

# Une construction de preuve en Coq

INF462 – Master S&T Informatique – Bordeaux 1

2008–2009

```
Coq < Variables P Q R:Prop.  
P is assumed  
Q is assumed  
R is assumed  
Coq < Lemma imp_perm:(P->(Q->R))->(Q->(P->R)).
```

```
1 subgoal
```

```
P : Prop
```

```
Q : Prop
```

```
R : Prop
```

```
=====
```

```
(P -> Q -> R) -> Q -> P -> R
```

```
Coq < Proof.
```

```
Coq <
```

$\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))$

```
Coq < Variables P Q R:Prop.
P is assumed
Q is assumed
R is assumed
Coq < Lemma imp_perm:(P->(Q->R))->(Q->(P->R)).

1 subgoal

P : Prop
Q : Prop
R : Prop
=====
(P -> Q -> R) -> Q -> P -> R
Coq < Proof.

Coq < intro H.
```

$\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))$

```
Coq < intro H.  
1 subgoal
```

```
P : Prop
```

```
Q : Prop
```

```
R : Prop
```

```
H : P -> Q -> R
```

```
=====
```

```
Q -> P -> R
```

```
Coq <
```

$$\text{Lam} \frac{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

```
Coq < intro H.
```

```
1 subgoal
```

```
P : Prop
```

```
Q : Prop
```

```
R : Prop
```

```
H : P -> Q -> R
```

```
=====
```

```
Q -> P -> R
```

```
Coq < intro q.
```

$$\text{Lam} \frac{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

```
Coq < intro q.  
1 subgoal
```

```
P : Prop
```

```
Q : Prop
```

```
R : Prop
```

```
H : P -> Q -> R
```

```
q : Q
```

```
=====
```

```
P -> R
```

```
Coq <
```

$$\text{Lam} \frac{\text{Lam} \frac{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

```
Coq < intro q.
```

```
1 subgoal
```

```
P : Prop
```

```
Q : Prop
```

```
R : Prop
```

```
H : P -> Q -> R
```

```
q : Q
```

```
=====
```

```
P -> R
```

```
Coq < intro p.
```

$$\text{Lam} \frac{\text{Lam} \frac{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

```
Coq < intro p.
```

```
1 subgoal
```

```
P : Prop
```

```
Q : Prop
```

```
R : Prop
```

```
H : P -> Q -> R
```

```
q : Q
```

```
p : P
```

```
=====
```

```
R
```

```
Coq <
```

$$\text{Lam} \frac{P \rightarrow (Q \rightarrow R), Q, P \vdash R}{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}$$
$$\text{Lam} \frac{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}$$
$$\text{Lam} \frac{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

```
Coq < intro p.  
1 subgoal
```

```
P : Prop
```

```
Q : Prop
```

```
R : Prop
```

```
H : P -> Q -> R
```

```
q : Q
```

```
p : P
```

```
=====
```

```
R
```

```
Coq < apply H.
```

$$\text{Lam} \frac{P \rightarrow (Q \rightarrow R), Q, P \vdash R}{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}$$
$$\text{Lam} \frac{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}$$
$$\text{Lam} \frac{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

Coq < apply H.  
2 subgoals

$P : Prop$   
 $Q : Prop$   
 $R : Prop$   
 $H : P \rightarrow Q \rightarrow R$   
 $q : Q$   
 $p : P$

=====

$P$   
subgoal 2 is:  
 $Q$   
Coq <

$$\begin{array}{c} \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad P \rightarrow (Q \rightarrow R), Q, P \vdash P \\ \text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad P \rightarrow (Q \rightarrow R), Q, P \vdash P \\ \text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q \rightarrow R} \quad P \rightarrow (Q \rightarrow R), Q, P \vdash Q \\ \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash R} \\ \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R} \\ \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)} \\ \text{Lam} \frac{}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))} \end{array}$$

Coq < apply H.  
 2 subgoals

$P : Prop$   
 $Q : Prop$   
 $R : Prop$   
 $H : P \rightarrow Q \rightarrow R$   
 $q : Q$   
 $p : P$

=====

$P$   
 subgoal 2 is:

$Q$   
 Coq < assumption.

$$\begin{array}{c}
 \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad P \rightarrow (Q \rightarrow R), Q, P \vdash P \\
 \text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad P \rightarrow (Q \rightarrow R), Q, P \vdash Q \\
 \text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q \rightarrow R} \quad P \rightarrow (Q \rightarrow R), Q, P \vdash Q \\
 \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash R} \\
 \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R} \\
 \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)} \\
 \text{Lam} \frac{}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}
 \end{array}$$

Coq < assumption.

1 subgoal

*P* : Prop

*Q* : Prop

*R* : Prop

*H* : *P* -> *Q* -> *R*

*q* : *Q*

*p* : *P*

=====

*Q*

Coq <

$$\begin{array}{c} \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P} \\ \text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad \text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q \rightarrow R} \quad \text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q} \\ \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash R} \\ \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R} \\ \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)} \\ \text{Lam} \frac{}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))} \end{array}$$

Coq < assumption.

1 subgoal

*P* : Prop

*Q* : Prop

*R* : Prop

*H* : *P* -> *Q* -> *R*

*q* : *Q*

*p* : *P*

=====

*Q*

Coq < assumption.

$$\begin{array}{c} \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P} \\ \text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q \rightarrow R} \quad \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q} \\ \text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q \rightarrow R} \quad \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash R} \\ \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R} \\ \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)} \\ \text{Lam} \frac{}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))} \end{array}$$

Coq < assumption.

*Proof completed.*

Coq <

$$\frac{\text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P}}{\text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q \rightarrow R}} \quad \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q}}{\text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash R}} \quad \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}}{\text{Lam} \frac{}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}} \quad \text{Lam} \frac{}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

Coq < assumption.

*Proof completed.*

Coq < Qed.

$$\frac{\text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P}}{\text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q \rightarrow R}} \quad \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q}}{\text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash R}} \quad \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}}{\text{Lam} \frac{}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}} \quad \text{Lam} \frac{}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

Coq < assumption.

*Proof completed.*

Coq < Qed.

*intro H.*

*intro q.*

*intro p.*

*apply H.*

*assumption.*

*assumption.*

*imp\_perm is defined*

Coq <

$$\text{App} \frac{\text{Var} \frac{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)}{\text{Var} \frac{P \rightarrow (Q \rightarrow R), Q, P \vdash P}}{\text{App} \frac{P \rightarrow (Q \rightarrow R), Q, P \vdash Q \rightarrow R} \text{Var} \frac{P \rightarrow (Q \rightarrow R), Q, P \vdash Q}}{\text{Lam} \frac{P \rightarrow (Q \rightarrow R), Q, P \vdash R}{\text{Lam} \frac{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}{\text{Lam} \frac{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}{\text{Lam} \frac{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}}}}}}}$$

Coq < assumption.

*Proof completed.*

Coq < Qed.

*intro H.*

*intro q.*

*intro p.*

*apply H.*

*assumption.*

*assumption.*

*imp\_perm is defined*

Coq < Print imp\_perm.

$$\frac{\text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P \rightarrow (Q \rightarrow R)} \quad \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash P}}{\text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q \rightarrow R}} \quad \text{Var} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash Q}}{\text{App} \frac{}{P \rightarrow (Q \rightarrow R), Q, P \vdash R}} \quad \text{Lam} \frac{}{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}}{\text{Lam} \frac{}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}} \quad \text{Lam} \frac{}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

```

Coq < Print imp_perm.
imp_perm =
fun (H : P -> Q -> R) (q : Q) (p : P) => H p q
: (P -> Q -> R) -> Q -> P -> R
Coq <

```

$$\frac{\frac{\frac{H:P \rightarrow (Q \rightarrow R), q:Q, p:P \vdash H:P \rightarrow (Q \rightarrow R)}{H:P \rightarrow (Q \rightarrow R), q:Q, p:P \vdash Hp:Q \rightarrow R} \quad \frac{H:P \rightarrow (Q \rightarrow R), q:Q, p:P \vdash p:P}{H:P \rightarrow (Q \rightarrow R), q:Q, p:P \vdash q:Q}}{H:P \rightarrow (Q \rightarrow R), q:Q, p:P \vdash (Hp)q:R}}{H:P \rightarrow (Q \rightarrow R), q:Q \vdash (\lambda p:P. (Hp)q):P \rightarrow R}}{H:P \rightarrow (Q \rightarrow R) \vdash (\lambda q:Q. \lambda p:P. (Hp)q):Q \rightarrow (P \rightarrow R)}}{\vdash (\lambda H:P \rightarrow (Q \rightarrow R). \lambda q:Q. \lambda p:P. (Hp)q):(P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$