathbb{R}^{2N}$. The ellipticity of F implies that $(c_{ij}(z))$ is a positive definite matrix $\forall z \in \mathbb{R}^{2N}$.

Moreover, if we assume that T_t is translation invariant, then

$$F(A, v, z) = \sum_{i,j=1}^2 c_{ij} \text{Tr} A_{ij},$$



The linear case: (3) The case where $(\mathcal{M}^r, g) = (\mathbb{R}^N, I)$

Finally, the PDE is

$$C_t = c_{11}(z)\Delta_x C + 2c_{12}(z)\text{Tr}(D_{xy}^2 C) + c_{22}(z)\Delta_y C + \langle b_1(z), D_x C \rangle + \langle b_2(z), D_y C \rangle \quad (14)$$

An example is

$$C_t = \Delta_x C + 2\text{Tr}(D_{xy}^2 C) + \Delta_y C. \quad (15)$$

Let $\mathbf{C}(t, \mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^N} \mathbf{g}_t(\mathbf{z})\mathbf{C}(0, \mathbf{x} + \mathbf{h}, \mathbf{y} + \mathbf{h}) d\mathbf{h}$, where \mathbf{g}_t is the Gaussian of scale t . Then $\mathbf{C}(t, \mathbf{x}, \mathbf{y})$ is a solution of (15) with initial condition $\mathbf{C}(0, \mathbf{x}, \mathbf{y})$.

- If $C(0, x, y) = I(x)J(y)$, then $C(t, x, y) := \int_{\mathbb{R}^N} g_t(z)I(x+h)J(y+h) dh$.
- If $C(0, x, y) = (I(x) - J(y))^2$, then $C(t, x, y) := \int_{\mathbb{R}^N} g_t(z)(I(x+h) - J(y+h))^2 dh$.
- Another example is $C(0, x, y) = \sum_{i=1}^N Z_i(x)Z_i(y)$, where $Z(x) = (Z_i(x))_{i=1}^N$ is the direction of the gradient of I .



The linear case: (3) The case where $(\mathcal{M}^r, g) = (\mathbb{R}^N, I)$

Some **examples** of functions $b(z)$. We can take L_1, L_2 so that $L_i(R \circ z) = L_i(z)$ for all $R \in O_z(N)$. Then

$$b(z) = \begin{pmatrix} L_1(z)x \\ L_2(z)y \end{pmatrix}$$

and

$$b(z) = \begin{pmatrix} L_1(z)(x - y) \\ L_2(z)(x - y) \end{pmatrix}$$

satisfy $R^t b_i(R \circ z) = b_i(z)$, $\forall R \in O(N)$, $\forall i = 1, 2$.



The linear case

Remark

Let A, B two $N \times N$ matrices,

$$\mathbf{C}(t, \mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^N} g_t(\mathbf{z}) \mathbf{C}(\mathbf{0}, \mathbf{x} + \mathbf{A}\mathbf{h}, \mathbf{y} + \mathbf{B}\mathbf{h}) d\mathbf{h}$$

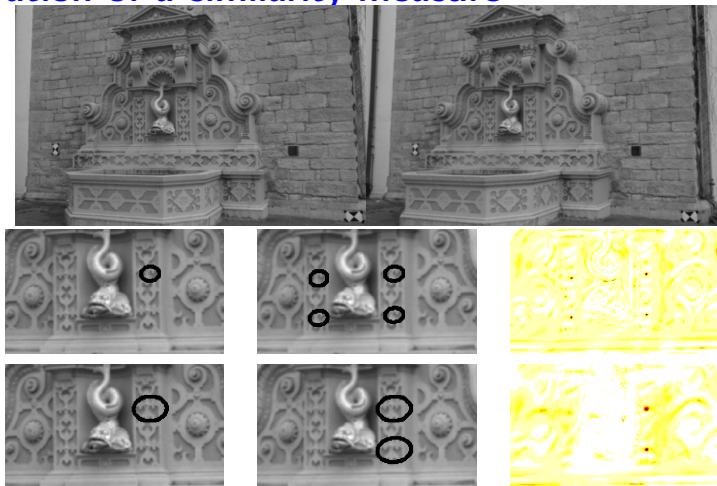
where g_t is the Gaussian of scale t , and $\mathbf{C}(\mathbf{0}, \mathbf{x}, \mathbf{y}) = (\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2$.
Then $\mathbf{C}(t, \mathbf{x}, \mathbf{y})$ satisfies the equation

$$C_t = \text{Trace}(A^t A D_x^2 C) + 2\text{Trace}(A B^t D_{xy} C) + \text{Trace}(B^t B D_y^2 C). \quad (16)$$

This equation corresponds to the models described in the Theorem for the linear case, and in particular when the metrics are constant in both images.



Illustration of a similarity measure



The values of the similarity measure are computed between one point x in the reference image (first row, left) and all the points y of the secondary image (first row, right), which is taken from a different viewpoint. In the second row we show closeup containing the point x , and the value of the similarity measure. Red pixels denote lower values of $C(t, x, y)$. Note that the minima of the similarity measure occur at the points where the structure of the secondary image is similar to the reference patch. The comparison windows are ellipses

The Morphological Axiom

By adding the axiom:

[Gray scale invariance] $T_t(f(C)) = f(T_t(C)) \forall t \geq 0$,
 $\forall C \in C_b^\infty(\mathcal{M}^1 \times \mathcal{M}^2)$, and for any strictly increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$.

It is also called the Morphological axiom.



The Morphological Axiom

Theorem

Let T_t be a multiscale analysis satisfying the axioms: all Architectural, Comparison principle, and the Morphological Axiom. Then

$$F(A, p, \xi, G, \Gamma^k) = F(Q_p^t A Q_p, p, \xi, G, \Gamma^k) \quad (17)$$
$$\forall A \in \text{SM}_\xi(\mathcal{N}), \forall p \in T_\xi^* \mathcal{N} \setminus \{0\}, \forall \xi \in \mathcal{N}.$$

A similar statement holds for \tilde{F} . Let B be such that $B^t G B = I$. In terms of \mathcal{H} we have

$$\mathcal{H}(B^t(A - \Gamma(p))B, B^t p, \xi) = \mathcal{H}(B^t Q_p^t(A - \Gamma(p))Q_p B, B^t p, \xi) \quad (18)$$
$$\forall A \in \text{SM}_\xi(\mathcal{N}), \forall p \in T_\xi^* \mathcal{N} \setminus \{0\}, \forall \xi \in \mathcal{N}.$$

We do not make precise the structure of the morphologically invariant operators, since we cannot simultaneously use the same rotation with respect to both ξ_1 and ξ_2 to extract curvatures as in [Alvarez-Guichard-Lions-Morel].



The Morphological Axiom

Some examples of morphological scale spaces in the paper. Their interpretation is much more complex because it probably reflects the correlations between directions of level lines of both image patches under comparison.



Some experiments

Reconstruction



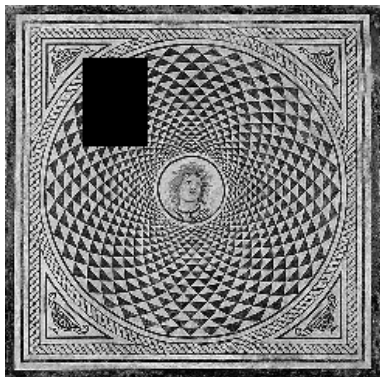
Two original images.



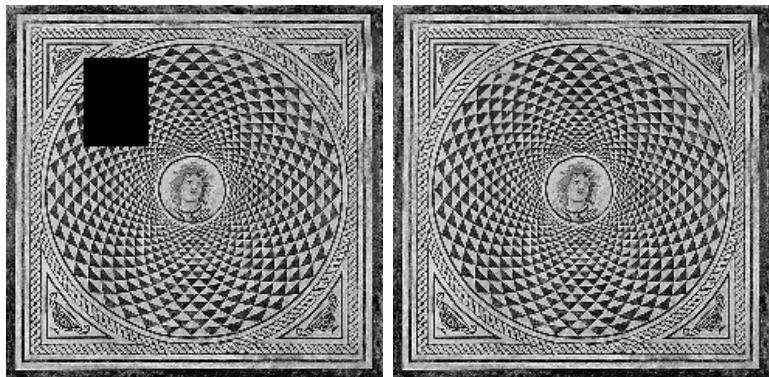
Reconstruction result obtained from the information in the left image.



Inpainting



Inpainting



Inpainting

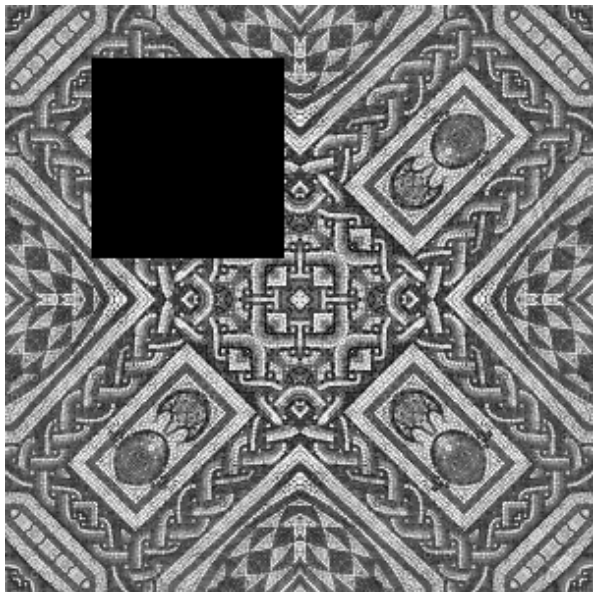


Image source: Vadim Fedorov



Inpainting

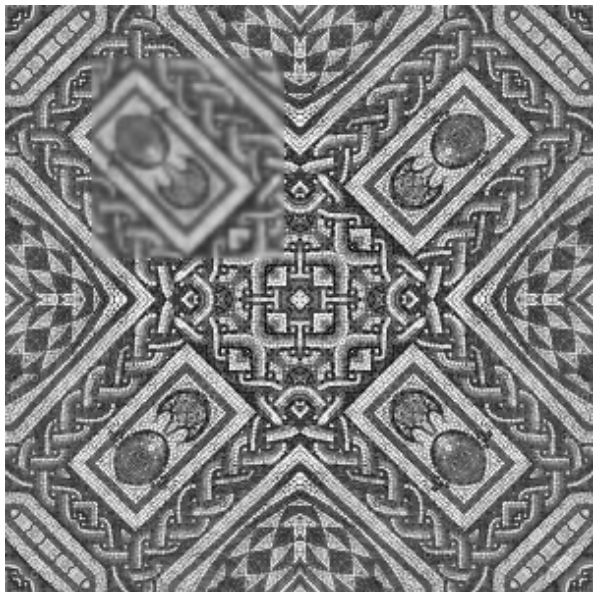


Image source: Vadim Fedorov



Inpainting

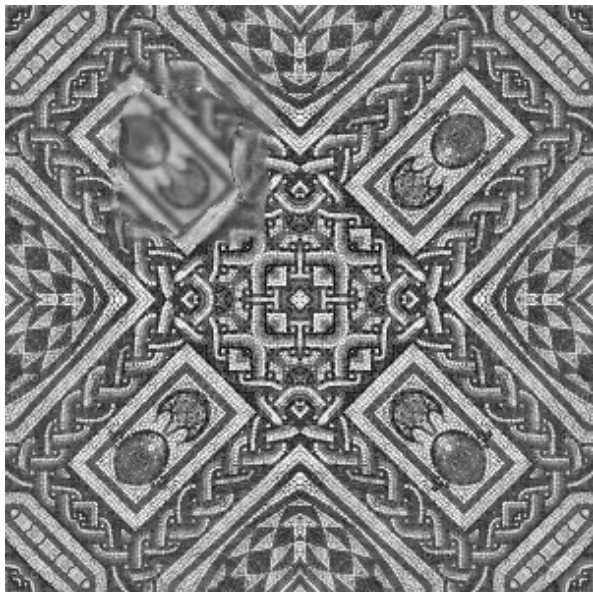


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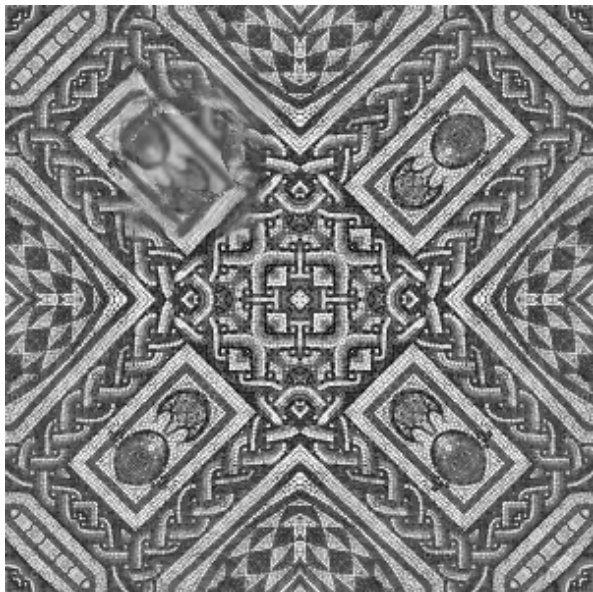


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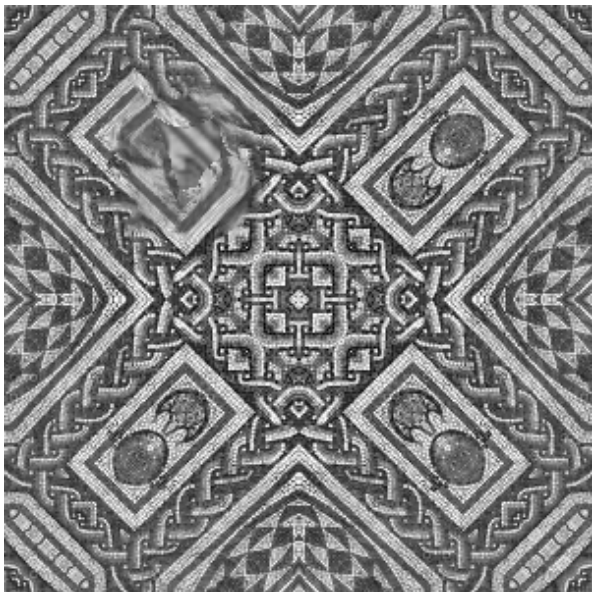


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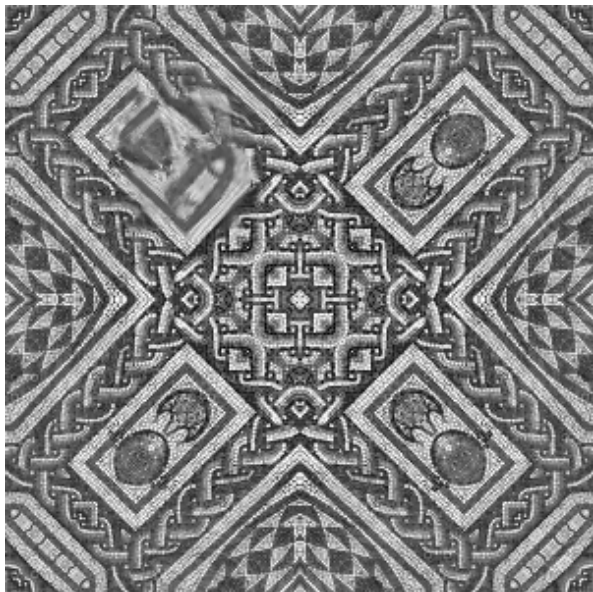


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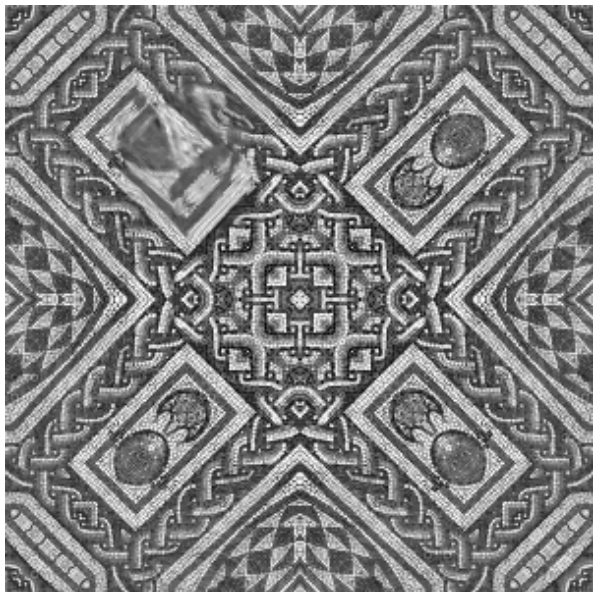


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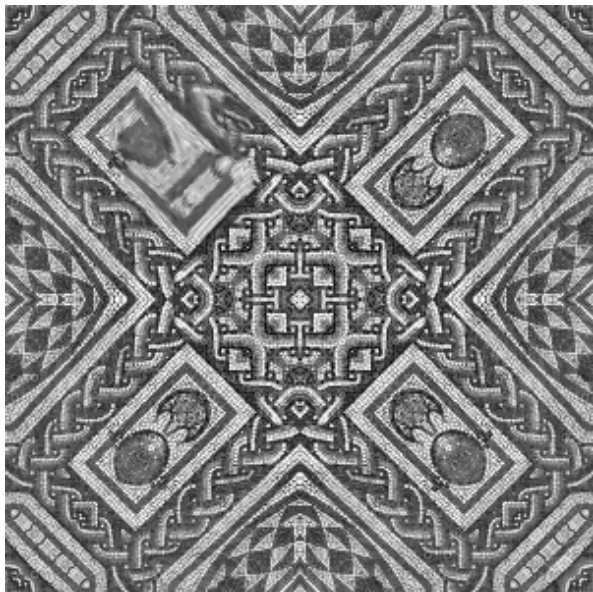


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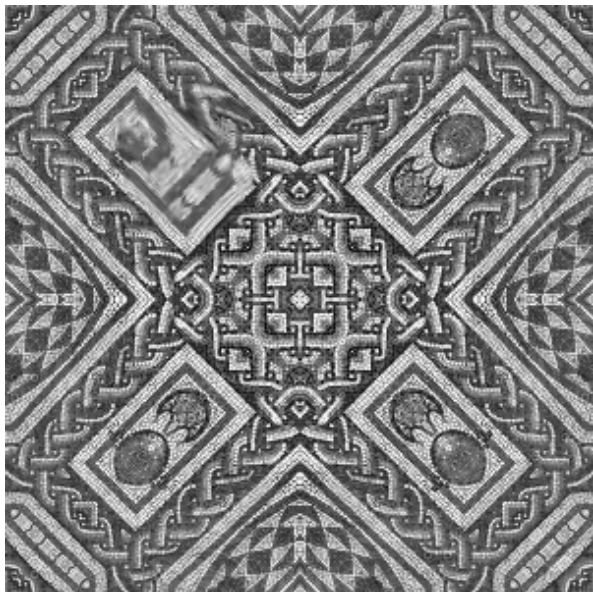


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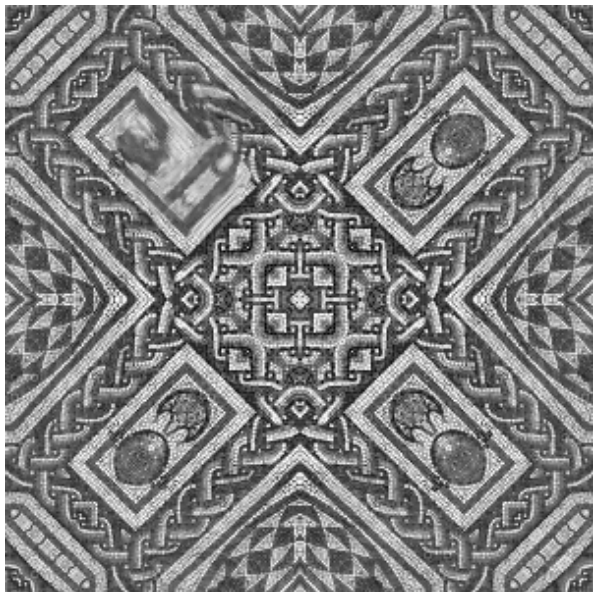


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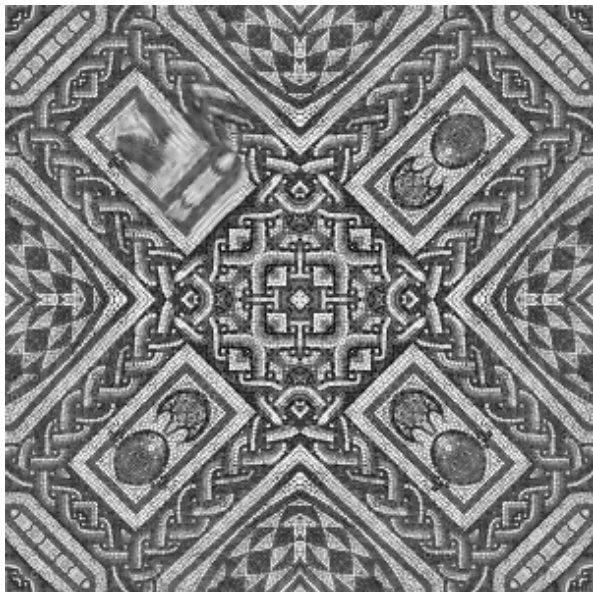


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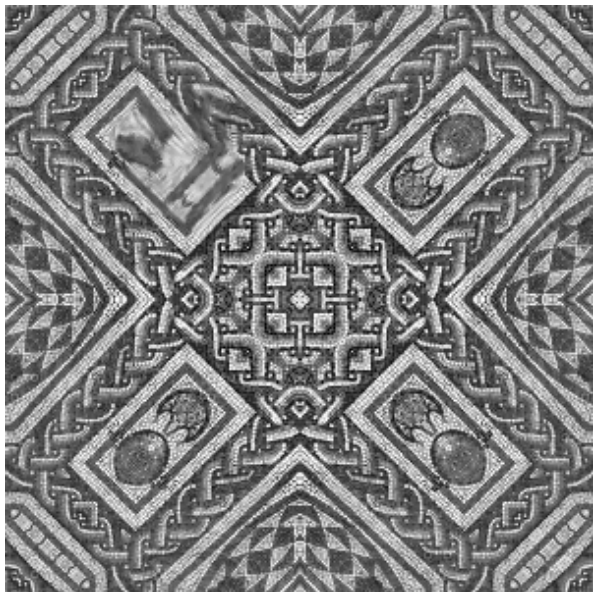


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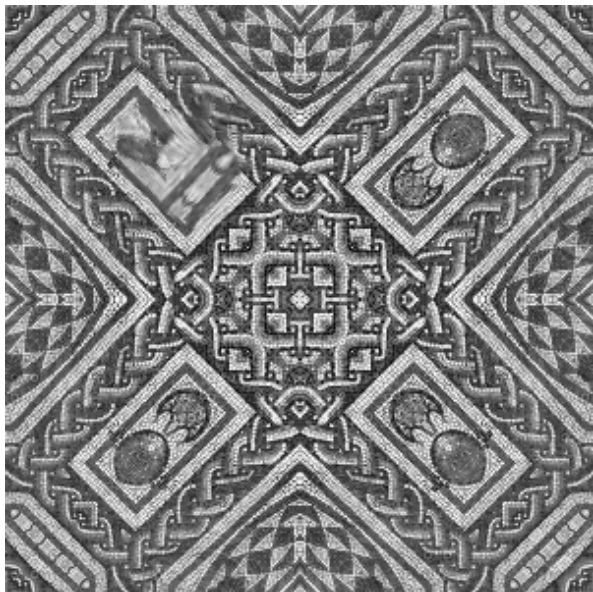


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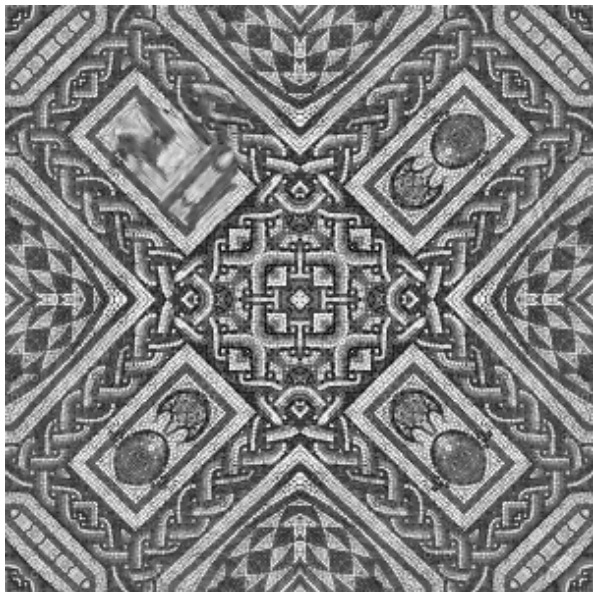


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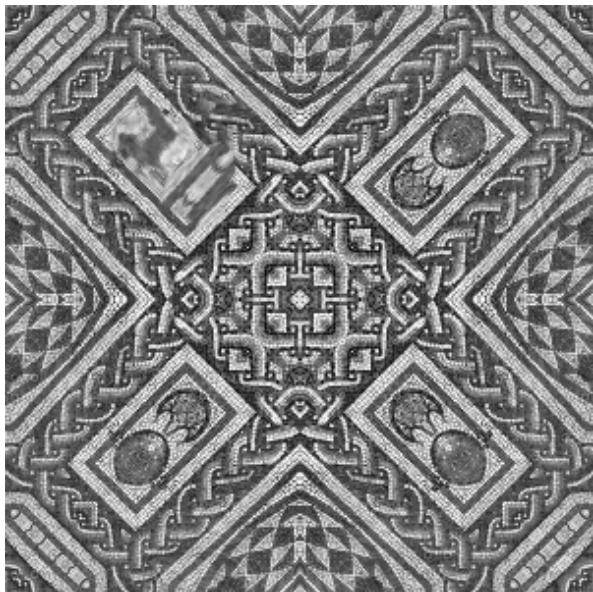


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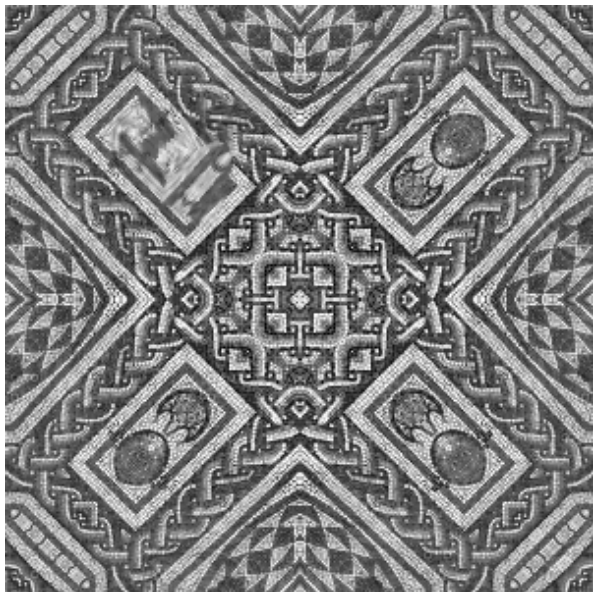


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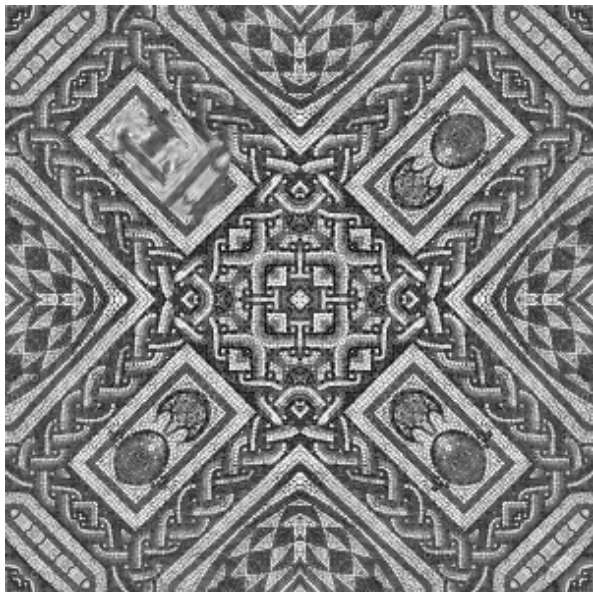


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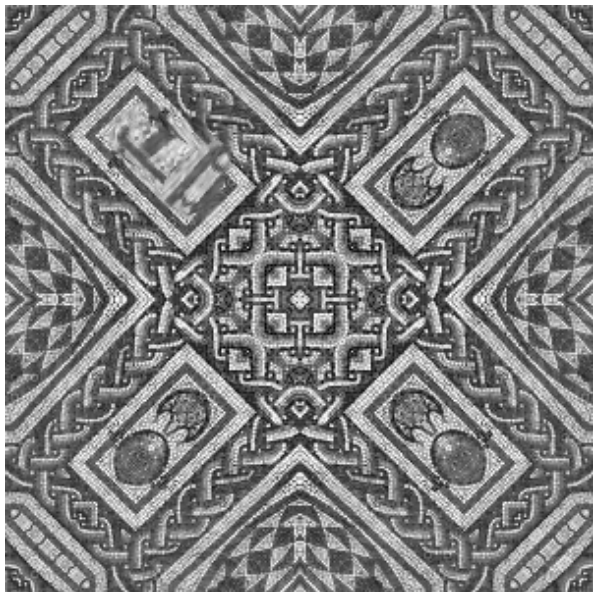


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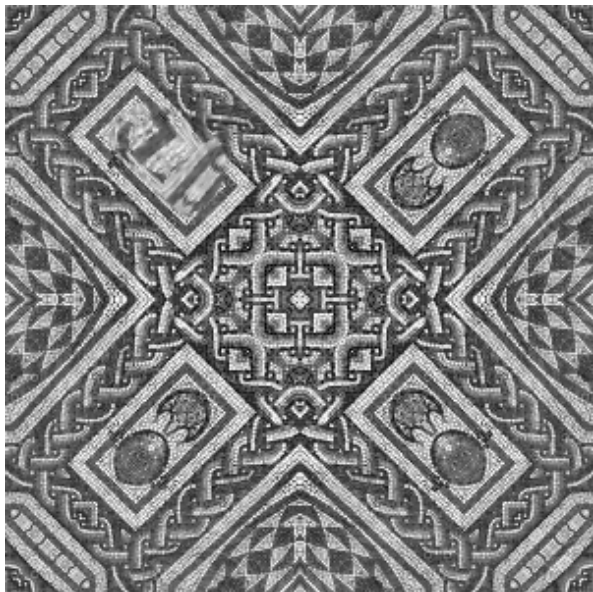


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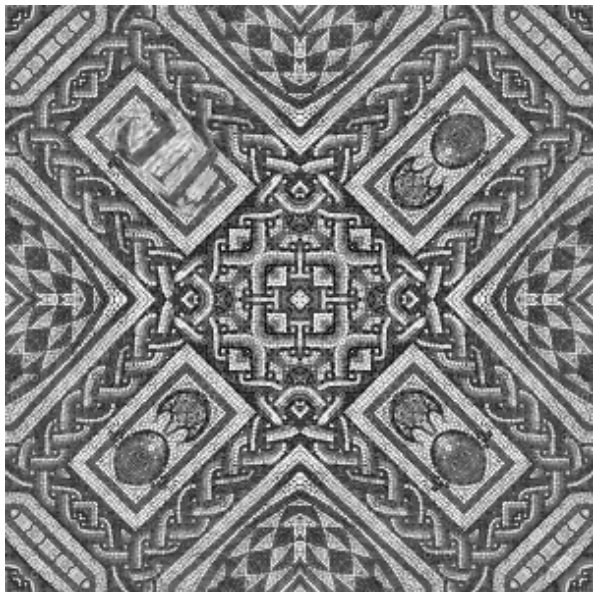


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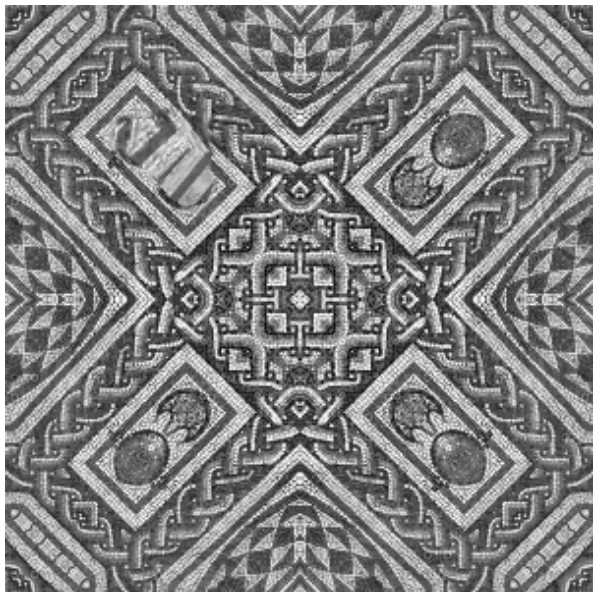


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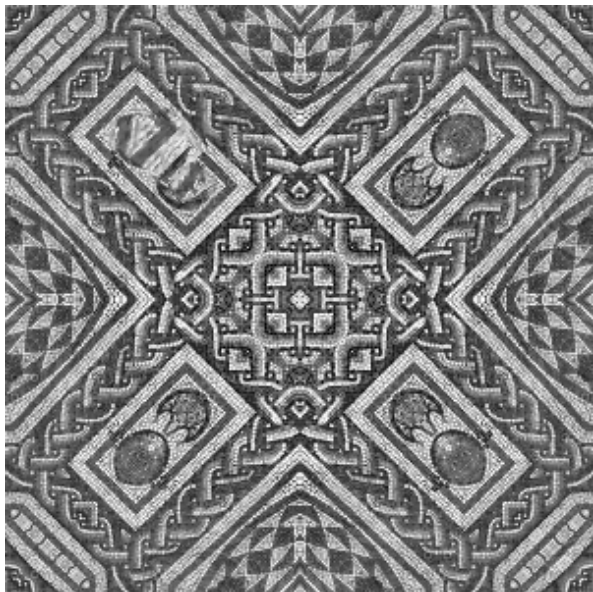


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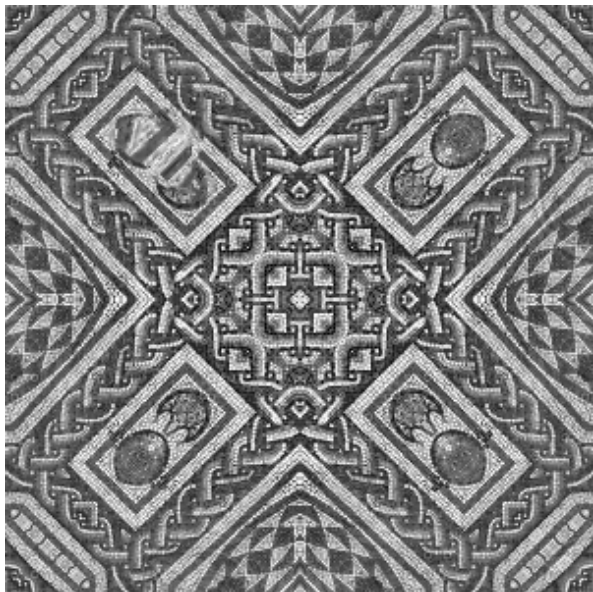


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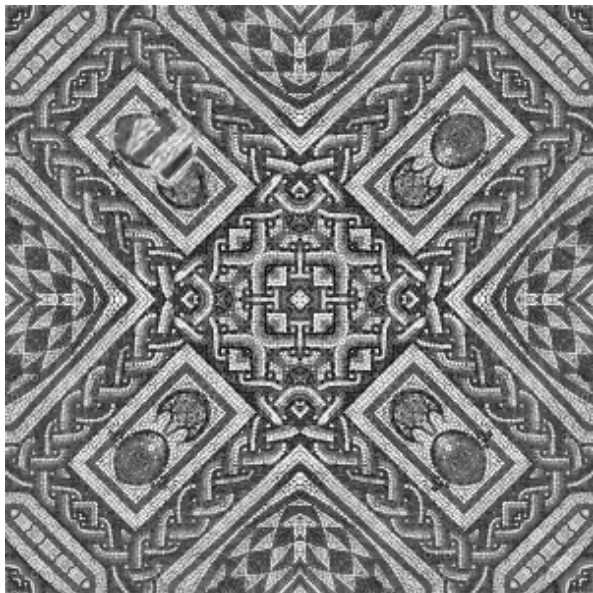


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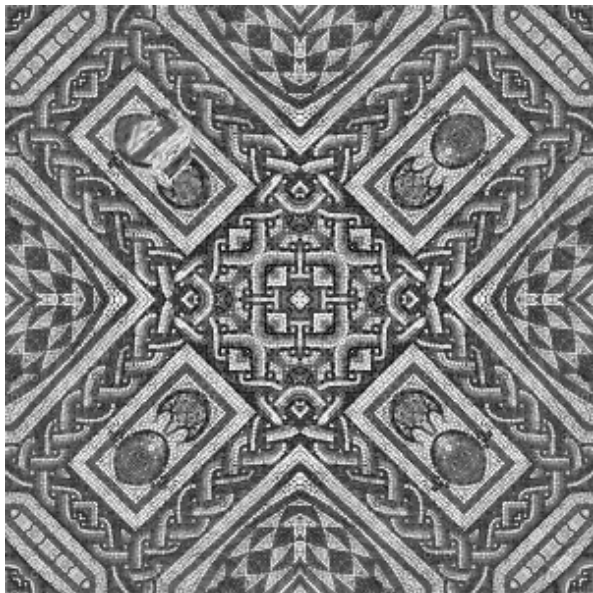


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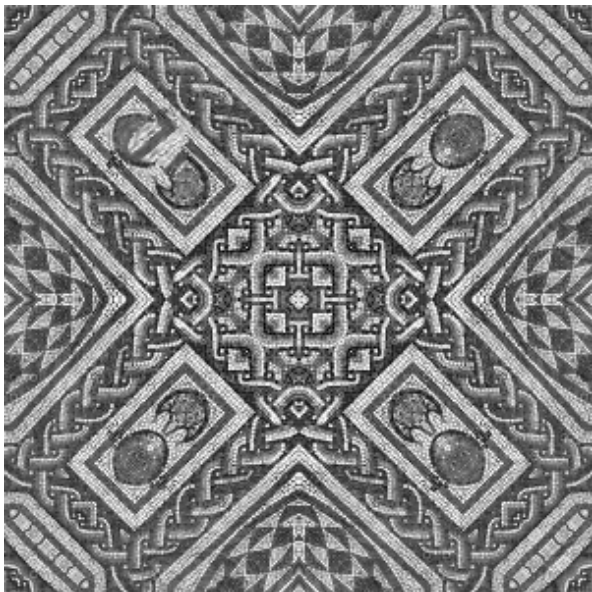


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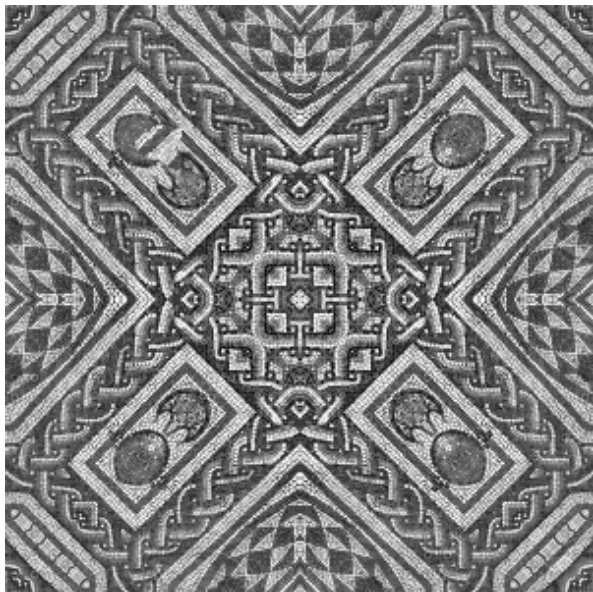


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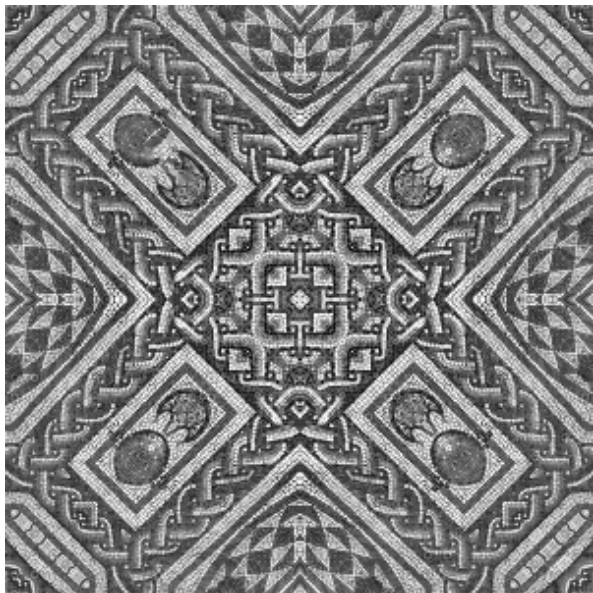


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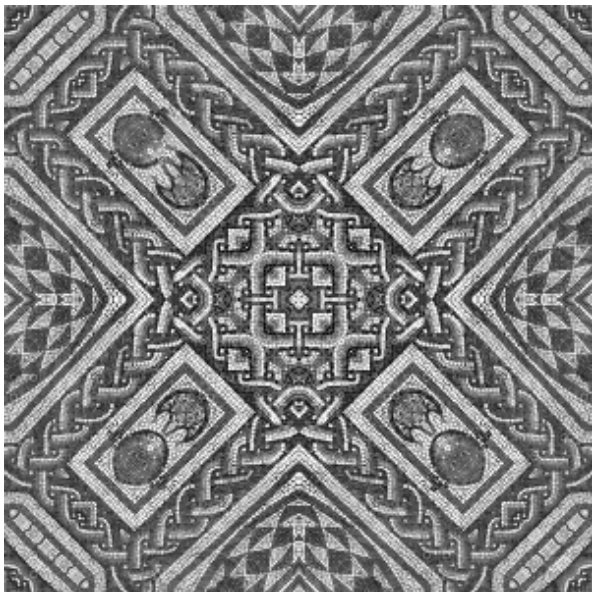


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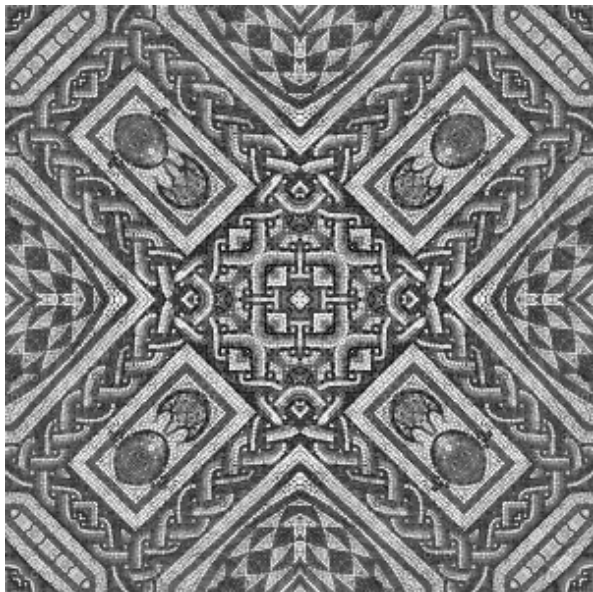


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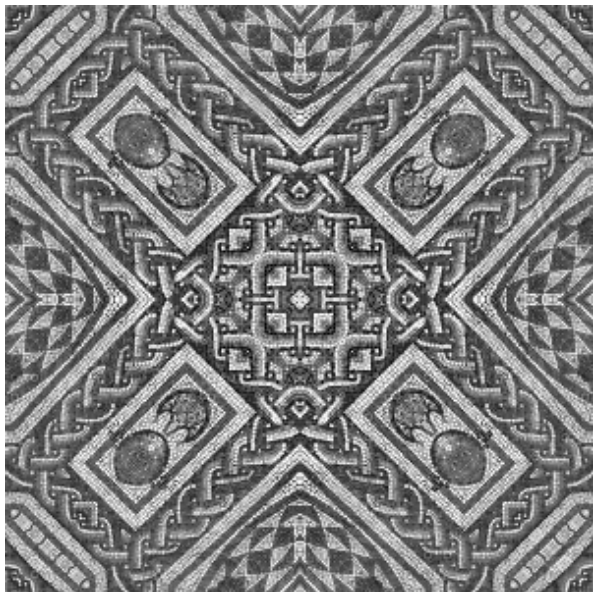


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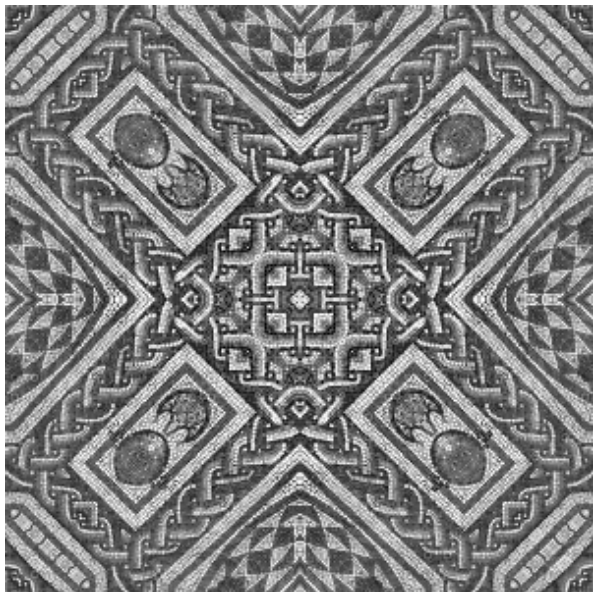


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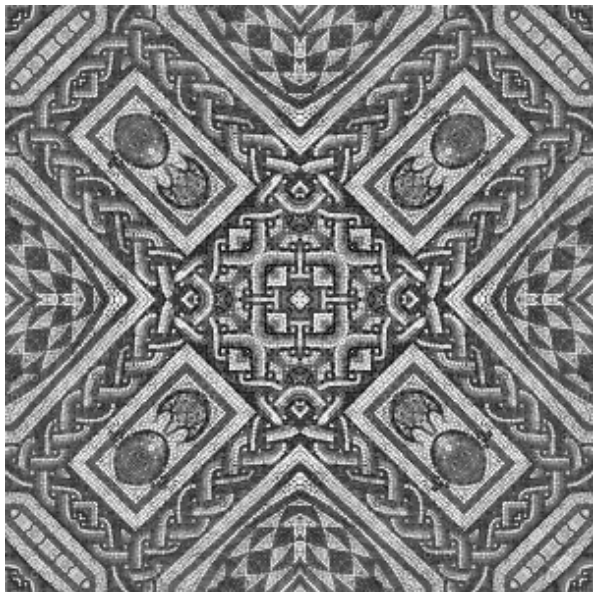


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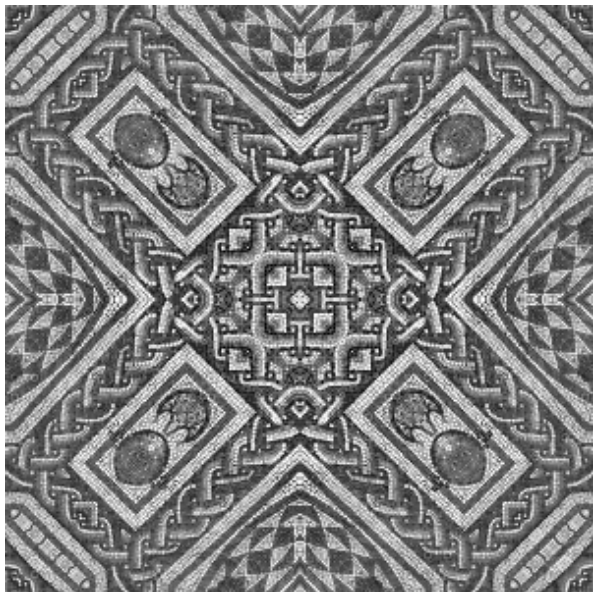


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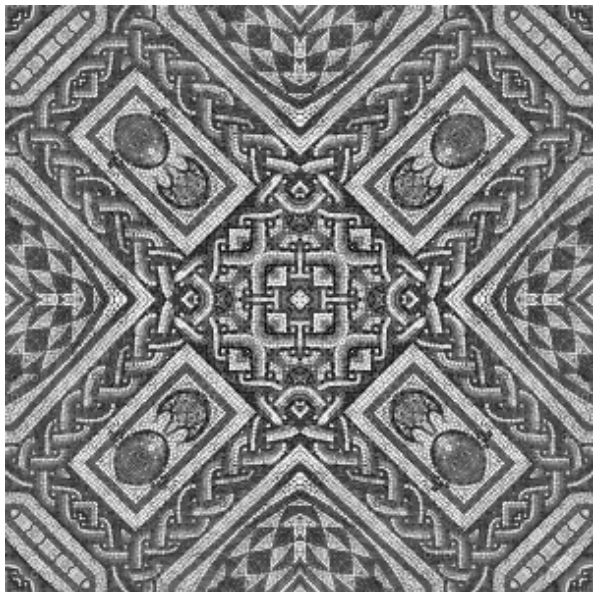


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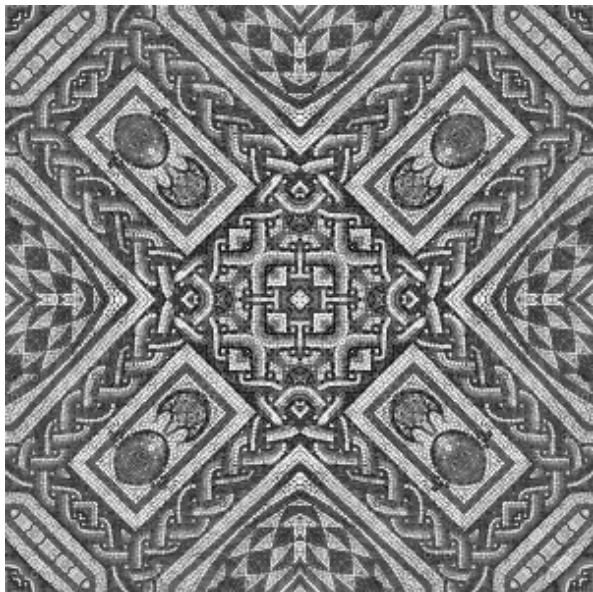


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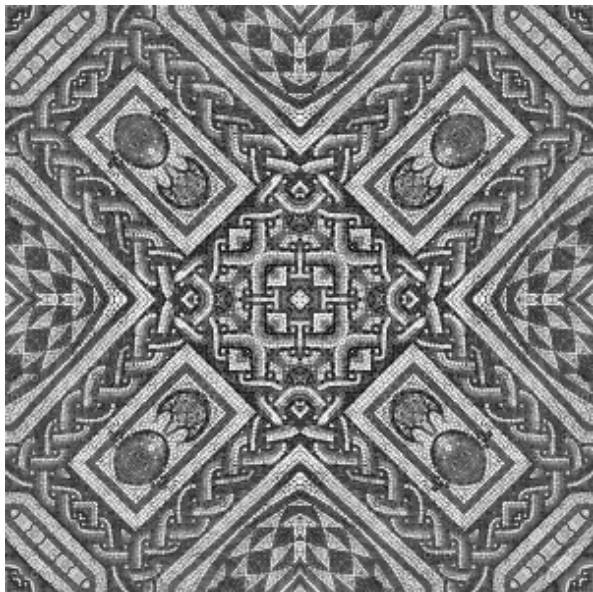


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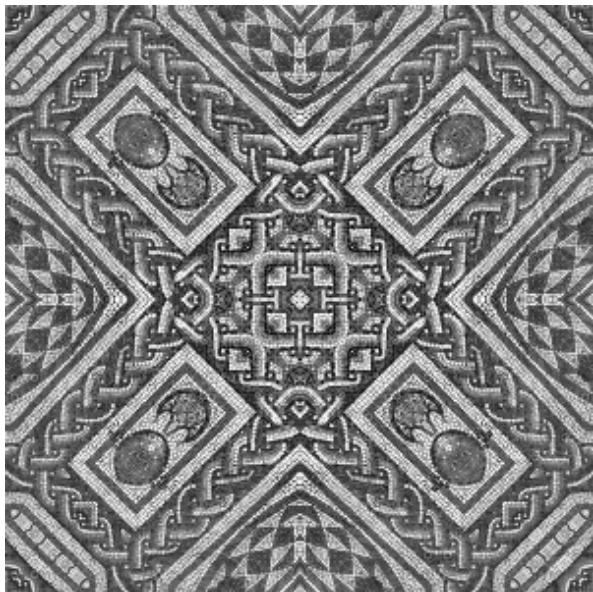


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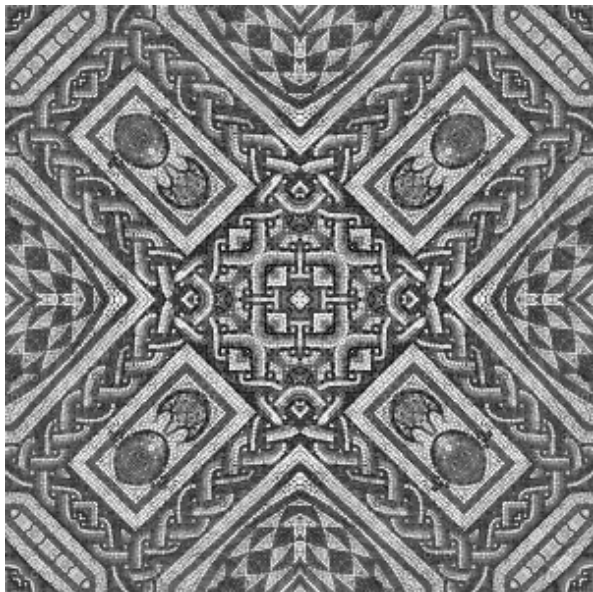


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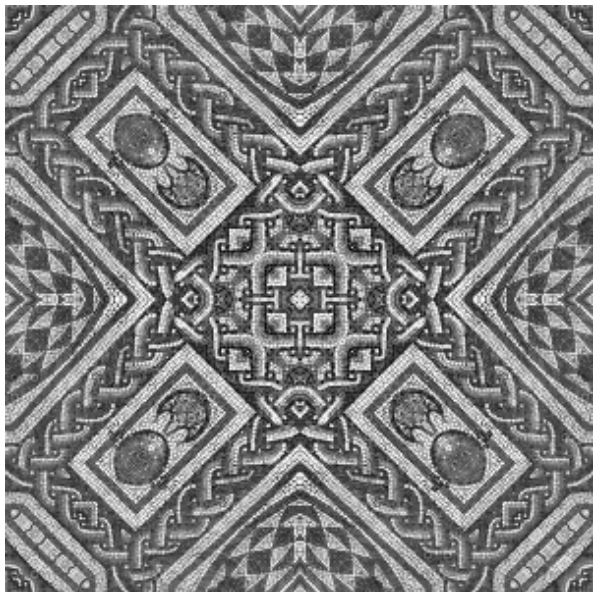


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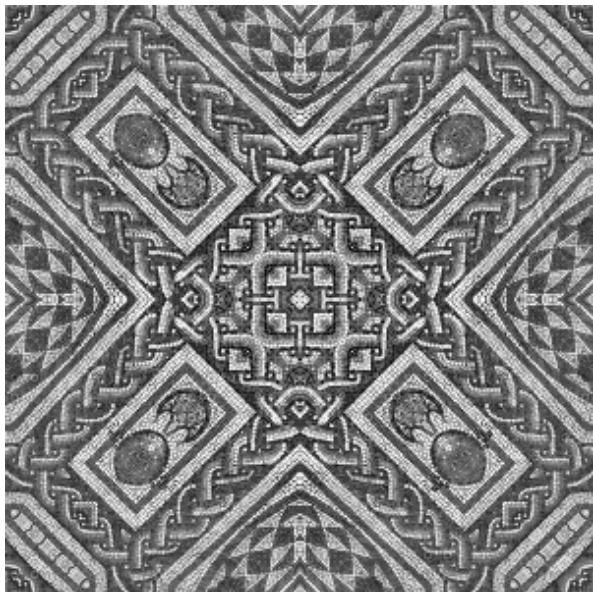


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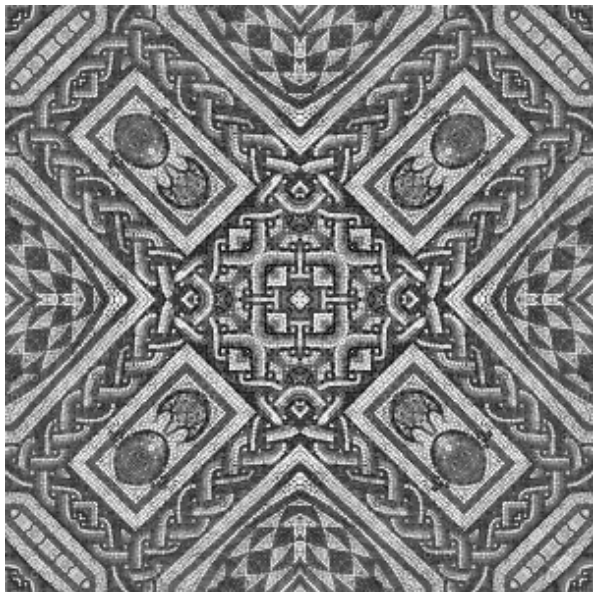


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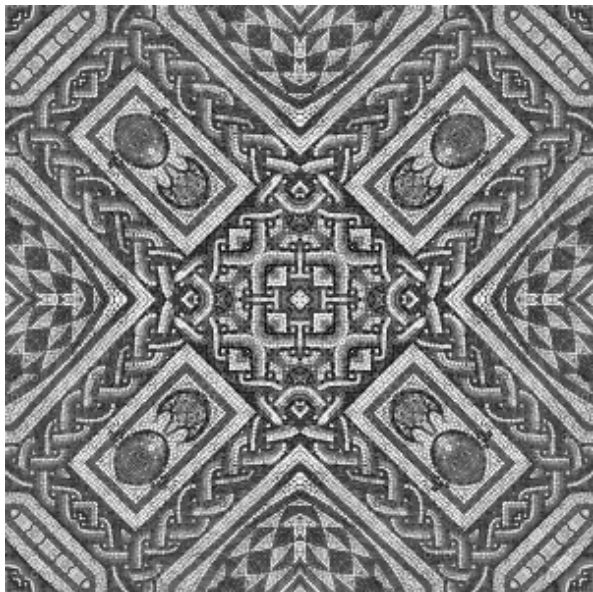


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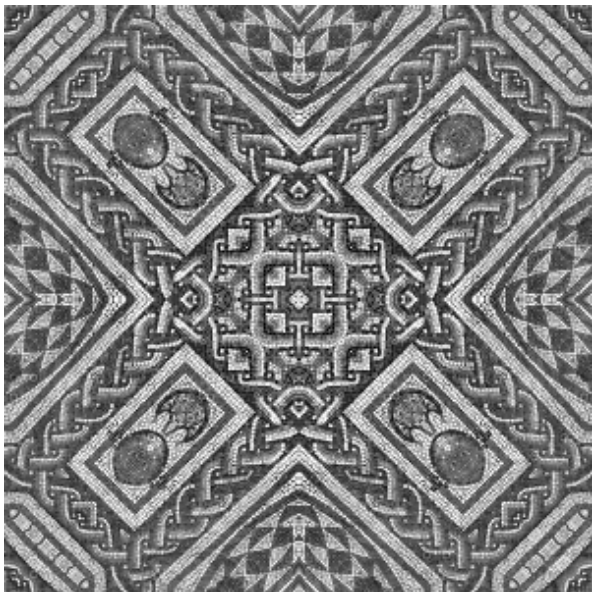


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Thank you

