Type Theory in Ludics

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Ludics may be seen as a non typed game semantics based on interaction.

Design: an innocent strategy (it may also be seen as a proof-like sequent structure).

Actions: moves with a polarity +,-.
- +: when a player talks
- -: when a player receives what the other player said
- there is a special positive action, called the daimon: $\rightarrow$ when one of the two player uses the daimon, he “gives up” and the other player wins.

Chronicle: play
an alternating sequence of actions.

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Design: a set of chronicles with some abrorescence-like properties.

Example

Strategy-like design

\[ D = \]

\[ \begin{array}{c}
\text{280} \\
\text{290} \\
\text{28} \\
\text{29} \\
\text{2} \\
\epsilon \\
\text{45} \\
\text{4} \\
\text{66} \\
\text{6} \\
\epsilon \\
\end{array} \]

Proof-like design

\[ D = \]

\[ \begin{array}{c}
\text{\textit{\varnothing}} & \text{\textit{\varnothing}} & \text{\textit{\varnothing}} & \text{\textit{\varnothing}} \\
\text{\vdash 280} & \text{\vdash 290} & \text{\vdash 450} & \text{\vdash 660} \\
\text{28} & \text{29} & \text{45} & \text{66} \\
\text{\vdash 2} & \text{\vdash 4} & \text{\vdash 6} \\
\epsilon & \vdash \\
\end{array} \]
Interaction, Normalization

- **Normalization** of $\{E, \mathcal{F}\}$ denoted by $[[E, \mathcal{F}]]$: the result of the interaction between $E$ and $\mathcal{F}$.
- **Interaction** between two designs: the travel which starts from the first positive action, moves to the corresponding negative one in the other design, moves upward to the (unique) positive action which follows, and so on.
  - If the corresponding negative action does not exist, normalization **fails**.
  - If the action $\ast$ is reached, $[[E, \mathcal{F}]] = \ast$, the designs are said **orthogonal** $E \perp \mathcal{F}$.
  - Interaction may also be **infinite**.

Given a set of designs $E$,

$$E^\perp = \{\mathcal{F} | \mathcal{F} \perp E, \text{ for all } E \in E\}$$
Let $\mathcal{E}, \mathcal{F}$ be the following designs

$$
\mathcal{E} = (+, \xi, \{1, 3\})
$$

$$
\mathcal{F} = (+, \xi116, \{1\})
$$

**Interaction** (dashed line) between two (orthogonal) designs

$$
\mathcal{E} \perp \mathcal{F}
$$

$$
[[\mathcal{E}, \mathcal{F}]] = \bullet
$$
Interaction is untyped

Designs can always interact with each other
→ there is only one arena of all the possible moves.
A set of designs closed by biorthogonal is called a behavior $\mathbf{G}^\perp = \mathbf{G}$.

→ a set of strategies that “behave in the same way”.

A design $\mathcal{D}$ can belong to several behaviors $\mathbf{E}^\perp, \mathbf{G}^\perp, ...$

- When we consider $\mathcal{D}$ in a behavior, there is a part of $\mathcal{D}$ which is **minimal** w.r.t. the behavior in which we consider it and is still a design.
- A design can be minimal in a behavior and not in another one.
- We call **principal** the subset of a behavior whose elements are $\chi$-free and minimal w.r.t. the behavior.
Martin-Löf Types Theory

A constructive set theory based on Curry-Howard isomorphism (proposition as types/sets, proofs as programs).

Types are defined by judgements of the form $\Gamma \vdash t : A$ where $t$ is a term of type $A$, w.r.t. a context $\Gamma$.

The type $\mathbb{N}$, the type $List$, the type **cartesian product** $(\Pi x \in A) B(x)$, the type **disjoint union** $(\Sigma x \in A) B(x)$.
There are four kinds of **rules** used to construct judgements:

- **Formation rule**: How to form a new **type A**
- **Introduction rule**: Explains what are the **canonical terms** of type A
- **Elimination rule**: How to define **functions** on the **terms** of type A
- **Equality rule**: How to define **functions** on the **canonical terms** of type A

In MLTT a proposition is characterized by the set of its proofs, in Ludics a behavior is characterized by the designs that generate it.
How to represent a Martin-Löf type in Ludics?

Designs are untyped objects, their type depends on the behavior in which we consider them.

\[ \text{TYPES} \rightarrow \text{BEHAVIORS} \]

canonical terms of type \( A \rightarrow \bot \).
\[
A \rightarrow A \rightarrow \bot
\]

Given a Martin-Löf type \( A \), it is represented by a behavior \( A = A \rightarrow \bot \rightarrow \bot \) generated by the principal set \( A \).
The designs of \( A \) represent the canonical terms of type \( A \).
The kind of rules used to construct judgements in Martin-Löf Type Theory are interpreted as follows:

- **Introduction rule** → define a set of designs which represent canonical terms
- **Formation rule** → prove that the set defined is principal
- **Elimination rule** → cut
- **Equality rule** → cut-elimination

In Ludics terms come before type, as to define a behavior we have to say what are the designs that belong to it.
Simple types

We represent following the simple types:

- The type $\text{Nat}$ with a principal set $\mathbb{N} \text{at}$ and the behavior that it generates.
- The type $\text{Bool}$ with a principal set $\mathbb{B} \text{ool}$ and the behavior that it generates.
- The type $\text{List}$ with a principal set $\mathbb{L} \text{ist}$ and the behavior that it generates.
- The type $\text{List}_n$ with a principal set $\mathbb{L}_n$ and the behavior that it generates.
Natural numbers

A natural number $n \in \mathbb{N}$ is represented by a design $n_\sigma$ ($\sigma$ is a fixed address):

\[
\begin{align*}
0_\sigma &= \vdash \sigma, \\
(n+1)_\sigma &= \vdash \sigma.
\end{align*}
\]

\[
\mathbb{Nat} = \{ n_\sigma | n \in \mathbb{N} \}
\]

**Proposition**

$\mathbb{Nat}$ is principal.

Thus $\mathbb{Nat}^\perp$ is the behavior which represent the type of the natural numbers and $\mathbb{Nat}$ represents the set of its canonical terms.
Lists

We represent the type of lists of length \( n \) following the recursive definition

\[
List_{n+1} ::= \epsilon + \mathbb{N} \times List_n
\]

\[
\mathcal{D}^\epsilon_{0,\xi} = \vdash \xi \text{ is the empty list,}
\]

\[
\mathcal{U}^{a_1}_{\xi 01} \quad \mathcal{D}^{<a_2,\ldots,a_n>}_{n-1,\xi 1} \\
\xi 0 \vdash \xi 0 \vdash \text{ for } n > 0,
\]

\[
\mathcal{D}^{<a_1,\ldots,a_n>}_{n,\xi} = \vdash \xi \text{ for } n > 0,
\]

We denote with \( \mathbb{L}_n \) the set of lists of a given length \( n \in \mathbb{N} \).

**Proposition**

\( \mathbb{L}_n \) is principal.

Thus we represent with \( \mathbb{L}_{n}^{\perp\perp} \), the type of lists of length \( n \) of natural numbers and with \( \mathbb{L}_n \) the set of its canonical terms.
Let the types $A, B$ be represented by $A \perp \perp, B \perp \perp$, then we represent the type $A \rightarrow B$ with the behavior $(A \Rightarrow B) \perp \perp$, where

$$A \Rightarrow B := \{ \emptyset \text{ minimal} \mid \forall A \in A, [[A, \emptyset]] \in B \}.$$  

**Proposition**  

$A \Rightarrow B$ is principal.  

Thus we represent the type $A \rightarrow B$ with the behavior $(A \Rightarrow B) \perp \perp$. 

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What are the elements of $\text{A} \Rightarrow \text{A}$, i.e., the canonical elements of type $A \rightarrow A$?

There is a design, called Pseudo-$\exists \alpha \xi$ that corresponds to the **copycat strategy**: each move is the copy of the last move of the opponent.

The elements of $\text{A} \Rightarrow \text{A}$ are the *Pseudo* – $\exists \alpha \xi$ which copy the branches of the elements of $\text{A}$. 
Simple types and functions on them

We represent the following simple types:

- The type $\text{Nat}$ with a principal set $\mathbb{N} \text{at}$ and the behavior that it generates.
  $\rightarrow$ **successor**, the **predecessor** and the **sum**.

- The type $\text{Bool}$ with a principal set $\mathbb{B} \text{ool}$ and the behavior that it generates.

- The type $\text{List}$ with a principal set $\mathbb{L} \text{ist}$ and the behavior that it generates.

- The type $\text{List}_n$ with a principal set $\mathbb{L}_n$ and the behavior that it generates.
  $\rightarrow$ **add an element in the head position** and **eliminate** the element in the position $i$ of a list.
The type \((\Pi x \in A)B(x)\)

Let \(A\) and a \((B(x))_{x \in A}\), respectively represented by means of the principal set \(A\) and the family of principal sets \((B(x))_{x \in A}\).

We represent the type \((\Pi x \in A)B(x)\) by the behavior generated by the set of designs

\[
(\Pi x \in A)B(x) = \{\mathcal{D} \text{ minimal} | \forall A \in A, [[A, \mathcal{D}]] \in B(A)\}.
\]

**Proposition**

*(\Pi x \in A)B(x)* is principal.

Thus *(\Pi x \in A)B(x)** represents the type \((\Pi x \in A)B(x)\).

**Proposition**

\[
((\Pi x \in A)B(x))^{\perp \perp} = \bigcap_{x \in A}(x \Rightarrow B(x))^{\perp \perp}
\]
The disjoint union and the product are two operations dual to each other, then we want the behavior which represents \((\Sigma x \in A)B(x)\) to be the dual of the behavior which represents \((\Pi x \in A)B(x)\).

Work in progress: can \((\Sigma x \in A)B(x)\) be represented by 
\[ \bigcup_{a \in A} \{(a, \tilde{F})| \tilde{F} \in B(a)\} \]?

Exploiting the usual duality between \(\cup\) and \(\cap\) through \((.)^\perp\).
THANK YOU!