

# Decomposing graphs with symmetries

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**Abstract.** While decomposing graphs in simpler items greatly helps to design more efficient algorithms, some classes of graphs can not be handled using the classical techniques.

We show here that a graph having enough symmetries can be factored into simpler blocks through a standard morphism and that the inverse process may be formalized as a pullback rewriting system.

## 1 Introduction

Several approaches have been used to try and make advantage of the structural properties of graphs, in order to devise more efficient algorithms or to prove properties for specific classes of graphs. To cite but a few, let us merely quote modular decomposition [3], 2-structures [4] or systems of recursive equations.

They normally come with a decomposition procedure/algorithm as well as a way to rebuild the graph from its basic blocks or at least to describe in a convenient way (such as a kind of syntax tree) the initial structure in terms of those blocks.

Another approach describes certain classes of graphs as algebras whose elements may be built from a set of basic blocks called prime factors [5].

In an earlier paper [1], we have tried to tackle the case of classes of graphs which are not well receptive to the previously cited methods, such as for instance, square grids. Indeed, we have shown how some classes of graphs endowed with a fairly regular structure were suitable for a new treatment based on a pair of quotient/product operations. The basic idea was to use the regularities of the graph structure to define a pair of "complementary" equivalence relations on the graph yielding two quotient graphs whose product (in a suitable category) would give back the initial graph (note that when we talk of product of graphs, we mean the categorical product of the graphs and *not* their cartesian product).

We present here a different approach, which makes use of some more intrinsic "geometrical" properties (although they do not rely on any embedding in a euclidian space) of graphs to provide a new kind of decomposition, namely the existence of symmetries of the graph .

Actually, we show that the presence of a symmetry in a graph defines a decomposition of the graph as the product of a factor which depends of the graph and of a canonical factor. As a consequence, a graph having enough symmetries

or local symmetries (those terms are defined below) may be factored through a standard morphism and the inverse process may be formalized as a (pullback) rewriting system [2].

This paper is organized in three sections. After the basics definitions given in section 2, we study symmetries of graphs (section 3) and described the rewriting process in section 4. For lack of space, proofs have been omitted. None of them is really difficult and we preferred to give complete examples. It must be noted that all pullback diagrams have been computed by program and not simply drawn since computing pullbacks "by hand" is a bit tricky !

## 2 Definitions

In this paper, we consider undirected, connected graphs, for which we shall use the following definition.

**Definition 1.** A graph is a pair of finite sets  $G = \langle V, E \rangle$  where  $V$  is the set of vertices or nodes and  $E$  is the set of edges, together with two mapping  $\sigma, \tau : E \rightarrow V$ . If  $(\sigma(e), \tau(e)) = (u, v)$ , we say that  $e$  is an edge between nodes  $u$  and  $v$  and write  $e = [u, v]$  (the order has no meaning). A node  $u$  is reflexive if there is at least one edge  $e = [u, u]$  in  $E$ , called a loop. A vertex with a loop is said to be reflexive. A graph is reflexive if all its vertices are.

Note that this definition allows multiple edges linking the same nodes and even multiple loops. This possibility will be needed in a technical way to define one of the essential ingredients of our computations, the graph  $A$ .

In this paper, we shall nevertheless be mainly interested in *simple graphs* i.e. graphs without multiple edges.

We shall use the word *item* to denote indifferently a node or an edge.

**Definition 2.** A graph morphism  $h : G \rightarrow G'$  is a pair  $h = \langle h_V, h_E \rangle$  with  $h_V : V \rightarrow V'$  and  $h_E : E \rightarrow E'$  such that  $h_E([u, v]) = [h_V(u), h_V(v)]$ .

It is well known that the good properties of graph morphisms turn the set of graphs into a category that we shall denote by  $\mathcal{G}$ . Let us briefly enumerate the main properties of this category that we shall need in the sequel.

**Proposition 1.** The category  $\mathcal{G}$  of graphs has arbitrary products and equalizers. The graph with one vertex and one edge (ie one reflexive vertex) is a terminal object simply denoted by  $\odot$ . It is a neutral element for the product. The category  $\mathcal{G}$  has arbitrary limits (is complete). In particular,  $\mathcal{G}$  has pullbacks.

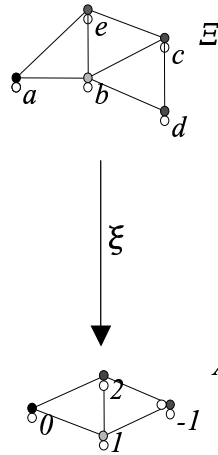
Since we shall use it intensively, let us recall here that the pullback of two graph morphisms  $f_i : G_i \rightarrow F, i = 1, 2$  is a pair of arrows  $h_i : H \rightarrow G_i, i = 1, 2$  where  $H$  is the subgraph of the product consisting of exactly those items (nodes and vertices) on which  $f_i \circ h_i$  coincide. Many examples will be provided in the rest of this paper.

We shall need in the sequel two special graphs :

- $\Xi$  is the reflexive graph with :
  - vertices :  $a, b, c, d, e$
  - edges :  $e_1 = [a, b], e_2 = [a, e], e_3 = [b, e], e_4 = [c, e], e_5 = [b, c], e_6 = [b, d],$   
 $e_7 = [a, a], e_8 = [b, b], e_9 = [c, c], e_{10} = [d, d], e_{11} = [e, e], e_{12} = [c, d]$
- $A$  is the graph with
  - vertices :  $0, 1, 2, -1$  and
  - edges :  $e_1 = [0, 1], e_2 = [0, 2], e_3 = [1, 2], e_4 = [2, -1], e_5 = [1, -1], e_6 = [0, 0],$   
 $e_7 = [1, 1], e_8 = [2, 2], e_9 = [-1, -1], e_{10} = [-1, -1],$

and a graph morphism  $\xi : \Xi \longrightarrow A$  defined

- on nodes by  $\xi(c) = \xi(d) = -1, \xi(a) = 0, \xi(b) = 1, \xi(e) = 2$
- on edges by
  - $\xi(e_1) = e_1,$
  - $\xi(e_2) = e_2,$
  - $\xi(e_3) = e_3,$
  - $\xi(e_4) = e_4,$
  - $\xi(e_5) = \xi(e_6) = e_5,$
  - $\xi(e_9) = \xi(e_{10}) = e_9,$
  - $\xi(e_{12}) = e_{10},$
  - all loops have obvious projections.



**Fig. 1.** The graph morphism  $\xi : \Xi \longrightarrow A$

These graphs together with the morphism  $\xi$  are depicted on figure 1 and will appear in the right hand column of all the pullback diagrams in the sequel (note

that  $A$  is not simple : it has two loops on node  $-1$ , which allow for a different projection for the edge  $[c, d]$  than for the loops on  $c$  and  $d$ .

As far as possible, we have tried to rely on the reader's intuition when drawing our morphisms : in this case,  $\xi$  may be understood as respecting the vertical projection.

### 3 Graph with symmetries

#### 3.1 Automorphisms

A graph automorphism is a graph endomorphism which is one to one on both vertices and edges.

The set of automorphisms of a graph  $G$  is a group under the composition of morphisms. The neutral element is the identity morphism  $Id$ .

Let  $e \in G$  be any item (vertex or edge) of  $G$ . The orbit of  $e$  under  $\varphi$  is the (finite) set  $o(e) = \{\varphi^n(e), n \geq 0\}$ . Let us set  $e \simeq e' \iff e' \in o(e)$ . This defines a congruence on  $G$ .

**Lemma 1.**  $\overline{G} = G / \simeq$  is a graph.

We let  $\overline{\varphi}$  denote the canonical projection  $G \longrightarrow \overline{G}$  and we call it a *folding of  $G$  along  $\varphi$* . Examples will be given later.

In the sequel, we shall concentrate on a very simple type of morphisms, and shall in general denote the image  $\varphi(u)$  of  $u$  by  $u'$ . Automorphisms satisfying other types of properties (such as  $\varphi^n = Id$  for  $n \geq 3$ ) will be studied in further works.

#### 3.2 Symmetries

**Definition 3.** A *symmetry* is an involution, ie a graph automorphism  $\varphi$  such that  $\varphi^2 = Id$ .

Clearly, if  $\varphi$  is a symmetry then for each orbit,  $\#o(e) \leq 2$ .

*Example 1.* Let us first consider a square grid  $G_n$  of size  $n$ . We shall discuss the situation without giving formal definitions of corners, diagonal and median axis or other notions which we hope to be intuitive enough (the same remark holds for the next example). We do not believe either that a picture is necessary.

It is clear from classical geometry (eg by representing the nodes of  $G_n$  as the set of points  $(i, j)$ ,  $1 \leq i, j \leq n$  in the euclidian plane) that it has at least six symmetries (the identity, a central symmetry, two symmetries w.r.t. the diagonals of the square and two w.r.t. the median axis of the square).

Now, let  $a$  be one of its 4 corners (i.e. its nodes of degree 2),  $b$  the opposite corner,  $c$  and  $d$  the two other corners. Under any automorphism, hence under any symmetry  $\varphi$ , the image of  $a$  has to be one of the four corners.

If  $a \rightarrow b$ ,  $c$  and  $d$  may be invariant. Then it is easy to see that  $\varphi$  is the symmetry with axis the diagonal "linking"  $c$  to  $d$  (a set of vertices which turns

out to be a cut set of the grid). Otherwise, ie if  $c \rightarrow d$ , then one can show that  $\varphi$  is the central symmetry (which has an invariant node iff  $n$  is odd).

If  $a \rightarrow c$ , then necessarily  $b \rightarrow d$ , and we find the symmetry whose axis is the median axis, "orthogonal" to  $ac$  (note that this axis may be a set of vertices or a set of edges, depending on the parity of  $n$ ). A similar conclusion holds if  $a \rightarrow d$ .

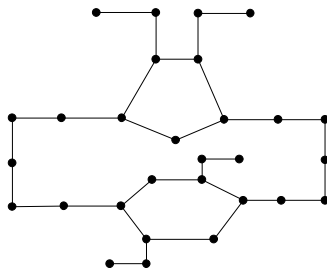
Last, if  $a$  is invariant, at least another corner must be, and one may show that  $b$  is. This leaves two possibilities : either  $c$  and  $d$  are invariant and  $\varphi$  is the identity or  $c \rightarrow d$ , and one gets the symmetry with respect to the axis  $ab$ .

Since all the possibilities have been considered, this shows that there are, as expected, exactly six symmetries.

In this example, we found that the symmetries of a square grid considered as a graph are exactly those which appear if we consider this graph as embedded in a natural way in the euclidian space. They are of two types : axial symmetries and central symmetries. The first ones have an invariant axis (a set of invariant nodes or edges which cut the graph in two isomorphic parts), the second ones have at most one invariant node which is the center of the symmetry.

This is often the case, but the intuition coming from euclidian geometry is not true in general : symmetries of a graph may be more complex since they can combine both types.

*Example 2.* This is easily seen on figure 2. The graph has two cycles, each of which has two "antennas". It is easy to check that it has only one symmetry, which has to be "axial" on the upper cycle and "central" on the lower one.



**Fig. 2.** A graph with a complex symmetry

This general situation is described by the next lemma.

A *path*  $P$  in  $G$  is a sequence  $\{u_1, \dots, u_n\}$  of nodes such that for each  $1 \leq i \leq n$ ,  $[u_i, u_{i+1}]$  is an edge in  $G$ . We say that  $P$  is *stable* if for each node  $u$  of  $P$ ,  $u'$  belongs to  $P$  as well.

Let  $u$  be a node of  $G$  and  $u' = \varphi(u)$  be its image under  $\varphi$  and let  $P = \{u, u_1, \dots, u_n, u'\}$  be a path from  $u$  to  $u'$  (there exist always one, since the graph is connected).

**Lemma 2.** *If there exists a node  $v$  of  $P$  such that either  $v = v'$  or  $[v, v']$  is an edge of  $P$ , then, one can find a path  $P'$  from  $u'$  to  $v'$  such that  $P' = \varphi(P)$ .*

*Otherwise, let  $f = u_i$  a node of  $P$  such that  $f' = u'_i$  belongs to  $P$  but  $u'_{i+1}$  does not. Then there exists a path  $P^*$  from  $u_i$  to  $u'_i$  such that the cycle obtained from  $P^*$  concatenated with the restriction of  $P$  to  $\{u_i, \dots, u'_i\}$  is stable, but has no invariant. Moreover, there exists an other path  $P^{**}$  from  $u'_i$  to  $u'$  such that the concatenation of  $P^*$  and  $P^{**}$  is exactly the image under  $\varphi$  of the restriction of  $P$  to  $\{u, \dots, u'_i\}$ .*

In other words, any path linking a node to its image either can be made symmetric or has to cross a stable cycle with no invariant. These two cases are illustrated in example 2 respectively in the upper and lower cycles.

The edges linking  $P'$  to  $f$  and their images under  $\varphi$  (as described in the lemma) will be said to *cut the cycle* and will be called *cut edges*.

**Lemma 3.** *Let  $G$  be a graph and  $\varphi$  be a symmetry of  $G$ . There exists a set  $C$  of pair of edges of the form  $(e, \varphi(e))$  such that each stable cycle with no invariant is cut exactly once by a pair in  $C$ .*

This set is not unique, since the choice of cut edges is to some extent arbitrary.

For the sake of simplicity, a node  $v$  such that  $v = \varphi(v)$ , an edge linking two invariant nodes or an edge of the form  $[v, \varphi(v)]$  will all be called *invariant* under  $\varphi$ . We let  $Cut(\varphi)$  denote the union of the (uniquely defined) set  $Inv(\varphi)$  of invariant items of the graph (containing the subgraph induced by the set of invariant nodes under the action of  $\varphi$ , and all edges of the form  $[v, \varphi(v)]$ ) and of a set  $C$  of cut edges as defined in the previous lemma.

**Lemma 4.** *The subgraph  $G'$  of  $G$  obtained by removing  $Cut(\varphi)$ , has two connected components  $G_1$  and  $G_2$ , such that  $G_2 = \varphi(G_1)$  (and conversely).*

By collapsing each component  $G_i$  to a different reflexive node, we get the following :

**Proposition 2.** *Let  $G$  be a graph and  $\varphi$  be a symmetry of  $G$ . Then, there exists a morphism  $\psi : G \longrightarrow \Xi$  sending  $G_1$  to  $c$ ,  $G_2$  to  $d$  and all cut nodes to  $b$ .*

The quotient graph  $\overline{G}$  defined in the previous section can be further analyzed to define a new morphism  $\tilde{\varphi}$ , which classifies items in  $G$  in terms of their behavior under the action of  $\varphi$ , ie which distinguishes "normal" vertices, invariant vertices, vertices linked by an invariant edge and path separated by a cut edge.

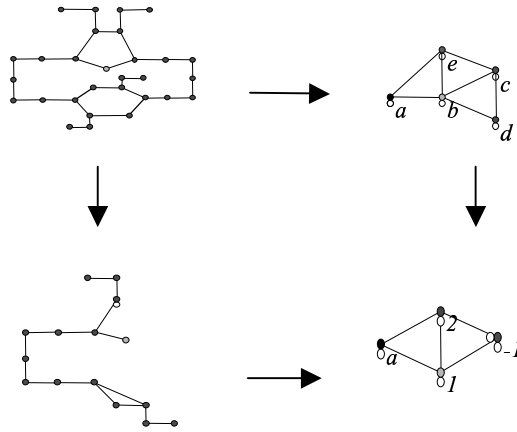
Let  $\tilde{\varphi} : \overline{G} \longrightarrow A$  be the unique graph morphism defined by

- $\tilde{\varphi}(v) = -1$  if  $\#o(v) = 2$ ,
- $\tilde{\varphi}(v) = 1$  if  $\#o(v) = 1$ ,
- $\tilde{\varphi}(e) = e_9 = [-1, -1]$  if  $e$  is a cut edge,
- $\tilde{\varphi}(e) = e_{10} = [-1, -1]$  if  $e$  is an invariant edge

The main result of this section is that the previous constructions of morphisms define both a way to analyze and synthesize the original graph  $G$ .

**Theorem 1.** Let  $G$  be a graph and  $\varphi$  be a symmetry of  $G$ . The pair of arrows  $\overline{G} \xleftarrow{\tilde{\varphi}} G \xrightarrow{\psi} \Xi$  is the pullback of the pair  $\overline{G} \xrightarrow{\tilde{\varphi}} A \xleftarrow{\xi} \Xi$  in the category of graphs.  $\square$

In other words, as was announced in the introduction, the existence of a symmetry  $\varphi$  on a graph  $G$ , gives a decomposition of  $G$  into two factors, one depending on  $G$  ( $\tilde{\varphi}$ ) and a generic one ( $\xi$ ) which depends only on the nature of the transformation (the fact that  $\varphi^2 = Id$ ). Then,  $G$  may be computed as a pullback, or even, if we work in the category of graphs over  $A$ , as a simple categorical product.

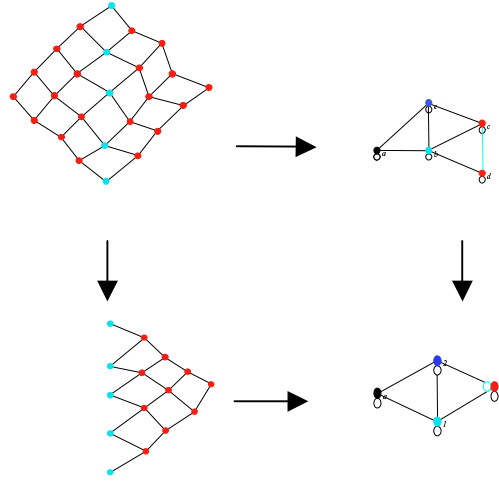


**Fig. 3.** Decomposition by symmetry

*Example 3.* Figure 3 shows a case where most difficulties appear, by providing a decomposition of the graph shown in figure 2. The folding of the graphs shows (left hand side column) : an invariant edge, collapsed to a loop and an invariant node on the upper cycle, two cut edges linking  $u$  to  $v'$  and  $u'$  to  $v$ , which both collapse to the same edge linking the image of  $u$  and  $u'$  to that of  $v$  and  $v'$ .

*Example 4.* An other example may be given with  $G_n$ , the square grid of size  $n$ . As already stated, such a graph has three kind of symmetries :

- with respect to the diagonals : these two symmetries have a set of invariant nodes which are cut sets of the graph
- with respect to the medians : these two symmetries have either a cut set or a set of invariant edges, depending on whether  $n$  is odd or even.
- with respect to the center of the grid, which may be a node.



**Fig. 4.** Decomposing  $G_5$  using a diagonal symmetry

Figure 4 shows a decomposition w.r.t. to one of the diagonal symmetries, while figure 5 shows a decomposition w.r.t. the central symmetry. The slightly irregular shape of the grid and half grid in figure 4 is intended to insist upon the fact that we do not rely on any euclidian embedding : our grid would not be symmetric !

The reader has probably noticed that nodes  $a$  and  $e$  of graph  $\Xi$  as well as nodes 0 and 2 of graph  $A$  did not play any role in the previous decomposition. They will only be useful in the next section, but including them from the beginning allows us to offer a more uniform treatment.

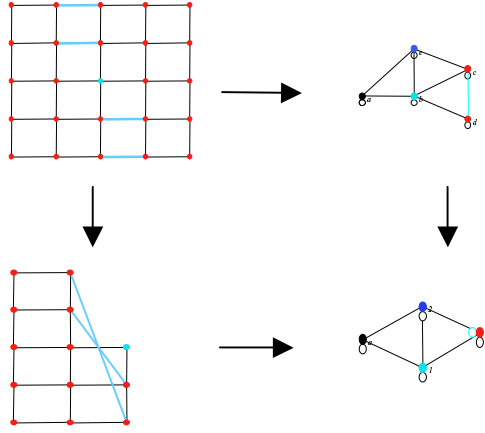
### 3.3 Local symmetries

**Lemma 5.** *Let  $\varphi$  and  $\psi$  be two symmetries of a graph  $G$  and assume that  $\varphi$  and  $\psi$  commute, ie  $\varphi \circ \psi = \psi \circ \varphi$ . Then,  $\psi$  defines a symmetry  $\bar{\psi}$  of the quotient graph  $G/\varphi$ .*

*Example 5.* Let  $G_n$  be the square grid of size  $n$ . Then the two diagonals define commuting symmetries  $\varphi_1$  and  $\varphi_2$  (the two medians define two other commuting symmetries). It is easily seen that applying first  $\varphi_1$  to get a quotient graph  $G_1$  and then  $\bar{\varphi}_2$  to  $G_1$  defines a two level decomposition of the initial grid.

A set of commuting symmetries of a graph can provide by successive folding a first kind of decomposition of a graph. Unfortunately, as is clearly shown by our running example of square grids, this does not go very far and a new analysis of the reduced graph is necessary if we want to reduce the graph to significantly smaller blocks : in our example of grids, using two commuting symmetries reduces





**Fig. 5.** Decomposing  $G_5$  using the central symmetry

the grid to something like one fourth of its original size. But then new symmetries appear and must be investigated. In the sequel we will not consider this approach, which would lead to slightly different developments.

A more general approach seems more fruitful, by considering *local symmetries* of graphs, that we shall define now.

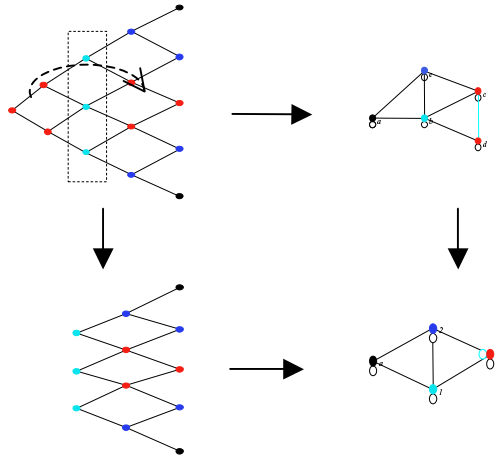
**Definition 4.** Let  $G$  be a graph. We say that an endomorphism  $\varphi$  of  $G$  is a *local symmetry*, if there exists a partial subgraph  $C(\varphi)$  separating  $G$  into two components  $G_1$  and  $G_2$  (not necessarily connected) such that  $\varphi$  is the identity on  $G_2 \cup C(\varphi)$  and  $\varphi(G_1)$  is isomorphic to a subgraph of  $G_2$ .

In other words,  $\varphi$  restricted to  $G_1 \cup C(\varphi) \cup \varphi(G_1)$  gives rise to a symmetry  $\phi$  which keeps invariant  $C(\varphi)$  and exchanges  $G_1$  and  $\varphi(G_1)$ . We shall say that  $G_1$  is the domain of  $\varphi$ . It is easily seen that inclusion of domains defines a partial order on the set of partial symmetries. Let us say that a local symmetry is maximal if it is a maximal element for that order.

We can then follow a line similar to the previous section on symmetries. The local symmetry  $\varphi$  defines an equivalence relation on  $G$  and therefore a canonical projection  $\bar{\varphi} : G \rightarrow \bar{G}$ . We can also define a morphism  $\psi : G \rightarrow \Xi$ , such that the pair of arrows  $\bar{G} \xleftarrow{\bar{\varphi}} G \xrightarrow{\psi} \Xi$  is the pullback of the pair of arrows  $\bar{G} \xrightarrow{\bar{\varphi}} A \xleftarrow{\xi} \Xi$ , yielding a result similar to that of theorem 1. The main difference is that the morphism  $\psi$  distinguishes now more components :  $G_1$  and  $\varphi(G_1)$  sent respectively to nodes  $c$  and  $d$ , neighbors of  $G_1$  and  $\varphi(G_1)$  sent to node  $e$  and other nodes sent to  $a$ , making the whole of graphs  $\Xi$  and  $A$  necessary. This is illustrated in the examples.

*Example 6.* Let us consider again the case of the square grid  $G_5$  and apply first a symmetry w.r.t. to a diagonal yielding a half-grid (as described in figure 4).

Then one may find several local symmetries by considering cuts of  $\overline{G_5}$  parallel to the symmetry axis. Let  $\varphi$  be the maximal local symmetry of  $\overline{G_5}$ . It is shown on figure 6, where the half grid appears on the top left corner. A dotted box indicates the cut used for the local symmetry and a dotted arrow shows the folding which yields the graph in the bottom left corner. It is easily checked that the diagram shown on the figure is actually a pullback.



**Fig. 6.** A local symmetry for the half grid of size 5

## 4 Reconstruction of the graph

In this section, we show how the previously described mechanism may be transformed into a generating system to build families of regular graphs.

### 4.1 Pullback rewriting

Let us recall here the basics of pullback rewriting as defined for instance in [2].

**Definition 5.** A production rule is a graph morphism  $p : R \longrightarrow A$ , where  $A$  is called the left-hand side of the rule, while  $R$  is the right-hand side. An occurrence of the left-hand side of  $p$  in the graph  $G$  is a morphism  $x : G \longrightarrow A$ . The rewriting of  $G$  by  $p$  at occurrence  $x$  is the pullback of  $x$  and  $p$  in the category of graphs.

The graph  $A$  of this definition plays the role of an alphabet in a classical labelling system, to specify the behavior of nodes of the graph  $G$  under a rewriting. The main difference is that the labelling mapping is now a graph morphism, which imposes strong constraints on all the possible labellings of a graph.

Actually, the alphabet graph has a very special form, allowing to distinguish between the nodes to be rewritten (called the *unknowns*), those which will not be affected (called *the context*) and those which lie in the zone between unknowns and context (called the *interface nodes* or *neighbors*).

The graph  $A$  defined in section 2 is an alphabet with one unknown (node  $-1$ ), two neighbors (nodes 1 and 2) and a context node (node 0).

It is well known that, in the classical situation of rewriting systems on words, the application of a rule to a word provides an automatic relabelling of the result, defining the occurrences of the unknowns in the resulting term, which allows to proceed to the application of any (applicable) rule to this result. This relabelling comes partly from the rules itself and partly from the initial labelling of the word.

Similarly, for each rewritten graph and each rewriting rule, we must specify the way the result will be relabelled, ie the way to build the new occurrence in the resulting graphs.

For that purpose, a rewriting rule will be defined by two components : the rule itself and a mechanism which specifies the way each node will be labelled in the resulting graph.

In a similar way, an occurrence of a rule will be defined by two components: the occurrence itself and a mechanism which specifies the behavior of the nodes which the occurrence considers as context nodes.

## 4.2 Rewriting as a generating system

Results of section 3 may be interpreted in terms of rewriting systems since each decomposition step (as described in theorem 1) was done through a pullback diagram which can be interpreted as well as a reconstruction step.

In fact, only three rules were used in the decomposition, all relying on  $\xi : \Xi \longrightarrow A$ . They differ only by the relabelling morphism :

1.  $R_1 = (\xi, \zeta_1)$  (where  $\zeta_1 : \Xi \rightarrow \mathcal{A}$  is the morphism defined by  $\{a \rightarrow 0, b \rightarrow 0, c \rightarrow 0, d \rightarrow 0, e \rightarrow 0\}$ ) is the terminal rule : it handles (local) symmetries by duplicating all nodes labelled by  $-1$  (since  $\Xi$  as two such items) and leaving nodes labelled by 1 invariant. It then relabels everything as 0.
2.  $R_2 = (\xi, \zeta_2)$  (where  $\zeta_2 : \Xi \rightarrow \mathcal{A}$  is the morphism defined by  $\{a \rightarrow 0, b \rightarrow -1, c \rightarrow 1, d \rightarrow 2, e \rightarrow 2\}$ ) handles (local) symmetries in general.
3.  $R_3 = (\xi, \zeta_3)$  (where  $\zeta_3 : \Xi \rightarrow \mathcal{A}$  is the morphism defined by  $\{a \rightarrow 0, b \rightarrow -1, c \rightarrow 1, d \rightarrow -1, e \rightarrow -1\}$ ) is necessary to handle the last local symmetry before applying a global symmetry  $R_1$ .

Note that  $R_1$  and  $R_2$  differ only in that  $R_1$  relabels everything as 0, stopping any possibility of further rewriting. In a way,  $R_1$  is the terminal version of  $R_2$ .

Now if  $G$  is a graph, an occurrence on  $G$  will be a pair  $x = (u, \psi)$  where  $u : G \rightarrow A$  is the occurrence itself (the morphism which will be used for computing the pullback) and  $\psi$  is a family of morphisms  $\psi_v : G \rightarrow A$ , for each  $v \in u^{-1}(0)$  which defines the relabelling mechanism on the context part of the graph.

Then the application of rule  $R_i$  to  $G$  will be simply done by computing the pullback of  $u$  and  $R_i$  in the category of graphs, i.e. the pair of arrows  $g : G' \rightarrow G$  and  $h : G' \rightarrow \Xi$ .

The relabelling of the resulting graph  $G'$  will be done in the following way :

- the new occurrence  $G'$  given by  $R_i$  will be  $\zeta_i \circ h$  on all nodes,
- except for context nodes whose behavior is defined by  $\psi : \psi_v$  is applied if  $v$  becomes a neighbor in the rewritten graph.

It is easily checked that :

**Lemma 6.** *The relabelling (extended in the canonical way to edges) is a well defined graph morphism which may be interpreted as an occurrence on  $G'$ .*

And we may state :

**Theorem 2.** *If  $G$  is decomposed via a succession  $\varphi_1, \varphi_2, \dots, \varphi_n$  of a starting (global) symmetry and of  $n - 1$  local symmetry, it may be reconstructed by applying a sequence of rewriting rules :*

- for a local symmetry following a global one, apply  $R_3$
- for all other local symmetries, apply  $R_2$ .
- for the starting (local) symmetry from the initial graph, apply  $R_1$  (which is yields a terminal graph),

We illustrate this result in the following example.

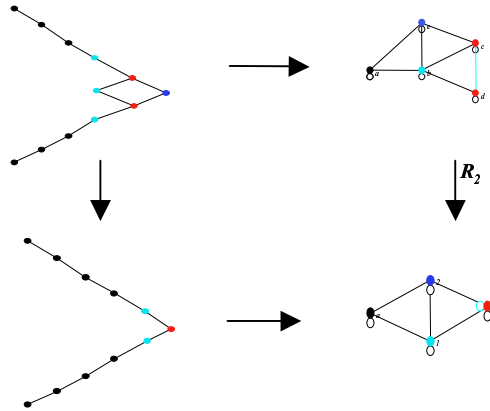
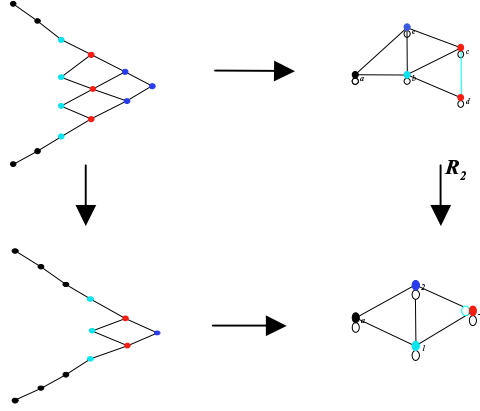


Fig. 7. Step 1

*Example 7.* Let  $G_6$  be the square grid of size 5 and  $P_{11}$  be the path of length 11, with nodes  $v_1, v_2, \dots, v_{11}$  in that order. Rather than running backwards the example described in the previous sections (the reader may check that this could as well be done, simply by choosing a different initial occurrence in the first step of the computation), we choose the occurrence of  $R_2$  on  $P_{11}$  which is described by the morphism  $\pi_{11}$  in the bottom of figure 7 :  $v_5, v_7 \rightarrow 1, v_6 \rightarrow -1$ , all other nodes 0. The relabelling morphisms  $\psi_v$  states that  $\psi_v(v) = 1$  when  $v$  becomes a neighbor, all other context nodes remaining untouched.

For simplicity, we shall identify  $P_{11}$  and  $\pi_{11}$ . Application of rule  $R_2$  to  $P_{11}$  is shown on that same figure. The relabelling computed after applying the rule is shown on the resulting graph.

Two more applications of  $R_2$  are shown on the next two figures 8 and 9, reconstructing stepwise the grid by reverse application of local symmetries.



**Fig. 8.** Step 2

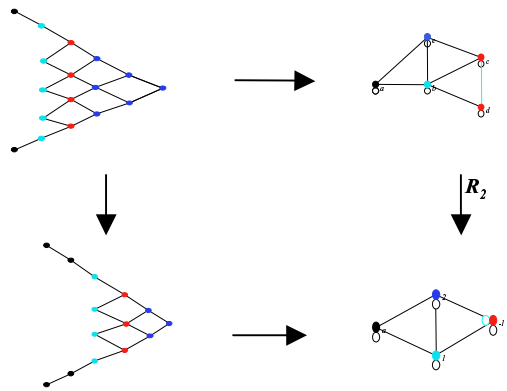
Two last steps are then necessary. First, an application of  $R_3$  as shown on figure 10 which does the same thing as  $R_2$ , but prepares a new relabelling which will allow to correctly apply rule  $R_1$  which has to take care of the final symmetry. The computation is shown on figure 11

This shows that :  $G_5 = R_1 \circ R_3 \circ R_2^3(P_{11})$ .

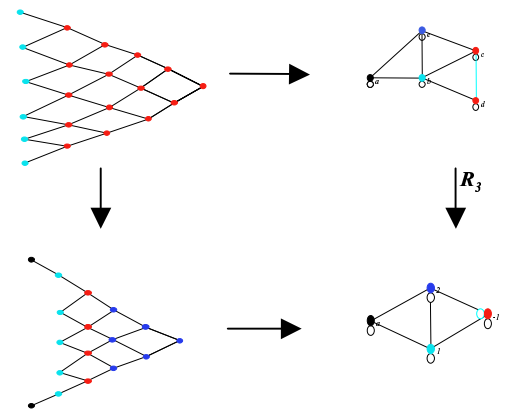
More generally, one could show that for  $n \geq 3$ ,  $G_n = R_1 \circ R_3 \circ R_2^{n-3}(P_{2n-1})$ .

As already mentioned, the same computation could have started from a different occurrence on  $P_{11}$ .

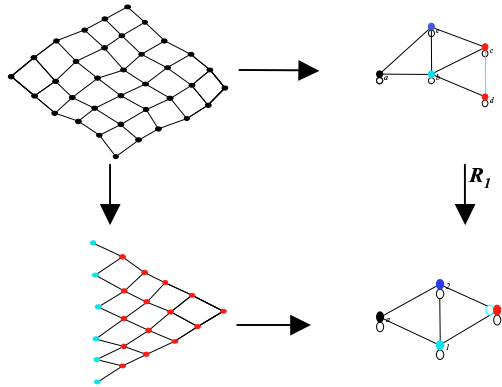
Yet another occurrence, (with a different relabelling mechanism) would lead for instance to the generation of other regular patterns such as hexagonal lattices (we don't have space here to develop that). A classification of all those possibilities remains to be done.



**Fig. 9.** Step 3



**Fig. 10.** Step 4



**Fig. 11.** Step 5

## 5 Conclusion

In this paper, we have shown how symmetries and local symmetries give rise to a natural factorization of a graph, and provide a way to decompose into simpler items graphs having a sufficiently regular structure. An inverse process is described through a rewriting system to reconstruct the initial graph.

The geometrical properties we have used here are but a very simple type of properties of a graph that may be described through the existence of automorphisms of the graph and which we believe are worth exploring.

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