Orthogonal Decomposition of Graphs

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1 Introduction

It is basic knowledge that using the internal structure of objects enables one to devise easier to write and/or more efficient algorithms and that intractable problems may become easier when restricted to classes of objects with appropriate properties.

Although at first sight, graphs may appear to be dramatically unstructured, many authors have endeavoured, with some success, to characterize graphs with properties which greatly reduce their apparent complexity. Modular decomposition [4] [5] [6], 2-structures [10], systems of equations, bounded tree width are but a few notions one should quote while trying to cover that topic.

We follow here a different path, pioneered in [7] and more recently in [9], where one aims at describing certain classes of graphs as algebras whose elements may be built from a set of basic building blocks called prime factors.

Our graphs will be undirected, edge-labelled graphs having a finite number of distinguished vertices called sources and our basic operation will be the pullback of arrows in the category of graphs ([1] [3]).

The main result of this paper is to show how some types of graphs may be decomposed into "orthogonal" components, whose pullback is precisely the original graph. Iterating the process, these graphs may be written as composition of some basic building blocks (or rather families of building blocks).

2 Coloured graph with sources

A *coloured graph with sources*, shortly a *graph*, is a four-tuple $G = \langle V, E, C, S \rangle$ where V is the set of *nodes* of G, $E \subset V \times V$ is the set of *edges* of G, with an edge $[u, v]$ considered to be the same as an edge $[v, u]$ for all edges $[u, v]$ in E, $C \subset \mathcal{P}(E)$ is a partition of E defining an equivalence relation on E, whose equivalence classes are the *colours* of G, and $S \subset V$ is the finite set of *sources* of G. We shall work in the category of graphs, whose morphisms are simply colour preserving usual morphisms. The *pullback* of a pair of graph morphisms

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$G_1 \xrightarrow{g_1} G \xleftarrow{g_2} G_2$ is another pair $G_1 \xleftarrow{h_1} H \xrightarrow{h_2} G_2$ of morphisms where H is the subgraph of the product $G_1 \times G_2$ consisting of those items (nodes and edges) on which $g_1 \circ \pi_1$ and $g_2 \circ \pi_2$ coincide.

In this paper, all graphs will be connected.

Definition 1. Let G be a graph and $c \in C$. We say that a set of nodes X is *uniform of colour* c iff any edge linking two nodes of X has colour c.

Definition 2. Let G be a graph. Let X and Y be two disjoint subsets of V. Y is an *X-fibre of colour c* iff any edge linking a node in X and a node in Y has colour c. Y is a *total X-fibre* (of colour c) iff Y is an X-fibre (of colour c) and each node of Y is linked with at least one node of X.

Example The set I of figure 1 is uniform of colour c_1. It is a J-fibre of colour c_1. but not total because the node s_1, for example, is not linked with J. The set K is a total J-fibre (of colour c_2). On this figure and the following, different types of lines indicate different edge colours. Square nodes shall represent the sources of a graph. There are no colours on the nodes.

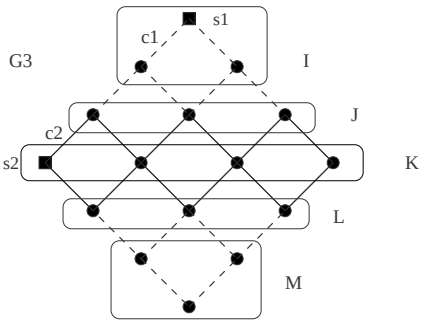

Fig. 1. J-fibres and uniform set

Lemma 3. *Let G be a graph. Let X, Y be subsets of V. One has:*

1. *if Y is an X-fibre then X is a Y-fibre,*
2. *if Y is a [total] X-fibre, every subset of Y is a [total] X-fibre,*
3. *if Y and Z are total X-fibres and $Y \cap Z \neq \emptyset$, then $Y \cup Z$ is a total X-fibre.*

□

2.1 s-width-fibres

For any source s of a graph G, let $B(s,0) = \{s\}$ and for all $r \geq 0$ $B(s,r+1) = B(s,r) \cup \{y \in V : \exists x \in B(s,r), [x,y] \in E(G)\}$. Let $D(s,0) = \{s\}$ and for all $r \geq 0$ $D(s,r+1) = B(s,r+1) - B(s,r)$.

Definition 4. Let G be a graph and $s \in S$. For any $x \in V - \{s\}$, the *set of coloured sequences from s to x* is the following set :
$SEQ(s,x) = \{(c_1, \ldots, c_n)$: there exists a shortest path $(s, x_1, \ldots, x_n)$ such that $x_n = x$ and $[s, x_1] \in c_1$, ..., $[x_{n-1}, x_n] \in c_n\}$.
$SEQ(s,s) = \{()\}$, () being the empty sequence.

Lemma 5. *Let G be a graph and $s \in S$. Let $z \in D(s,r)$, for some $r \geq 1$.*
$SEQ(s,z) = \bigcup_{x \in D(s,r-1)} \{(u,c) : u \in SEQ(s,x)$ *and* $[x,z] \in c\}$. □

Definition 6. Let G be a graph and $s \in S$. We say that two nodes, x and y, are *width-equivalent for s*, denoted $x \sim_s y$, iff $SEQ(s,x) = SEQ(s,y)$.
An *s-width-fibre* of G is an equivalence class of $\sim_s$.

Example In figure 2, the sets $\{s\}$, Y, P, Q, T, U, Z, W, M_1, M_2, M_3 are the s-width-fibres of G.
In particular, one has for all u in U, $SEQ(s,u) = \{(c_1, c_2, c_3, c_1), (c_1, c_3, c_2, c_1)\}$.

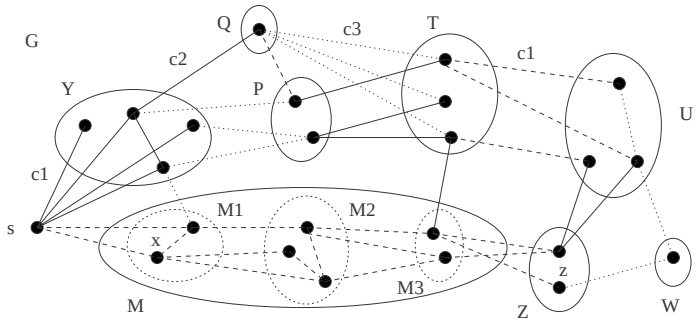

Fig. 2. The s-width-fibres

We say that a graph G is *coherent* from $s \in S$ iff for all edges $[u,v]$ and $[x,y]$ in G with $u \sim_s x$ and $v \sim_s y$, $[u,v]$ and $[x,y]$ have the same color. Then $\widetilde{G} = G/\sim_s$ is well defined as a graph.

In the following we shall always deal with coherent graphs.

Proposition 7. *Let G be a graph and $s \in S$ such that G is coherent from s. Let $M \subset D(s,r)$, for some $r \geq 1$. M is an s-width-fibre of G iff M is a maximal total X-fibre, for all s-width-fibres X of G such that $X \subset D(s,r-1)$.* □

2.2 s-depth-fibres

Definition 8. Let G be a graph and $s \in S$ such that G is coherent from s. Let $\widetilde{G} = G/\sim_s$ be the quotient graph for $\sim_s$. Let $x, y \in V$ and $\tilde{x}, \tilde{y}$ be the equivalence classes of x and y modulo $\sim_s$. Let $\widetilde{G}'$ be the partial subgraph of $\widetilde{G}$,

obtained by deleting the edges of $D(\tilde{s},r) \times D(\tilde{s},r)$, for all $r \geq 0$. We say that x and y are *depth-equivalent* for s, denoted by $x \equiv_s y$, iff either $\tilde{x} = \tilde{y}$, or the following holds:

- $\tilde{x}$ and $\tilde{y}$ belong to a uniform path P of $\widetilde{G}'$, such that each vertex of P is of degree at most 2 in $\widetilde{G}'$.
- if $\tilde{x}$ [resp. $\tilde{y}$] is an extremity of the uniform path P, then $\tilde{x}$ [resp. $\tilde{y}$] is of degre 1 in $\widetilde{G}'$.

Let $\overline{G} = \widetilde{G}/\equiv_s$ be the quotient graph for $\equiv_s$. $\overline{V}$ is the set of *s-depth-fibres* of G.

Example In figure 2, the set $M = M_1 \cup M_2 \cup M_3$ is an s-depth-fibre of G. The set $M_1 \cup M_2 \cup M_3 \cup Z$ is not an s-depth-fibre of G, because $z \not\equiv_s x$: in $\widetilde{G}'$, $\tilde{z}$ is an extremity of a uniform path P (of colour c_1), whose vertices are of degree at most 2, but $\tilde{z}$ is not of degree 1 in $\widetilde{G}'$.

3 Product and decomposition

Definition 9. Let G be a graph. For $i = 1, 2$, let s_i be an element of S such that G is coherent from s_i. Let us denote the depth-equivalence for s_1 by $\equiv_1$, and the width-equivalence for s_2 by $\sim_2$. We say that $\equiv_1$ and $\sim_2$ are *orthogonal* iff the following holds: if $x \equiv_1 y$, then $x \not\sim_2 y$, and if $x \sim_2 y$ then $x \not\equiv_1 y$.
We say that $\equiv_1$ and $\sim_2$ are *balanced* iff the following holds:

- if $x \equiv_1 y$, then for all $x' \sim_2 x$ there exists $y' \sim_2 y$ such that $x' \equiv_1 y'$,
- if x and y are not equivalent for $\equiv_1$ and $\sim_2$, if $[x, y] \in c$, then for all u, v such that $u \equiv_1 x$, $u \sim_2 y$ and $v \sim_2 x$, $v \equiv_1 y$, one has $[u, v] \in c$.

We say that G is *regular* with respect to the two sources s_1 and s_2 iff $\equiv_1$ and $\sim_2$ are orthogonal and balanced.

For two graphs H, F, containing sources s_1, s_2 respectively, we shall denote by $H \bowtie F$ the pullback of the two morphisms $(H \longrightarrow H/\equiv_{s_1})$, $(F \longrightarrow F/\sim_{s_2})$, if $H/\equiv_{s_1} = F/\sim_{s_2}$.

Theorem 10. *Let G be a graph. For $i = 1, 2$, let s_i be an element of S such that G is coherent from s_i. Let us denote the depth-equivalence for s_1 by $\equiv_1$, and the width-equivalence for s_2 by $\sim_2$.*
G is regular with respect to the two sources s_1 and s_2 iff $G = (G/\sim_2) \bowtie (G/\equiv_1)$. □

4 Example: Grids

Let us say that a pair of edges of a graph G is *admissible* if they are not loops, they have the same colour and exactly one extremity in common. Let

$(e_1, e'_1), (e_2, e'_2), \ldots, (e_n, e'_n)$ be a sequence of admissible pairs , the graph $G/[e_1 = e'_1, e_2 = e'_2, \ldots, e_n = e'_n]$, obtained by identifying e_1 with e'_1, e_2 with e'_2, $\ldots$, e_n with e'_n, will be called the result of a *folding* of G (see [11] for a definition in the non coloured case).

Let G_3 be the graph of figure 1. The sets I, J, K, L and M are the depth-fibers of G_3 for s_1 (i.e. the equivalence classes for the relation $\equiv_{s_1}$). Let $Y_3 = G_3/\equiv_{s_1}$ be the quotient graph. Let D be the set of vertices $\{s_2, x, y, z\}$. G_3 is such that the edges which are symmetrical relatively to D, are coloured with the same colour. Thus two vertices which are symmetrical relatively to D, will belong to the same width-fibers of G_3 for s_2, and they will be identified in $G_3/\sim_{s_2}$.
The width-fibers of G_3 for s_2 (i.e. the equivalence classes for the relation $\sim_{s_2}$), and the quotient graph $\widetilde{G_3} = G_3/\sim_{s_2}$ are shown in figure 3. $\widetilde{G_3}$ is the result of three consecutive foldings of G_3 (starting from the nodes s_2, x, y and z).
Figure 4 shows that $\widetilde{G_3} \bowtie Y_3 = G_3$. One easily obtains the decomposition of the

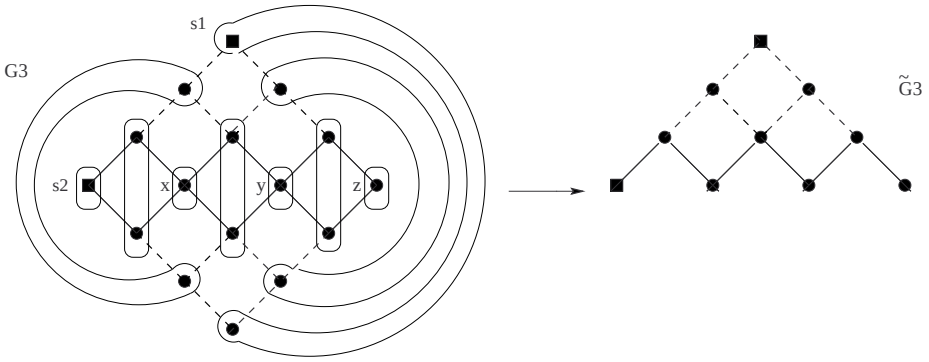

Fig. 3. The width-fibers of G_3 for s_2

graph G_2 of figure 5. $\widetilde{G_2}$ is the result of two consecutive foldings of G_2. On the other hand, the only difference between $\widetilde{G_3}$ and G_2 is the coloring of edges. In fact, $\widetilde{G_3}$ is obtained from G_2 by translating the colours "downward" using the following translation t.
As for all r, $1 \leq r \leq 3$, each $z \in D(s_1, r)$ is a $D(s_1, r-1)$-fiber, we can define $t(G)$ as follows : for $r = 2$ and $r = 1$, translate the colour of any edge $[x, y] \in D(s_1, r-1) \times D(s_1, r)$, on the edges $[y, z] \in D(s_1, r) \times D(s_1, r+1)$ and colour all the edges of $D(s_1, 0) \times D(s_1, 1)$ with the colour c_1.
$\widetilde{G_3}$ is obtained by applying t to G_2 (figure 6), so $\widetilde{G_3} = t(\widetilde{G_2} \bowtie Y_2)$.
The decomposition of the graph G_1 is shown in figure 7. Let us call A the quotient graph $G_1/\sim_{s_2}$. A is a folding of G_1 and $A \bowtie Y_1 = G_1$. On the other hand, $\widetilde{G_2}$ is obtained by applying the translation of colours t, to the graph G_1. Consequently, $\widetilde{G_2} = t(A \bowtie Y_1)$.

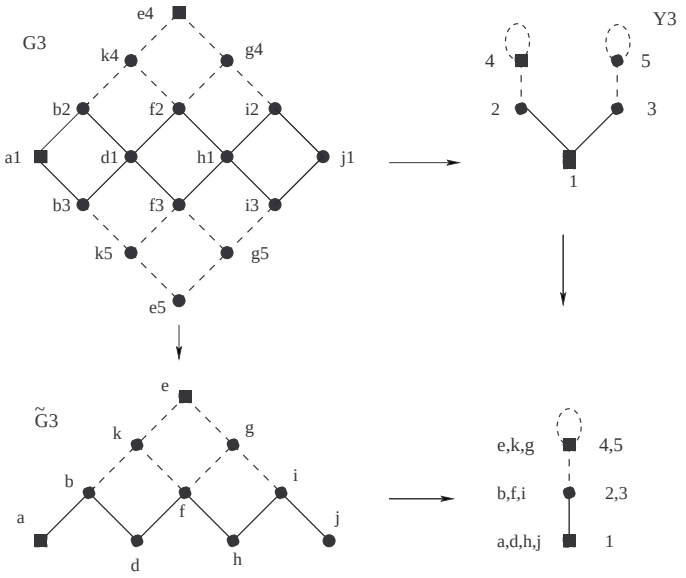

Fig. 4. The pullback of $\widetilde{G_3}$ and Y_3

Finally one obtains the following factorization of G_3:
$G_3 = \widetilde{G_3} \bowtie Y_3 = t(t(A \bowtie Y_1) \bowtie Y_2)) \bowtie Y_3$.

This sequence of folding can be done in a similar way, starting from a grid G_n of size n. One would obtain G_n as the result of n pullbacks of very simple graphs as shown in figure 8.

5 Conclusion

This paper describes some ongoing research on the structure of several types of graphs which we call regular just because we feel they look that way, although they are not in any language theoretic sense (even worse, grids are commonly presented as the basic obstruction to context-freeness). Using the pullback as a basic operation, we show how they may be decomposed into some basic building blocks. Further work is currently done in several directions :

- try to characterize more clearly what classes of graphs are adequate for such an approach,
- make precise the nature of the building blocks, the "prime factors": how many do we need to build a regular graph of a given size (in some family), do they have a predictable shape when size grows (or not, as usual prime numbers),

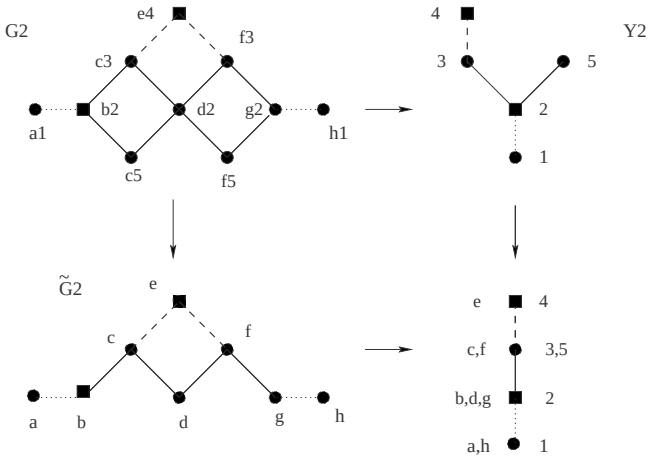

Fig. 5. The pullback of $\widetilde{G_2}$ and Y_2

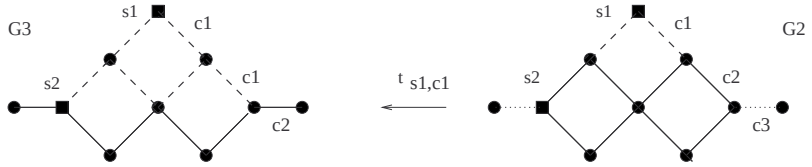

Fig. 6. The transformation t applied to G_2

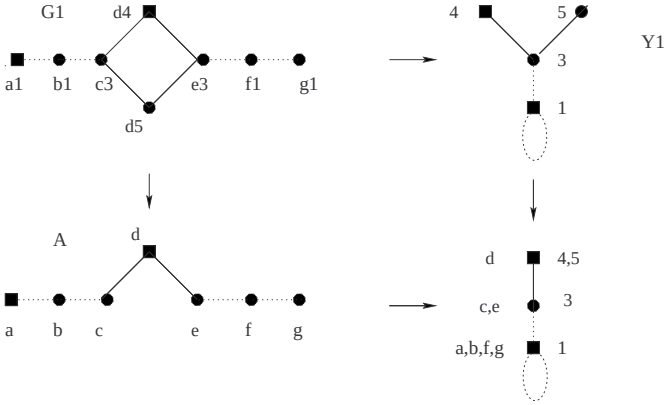

Fig. 7. The pullback of A and Y_1

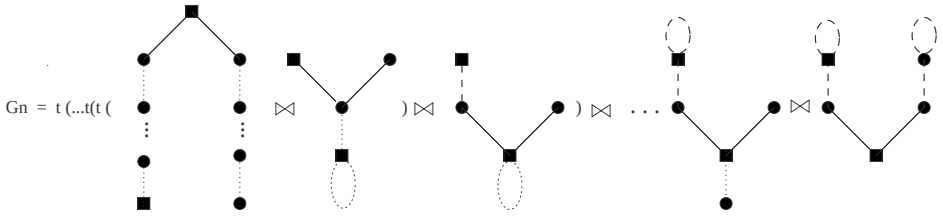

Fig. 8. The factorization of G_n

- given a set of prime factors (coming from the decomposition of grids for instance), decide whether we can use them to build something different (i.e. are there any closure properties),
- extend this decomposition to classes of graphs whose intrinsic regularity has to be understood in a different way (like equational graphs, automatic graphs or covering of graphs)

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