2-distance coloring of not-so-sparse graphs

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Abstract

The square $G^2$ of a graph $G$ is the graph obtained from $G$ by adding an edge between every pair of vertices having a common neighbor. A proper coloring of $G^2$ is also called a 2-distance coloring of $G$.

The maximum average degree $\text{Mad}(G)$ of a graph $G$ is the maximum among the average degrees of the subgraphs of $G$, i.e. $\text{Mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|} : H \subseteq G \right\}$. Graphs with bounded maximum average degree are often called sparse graphs. There are results about the chromatic number of $G^2$ with $\text{Mad}(G) < m$ for several values of $m$, all strictly lesser than 4.

We provide upper and lower bounds for the chromatic number of $G^2$ when $\text{Mad}(G) < 2k$ for every $k \geq 2$, and conjecture that our lower bounds are best possible.

1 Introduction

In this paper, all the graphs are finite, simple and undirected. For a graph $G$, we denote by $V(G)$, $E(G)$, $g(G)$, $\Delta(G)$ and $\delta(G)$ its vertex set, edge set, girth (i.e. the length of a smallest cycle), maximum degree and minimum degree respectively. A vertex with degree $k$ (at most $k$, at least $k$) is called a $k$-vertex (a $k^-$-vertex, a $k^+$-vertex). We define similarly a $k$-neighbor ($k^-$-neighbor, $k^+$-neighbor) of a vertex. The distance between two vertices is the length of a shortest path connecting them.

The $k$-th power $G^k$ of a graph $G$ is the graph having $V(G)$ as vertex set, and such that $xy \in E(G^k)$ if and only if $x$ and $y$ are at distance at most $k$ in $G$. The chromatic number of the power of a graph was first introduced by Kramer and Kramer in 1969 [15], and was widely studied since this date (for a survey, see [16]). The second power of a graph is also called the square. A proper coloring of $G^2$ is also called a 2-distance coloring of $G$.

In a proper coloring of $G^2$, a vertex and its neighbors in $G$ have different colors, so $\chi(G^2) \geq \Delta(G) + 1$. Since every vertex of $G^2$ has at most $\Delta(G)$ neighbors and at most $\Delta(G)^2 - \Delta(G)$ vertices at distance 2, we have $\chi(G^2) \leq \Delta(G)^2 + 1$. There are graphs reaching that lower and upper bounds, like the stars for the lower bound, and
the cycle of length 5 for the upper bound. For large values of $\Delta$, the best known lower bound, obtained by Alon and Mohar [1], is $\Delta(G)^2 - 2\Delta(G) + 1$ when $\Delta(G) - 1$ is a prime power. For a deep survey on the 2-distance coloring and one of its generalization, the $L(p,q)$-labeling, see [11].

Studying the chromatic number of the square of planar graphs, Wegner proposed the following conjecture:

**Conjecture 1** (Wegner 1977 [20]). Let $G$ be a planar graph with $\Delta(G) = \Delta$. Then

$$\chi(G^2) \leq \begin{cases} 7 & \text{if } \Delta = 3, \\ \Delta + 5 & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1 & \text{if } \Delta \geq 8 \end{cases}$$

In [20], Wegner proved that the upper bounds are sharp if the conjecture is true, and that $\chi(G^2) \leq 8$ if $\Delta(G) = 3$. Wang and Lih [17] established this conjecture for outerplanar graphs. For planar graphs, all three parts of the conjecture remains open (although part 1 was believed established for some time, see [16]).

In [19], Wang and Lih provides better bounds for planar graphs with given girth:

**Theorem 1** (Wang and Lih 2003 [19]). Let $G$ be a planar graph with $\Delta(G) = \Delta$:

- If $g(G) \geq 5$, then $\chi(G^2) \leq \Delta + 16$.
- If $g(G) \geq 6$, then $\chi(G^2) \leq \Delta + 10$.
- If $g(G) \geq 7$, then $\chi(G^2) \leq \Delta + 5$.

Recently, Bu and Zhu improved the second result of Wang and Lih:

**Theorem 2** (Bu and Zhu, 2012 [10]). Let $G$ be a planar graph. If $g(G) \geq 6$, then $\chi(G^2) \leq \Delta(G) + 5$.

For greater values of $\Delta$, Wang and Lih established a conjecture:

**Conjecture 2** (Wang and Lih 2003 [19]). For every $g \geq 5$, there is an integer $M(g)$ such that $\chi(G^2) = \Delta(G) + 1$ for every planar graph $G$ of girth $g$ and with $\Delta(G) \geq M(g)$.

This conjecture was disproved by Borodin et al. in 2004 [5, 8], since they provided examples of planar graphs $G$ with girth 5 and 6 and with $\chi(G^2) = \Delta(G) + 2$ for every value of $\Delta(G)$. Therefore they proved this conjecture for planar graphs with girth at least 7 [9]. The disproval of this conjecture inspired this result:

**Theorem 3** (Dvořák et al. 2008 [13]). For every planar graph $G$, $\chi(G^2) \leq \Delta(G) + 2$ if $g(G) \geq 6$ and $\Delta(G) \geq 8821$. 

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The study of 2-distance coloring of planar graph was generalized toward two directions we describe below: generalization to list coloring and to classes of graphs bounded by their maximum average degree.

Let $G$ be a graph and $C$ a set of colors. Let $L$ be a function assigning to every vertex $v$ of a graph $G$ a list $L(v)$ of colors of $C$. If $G$ has a proper coloring where every vertex $v$ is colored with a color of $L(v)$, then $G$ is said to be $L$-colorable. The list chromatic number or choosability of a graph $G$, denoted by $\chi_l(G)$, is the lesser integer $k$ such that $G$ is $L$-colorable for every function $L$ such that $|L(v)| \geq k$ for every vertex $v$ of $G$. We have easily $\chi_l(G) \geq \chi(G)$.

The maximum average degree $\text{Mad}(G)$ of a graph $G$ is the maximum among the average degrees of the subgraphs of $G$, i.e. $\text{Mad}(G) = \max \{ \frac{|2|E(H)|}{|V(H)|} \mid H \subseteq G \}$. The maximum average degree of a graph can be computed in polynomial time by an algorithm from Jensen and Toft [14]. The following lemma is well known:

**Lemma 4.** From every planar graph $G$, $\text{Mad}(G) \leq \frac{2g(G)}{g(G)-2}$.

The following table gives the upper bound for $\text{Mad}(G)$ depending on $g(G)$ in a planar graph $G$:

<table>
<thead>
<tr>
<th>maille $g(G)$</th>
<th>$\text{Mad}(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$&lt; 6$</td>
</tr>
<tr>
<td>4</td>
<td>$&lt; 4$</td>
</tr>
<tr>
<td>5</td>
<td>$&lt; \frac{10}{3}$</td>
</tr>
<tr>
<td>6</td>
<td>$&lt; 3$</td>
</tr>
<tr>
<td>7</td>
<td>$&lt; \frac{14}{5}$</td>
</tr>
<tr>
<td>8</td>
<td>$&lt; \frac{8}{3}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Theorem 1 was recently improved and/or extended to maximum average degree (even if it is not explicitely stated, results of Theorems 1 and 5 holds for list coloring):

**Theorem 5** (C., Montassier, Raspaud 2013 [12]). Let $G$ be a graph with $\Delta(G) = \Delta$ :

- If $\text{Mad}(G) < \frac{10}{3}$ then $\chi(G^2) \leq \Delta + 10$, and $\chi(G^2) \leq \Delta + 9$ if $\Delta \leq 8$ or $\Delta \geq 12$.
- If $\text{Mad}(G) < 3$ and $\Delta \neq 5$, then $\chi(G^2) \leq \Delta + 5$. If $\text{Mad}(G) < 3$ and $\Delta = 5$, then $\chi(G^2) \leq 11$.
- If $\text{Mad}(G) < \frac{14}{3}$, then $\chi(G^2) \leq \Delta + 5$.

Theorem 3 was improved and extended to list coloring by Borodin et al., then improved further by Bonamy et al. :

**Theorem 6** (Borodin and Ivanova 2009 [6]). Let $G$ be a planar graph. If $g(G) \geq 6$ and $\Delta(G) \geq 18$, then $\chi(G^2) \leq \Delta(G) + 2$. 

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Theorem 7 (Borodin and Ivanova 2009 [7]). Let $G$ be a planar graph. If $g(G) \geq 6$ and $\Delta(G) \geq 24$, then $\chi_l(G^2) \leq \Delta(G) + 2$.

Theorem 8 (Bonamy, Lévêque, Pinlou 2014 [2]). Let $G$ be a graph. If $\text{Mad}(G) < 3$ and $\Delta(G) \geq 17$, then $\chi_l(G^2) \leq \Delta(G) + 2$.

The following question remains open:

**Question 1.** Is there an integer $\Delta_0$ such that every planar graph $G$ with $g(G) \geq 5$ and $\Delta(G) \geq \Delta_0$ has $\chi(G^2) \leq \Delta(G) + 2$?

In [3], Bonamy et al. proved the following:

Theorem 9 (Bonamy, Lévêque, Pinlou 2014 [3]). Every graph $G$ with $\text{Mad}(G) < 4 - \epsilon$ verifies $\chi_l(G^2) \leq \Delta(G) + h(\epsilon)$, where $h(\epsilon) \sim \frac{40}{\epsilon}$ when $\epsilon \to 0$.

This result may not be best possible in term of value of $h(\epsilon)$. In term of maximum average degree, we proposed, as reported in [4] a construction showing that there is no integer $d$ such that every graph $G$ with $\text{Mad}(G) < 4$ has $\chi(G^2) \leq \Delta + d$, and so Theorem 9 is sharp in terms of Mad($G$). We give this construction further.

This construction led us to propose in this paper a conjecture about 2-distance coloring of graphs $G$ with $\text{Mad}(G) < 4$ and, furthermore, a conjecture about 2-distance coloring of graphs with larger values of Mad($G$):

**Conjecture 3.** There exists an integer $D$ such that every graph $G$ with $\Delta(G) \geq D$ and $\text{Mad}(G) < 4$ has $\chi(G^2) \leq 2\Delta(G)$.

**Conjecture 4.** Let $k \geq 3$ be an integer. There exists an integer $D_k$ such that every graph $G$ with $\Delta(G) \geq D_k$ and $\text{Mad}(G) < 2k$ has $\chi(G^2) \leq k\Delta(G) - k$.

We provides, in this paper, graphs reaching these bounds for every considered value of Mad($G$) and for arbitrary large values of $\Delta(G)$.

These classes of graphs are not studied yet, mainly because of limitations of the discharging procedure, which is the standard method for studying 2-distance coloring (used in every proof of Theorems 1 to 9). This limitation holds for other coloring problems studied with discharging procedure (as strong edge coloring, for example). We propose here a way to deal with these limitations, proving the following theorem:

**Theorem 10.** Let $k \geq 2$ be an integer.

There is an integer $D_k$ such that every graph $G$ with $\text{Mad}(G) < 2k$ and $\Delta(G) \geq D_k$ has $\chi(G^2) \leq (2k - 1)\Delta + f(k) + 1$, with

$$f(k) = \max\left\{ \frac{k^4 - 2k^3 - 4k^2 + 4k}{k + 2}, \frac{k^5 - k^4 - k^3 + k^2 + 6k + 2}{3k + 1} \right\}$$

Observe that our theorem has the following corollary:
Theorem 11. There is an integer $\Delta_k$ such that every graph $G$ with $\text{Mad}(G) < 4$ has $\chi(G^2) \leq 3\Delta + 3$.

It seems possible to improve our discharging procedure to obtain a more refined value of $f(k)$, what we did not in order to keep the proof easily readable, as we wanted to focus on the multiplier of $\Delta$ and the global approach. Observe that discharging procedure is used to study a large number of coloring problems, we believe it would be interesting to see what problems our approach could be adapted for.

In the next section, we give the constructions supporting Conjectures 3 and 4. In Section 3, we give the proof of Theorem 11.

2 Lower bounds

In this section we prove both Conjectures 3 and 4 are best possible. For Conjecture 3, we use a construction we proposed to Bonamy in a private communication (see [4]). Let $G_k$ be a graph constructed as follows (see Figure 1):

- We take the complete bipartite graph $K_{k,k}$. Recall that this graph has two sets of $k$ vertices, say $V_0$ and $V_1$, with an edge between a pair of vertices is one is in $V_0$ and the other in $V_1$.
- We subdivide the graph (i.e. we replace every edge $uv$ by a 2-vertex adjacent to $u$ and $v$).
- We add a vertex $w_0$ adjacent to every vertex of $V_0$, and a vertex $w_1$ adjacent to every vertex of $V_1$.
- We add an edge between $w_0$ and $w_1$.

Observe that $\Delta(G_k) = k + 1$ and $\chi(G^2) = 2k + 2 = 2\Delta(G_k)$. Moreover, we prove $\text{Mad}(G_k) < 4$, proving Conjecture 3, if true, is best possible.

Let $H_k$ be a subgraph of $G_k$, we denote by $E_{2-}(H_k)$ and $E_{3+}(H_k)$ being and not being adjacent to a $2^-$-vertex respectively. Let $V_K(H_k)$, $V_2(H_k)$ and $V_w(H_k)$ be respectively the sets of vertices of $H_k$ being in $V_0$ or $V_1$, the set of vertices of $H_k$ created by a
subdivision, and the largest subset of \( \{w_0, w_1\} \) in \( H_k \). We have \(|E_2-(H_k)| \leq 2|V_2(H_k)|\) and \(|E_3+(H_k)| \leq |V_K(H_k)| + |V_w(H_k)|\), so:

\[
\frac{2|E(H_k)|}{|V(H_k)|} \leq 2 \times \frac{|E_2-(H_k)| + |E_3+(H_k)|}{|V(H_k)|} \leq 2 \times \frac{2|V_2(H_k)| + |V_K(H_k)| + |V_w(H_k)|}{|V_2(H_k)| + |V_K(H_k)| + |V_w(H_k)|} < 4
\]

For Conjecture 4, we use a more complicated construction, for which we need some observations on a seemingly-unrelated problem: orthogonal latin squares.

### 2.1 Orthogonal latin squares

A latin square is a matrix \( n \times n \) filled with \( n \) different symbols, each of them occuring exactly once in each row and each column.

**Definition 12.** Two latin squares of the same order \( L_1 : R \times C \rightarrow S_1 \) and \( L_2 : R \times C \rightarrow S_2 \) are said to be orthogonal if, for each set of ordered pair \((s_1, s_2) \in S_1 \times S_2\), there is a unique cell \((x,y) \in R \times C\) so that \( L_1(x,y) := s_1 \) and \( L_2(x,y) := s_2 \).

For example, consider the two latin square of Figure 2: we have \( S_A = \{A,K,Q,J\} \) and \( S_B = \{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\} \). The orthogonality of \( L_A \) and \( L_B \) is evident when the two squares are superposed and we see that each element of \( S_A \times S_B \) appears exactly once (Figure 3). A matrix obtained by the superposition of two orthogonal latin squares, like the one in Figure 3, is called a graeco-latin square or an Euler square.

For more details about orthogonal latin squares, see Chapter 22 of *A Course in Combinatorics*, by van Lint and Wilson [18], where the following result is also given:

**Lemma 13** (Folklore). For each prime power \( q \), there exist \( q - 1 \) pairwise-orthogonal latin squares.
As an illustration, we give the construction of these latin squares, though we will not demonstrate this construction is good (as it is not important for our proof). For a prime power $q$, they are $L_1$ to $L_q$ with, for every $1 \leq i \leq q$,

$$L_i(x, y) := ax + y \pmod{q}$$

We give in Figure 4 the four pairwise-orthogonal latin squares of order 5.

In [18], the authors say that ”the use of of the word ”orthogonal” is perhaps unfortunate since it has other meanings in mathematics, but it has become far too commonplace to try to change now”. We would like to generalise this notion for all matrices, but orthogonal matrix has another meaning in mathematics, so we will use l-orthogonal instead (for ”orthogonal in a latin way”).

**Definition 14.** Let be two matrices of order $n$ denoted $M_1 : R \times C \rightarrow S_1$ and $M_2 : R \times C \rightarrow S_2$, each one filled with $n$ symbols occuring $n$ times. We say that $M_1$ and $M_2$ are l-orthogonal if, for each for each set of ordered pair $(s_1, s_2) \in S_1 \times S_2$, there is a unique cell $(x, y)$ so that $M_1(x, y) := s_1$ and $M_2(x, y) := s_2$.

Obviously, two orthogonal latin squares are l-orthogonal matrices. Also are l-orthogonal the matrices $M_x$ and $M_y$ of the same order defined as follows (see Figure 5):

$$M_x(x, y) := x \quad \text{and} \quad M_y(x, y) := y$$

Moreover, all matrices of order $n$ are latin squares if and only if they are l-orthogonal to both the matrices $M_x$ and $M_y$ of order $n$. This gives us the following corollary:

**Corollary 15.** For every prime power $q$, there exist $q + 1$ pairwise-l-orthogonal matrices of order $q$.

This observation will be needed for our construction.
2.2 Sharpness of Conjecture 4

The following theorem proves that Conjecture 4, if true, is best possible.

**Theorem 16.** Let $k, q$ be two integers with $3 \leq k \leq q + 1$ and $q$ is a prime power. There is a graph $G$ with $\text{Mad}(G) < 2k$, $\Delta(G) = q + 1$ and $\chi(G^2) = k \times q = k\Delta(G) - k$.

Such a graph has $\Delta$ as large as we set $q$. We prove its existence by giving his construction. This construction, in the case $k = q - 1$, is similar to the construction given by Alon and Mohar in [1], even if they define it with different concepts than orthogonal matrices. Let $k, q$ be two integers with $3 \leq k \leq q + 1$ and $q$ is a prime power.

We create a graph $G_{q,k}$ the following way:

**Step 1:** Let $V_0, V_1, \ldots, V_{k-1}$ be $k$ sets of $q$ vertices each. For each set $V_i$ in $V_0, \ldots, V_{k-1}$, we denote the vertices of $V_i$ by $v_{i,0}, v_{i,1}, \ldots, v_{i,q-1}$. We then add a vertex $w_i$ adjacent to every vertex of $V_i$, all the vertices $w_i$ form the set $W$ (see Figure 6).

**Step 2:** Let $U$ be a set of $q^2$ vertices, sorted in an $q \times q$ array: we denote by $u_{x,y}$ each one of these vertices, with $0 \leq x \leq q - 1$ and $0 \leq y \leq q - 1$. Since $q$ is a prime power and $k \leq q + 1$, there exists by Corollary 15 a set of $k$ pairwise-l-orthogonal matrices of order $q$ filled with integers from 0 to $q - 1$, we denote them $L_0, \ldots, L_{k-1}$. For each $0 \leq i \leq k - 1$ and each $0 \leq j \leq q - 1$, we add an edge between $v_{i,j}$ and $u_{x,y}$ if $L_i(x, y) := j$ (see Figure 7).

Now we will do some observations about the graph we constructed.

**Claim 1.** $\chi(G_{q,k}^2) = k \times q$

**Proof.** We want to show that two vertices $v_{i,j}$ and $v_{i',j'}$ are at distance 2 if they both belong to one of the sets $V_0$ to $V_{k-1}$. If this is true, then all of these vertices must have different colors in every proper coloring of the square of $G_{q,k}$ and so $\chi(G_{k,q}^2) = k \times q$. 

Figure 6: Step 1 in $G_{5,3}$
We have two cases: if $i = i'$ then $w_i$ is adjacent to both $v_{i,j}$ and $v_{i',j'}$; if $i \neq i'$ then, since $L_i$ and $L_i'$ are $l$-orthogonal, there is a unique pair $(x, y)$ such that $L_i(x, y) := j$ and $L_i'(x, y) := j'$. The vertex $u_{x,y}$ is both adjacent to $v_{i,j}$ and $v_{i',j'}$.

Claim 2. $\Delta(G_{q,k}) = q + 1$

Proof. For each vertex $w_i \in W$, $d(w_i) = q$. For each vertex $v_{i,j} \in V_0 \cup \ldots \cup V_{k-1}$, $d(v_{i,j}) = q + 1$. For each vertex $u_{x,y} \in U$, $d(u_{x,y}) = k$. By hypothesis, $k \leq q + 1$.

Claim 3. Mad($G_{q,k}$) < 2k

Proof. We prove that the average degree of every subgraph of $G_{q,k}$ is strictly lesser than $2k$. We consider a subgraph $G'_{q,k}$ of $G_{q,k}$. We denote by $W'$, $V'_0, \ldots, V'_{k-1}$ and $U'$ the sets of vertices of $W, V_0, \ldots, V_{k-1}$ and $U$ respectively in $G'_{q,k}$.

All the edges of $G'_{q,k}$ are adjacent to a vertex of $W'$ or to a vertex of $U'$: let $E'_W$ and $E'_U$ these sets of edges. We have $|E'_W| \leq |V'_0| + \ldots + |V'_{k-1}|$ and $|E'_U| \leq k|U'|$, donc $|V(G'_{q,k})| \geq |E'_W| + |E'_U|$.

The average degree of $G'_{q,k}$ is at most equal to:

$$\frac{2|E(G'_{q,k})|}{|V(G'_{q,k})|} \leq 2 \times \frac{|E'_W| + |E'_U|}{|E'_W| + \frac{|E'_U|}{k}} = 2k \times \frac{|E'_W| + |E'_U|}{k|E'_W| + |E'_U|} < 2k$$

These three claims prove Theorem 16. We remark that this construction has girth 6, as one can observe.
3 Upper bounds

Most of the discharging arguments use a structural lemma about the class of graphs they are used on. In our proof, we obtain our structural lemma on two steps, required due to the relative non-spariness of the graphs we study. In order to facilitate some calculus, we make the following observation:

Observation 17. We consider a fixed number \( y \in \mathbb{R}^+ \) and a set \( S \subset \mathbb{R}^+ \). We have

\[
\min \left\{ \frac{x-y}{x}; x \in S \right\} = \frac{\min \{x; x \in S\} - y}{\min \{x; x \in S\}}
\]

Proof. For three numbers \( x_1, x_2, y \in \mathbb{R}^+ \), we have

\[
x_1 < x_2 \iff \frac{y}{x_1} > \frac{y}{x_2} \iff 1 - \frac{y}{x_1} > 1 - \frac{y}{x_2} \iff \frac{x_1 - y}{x_1} > \frac{x_2 - y}{x_2}
\]

\( \blacksquare \)

3.1 Structural lemma

In our first step, we study graphs with \( \text{Mad} < 2k \) and \( \Delta > k \).

Definition 18. Let \( k \) be an integer with \( k \geq 2 \). A \( k \)-light vertex \( v \) is a vertex of degree \( d \) corresponding to one of the following descriptions:

(i) \( d = k + 1 \).

(ii) \( k + 2 \leq d \leq 2k \) and \( v \) has \( k \) or less \( \left( \frac{k(k+1)^2}{k+2} \right)^+ \)-neighbors.

(iii) \( 2k + 1 \leq d \leq k(k+1) - 1 \) and \( v \) has \( k \) or less \( \left( \frac{2k^2(k+1)}{3k+1} \right)^+ \)-neighbors.

Lemma 19. Let \( k \) be a integer with \( k \geq 2 \). Every graph \( G \) with \( \text{Mad}(G) < 2k \) and \( \delta(G) > k \) contains at least one \( (2k+1)^- \)-vertex \( v \) adjacent to \( k - 1 \) or less \( (2k^2)^+ \)-vertices and to at least one \( k \)-light vertex.

Proof. We prove this lemma with a discharging argument. Suppose that this lemma is false, and let \( G \) be a counterexample, we have \( \text{Mad}(G) < 2k \) and \( \Delta(G) > k \). We give to each vertex \( v \) of \( G \) an initial charge equals to \( \mu(v) = d(v) - 2k \). As \( \text{Mad}(G) < 2k \), the sum of the initial charges of the graph is negative.

Then we redistribute these charges, according to several rules given below, verifying that the whole charge on the graph does not change. After this discharging, we observe that every vertex \( v \) have a non-negative final charge \( \mu^*(v) \). This contradiction completes the proof.

Here are the discharging rules; we apply them successively.

R1 - Every \( (2k+1)^+ \)-vertex has a positive charge. It repartites equally its charge to its \( (k(k+1)-1)^- \)-neighbors.
R2 - Every vertex having a positive charge after R1 repartites it equally to its neighbors with a negative charge.

Let \( v \) be a vertex of \( G \). We will now prove that \( \mu^*(v) \geq 0 \), considering several cases according to the degree of \( v \). Since \( \Delta(G) \geq k \), we have \( d(v) \geq k \). If \( d(v) \geq 2k \), then \( \mu(v) \geq 0 \) and so \( \mu^*(v) \geq 0 \).

Suppose now \( k + 1 \leq d(v) \leq 2k - 1 \). We denote \( d(v) = d \) and we have \( \mu(v) = d - 2k \).

By definition of \( G \), vertex \( v \) is in at least one of these two cases:

- **First case: Vertex \( v \) is adjacent to \( k \) or more \((2k^2)^\pm\)-vertices.** In this case, each neighbor of \( v \) of degree \( d' \geq 2k^2 \) gives charge to \( v \) by Rule R1. Since this vertex repartites its charge to some its neighbors, \( v \) receives at least \( \frac{d' - 2k}{d^2} \) to \( v \). By Observation 17,

\[
\frac{d' - 2k}{d^2} \geq \frac{2k^2 - 2k}{2k^2} = \frac{2k(k - 1)}{2k^2} = \frac{k - 1}{k}
\]

Vertex \( v \) receives at least \( \frac{k - 1}{k} \) from each one of its \((2k^2)^\pm\)-neighbors, \( v \) receives at least \( k \times \frac{k - 1}{k} = k - 1 \) and \( \mu^*(v) \geq (d - 2k) + k - 1 \geq 0 \).

- **Second case: Vertex \( v \) is adjacent to no \( k \)-light vertex.** Let \( v' \) be a neighbor of \( v \) and let \( d' = d(v') \). By Definition 18, we have \( d' \geq k + 2 \). Depending on the value of \( d' \), we have three subcases:

  - If \( k + 2 \leq d' \leq 2k \), then \( v' \) is adjacent to at least \( k + 1 \) vertices with degree at least \( \frac{k(k + 1)^2}{k + 2} \). Each one of these neighbors \( v'' \) with degree \( d'' \) gives at least \( \frac{d'' - 2k}{d^2} \) to \( v' \) by Rule R1. We have by Observation 17:

\[
\frac{d'' - 2k}{d''} \geq \frac{\frac{k(k + 1)^2}{k + 2} - 2k}{\frac{k(k + 1)^2}{k + 2}} = \frac{k(k + 1)^2 - 2k(k + 2)}{k + 2} \times \frac{k + 2}{k(k + 1)^2} = \frac{k(k + 1)^2 - 2k(k - 4)}{k(k + 1)^2} = \frac{(k + 1)(k - 2) + 3(k + 1) - 2k - 4}{(k + 1)^2} = \frac{k - 2}{k + 1} + \frac{k - 1}{(k + 1)^2}
\]

So vertex \( v' \) receives at least \( k - 2 + \frac{k - 1}{k + 1} \) by Rule R1. After this rule, the charge on \( v' \) is greater than 0 (since \( (d' - 2k) + (k - 2) \geq 0 \)) and so \( v' \) will give charge to its neighbors by Rule R2. There are at most \( d' - (k + 1) \) vertices needing charge, so \( v' \) gives at least \( \frac{d' - 2k + (k - 2) + \frac{k - 1}{k + 1}}{d' - (k + 1)} \) to \( v \). By Observation 17 (with \( x = d' - (k + 1) \)), \( v' \) gives to \( v \) by Rule R2 at least:

\[
\frac{d' - 2k + (k - 2) + \frac{k - 1}{k + 1}}{d' - (k + 1)} \geq \frac{(k + 2) - 2k + (k - 2) + \frac{k - 1}{k + 1}}{(k + 2) - (k + 1)} = \frac{k - 1}{k + 1}
\]
If \( 2k + 1 \leq d' \leq k(k + 1) - 1 \), then \( v' \) is adjacent to at least \( k + 1 \) vertices with degree at least \( \frac{2k^2(k+1)}{3k+1} \). Each one of these neighbors \( v'' \) with degree \( d'' \) gives at least \( \frac{d''-2k}{d''} \) to \( v' \) by Rule R1. We have by Observation 17:

\[
\frac{d'' - 2k}{d''} \geq \frac{2k^2(k+1) - 2k}{2k^2(k+1)} \geq \frac{2k^2(k+1) - 2k(3k+1)}{2k^2(k+1) + 1} = \frac{k(k+1) - (3k+1)}{k(k+1)} = \frac{k-1}{k+1}
\]

So vertex \( v' \) receives at least \( k \frac{k+1}{k+1} - 1 \) by Rule R1. If we add the initial charge of \( v' \) and the charge received on Rule R1, we obtain the charge \( v' \) gives to its neighbors by Rules R1 and R2. This charge is at least \( d' - 2k + k \frac{k-1}{k+1} - 1 \). Vertex \( v' \) gives charge to at most \( d' - (k + 1) \) vertices, so \( v' \) gives at least \( \frac{d' - 2k + k \frac{k-1}{k+1} - 1}{d' - (k + 1)} \) to vertex \( v \) by Rule R1 and R2. By Observation 17 (with \( x = d' - (k + 1) \)):

\[
\frac{d' - 2k + k \frac{k-1}{k+1} - 1}{d' - (k + 1)} \geq \frac{2k + 1 - 2k + k \frac{k-1}{k+1} - 1}{2k + 1 - (k + 1)} = \frac{k - 1}{k + 1}
\]

If \( d' \geq k(k + 1) \), then \( \mu(v') \geq d' - 2k \). Vertex \( v' \) gives at least \( \frac{d' - 2k}{d'} \) to \( v \) by Rule R1. By Observation 17, we have:

\[
\frac{d' - 2k}{d'} \geq \frac{k(k+1) - 2k}{k(k+1)} = \frac{k - 1}{k + 1}
\]

So every neighbor of \( v \) gives him at least \( k \frac{k+1}{k+1} - 1 \), and \( v \) receives at least \( k - 1 \). We have \( \mu^*(v) \geq d - 2k + k - 1 \geq 0 \).

This completes the proof of our lemma. \( \square \)

### 3.2 Structural corollary

Now we use our previous lemma to study the structural properties of every graphs with Mad < 2k. For every vertex \( v \) of a graph \( G \), we denote by \( d^{k^+}(v) \) the number of \( k^+ \)-vertices of \( v \) in \( G \).

**Definition 20.** Let \( k \) be an integer with \( k \geq 2 \). A \( k \)-quasilight vertex \( v \) is a vertex of degree \( d \) corresponding to one of the following descriptions:

1. \( d^{k^+}(v) = k + 1 \).
2. \( k + 2 \leq d^{k^+}(v) \leq 2k \) and \( v \) has at most \( k \) neighbors with \( d^+ \geq \frac{k(k+1)^2}{k+2} \).
Corollary 21 (Lemma 19). Let \( k \) be an integer with \( k \geq 2 \). Every graph \( G \) with \( \text{Mad}(G) < 2k \) contains at least one of the following configurations:

(C1) A \( k^- \)-vertex adjacent to a vertex \( v \) with \( d^{k^+}(v) \leq k \).

(C2) A \( k^- \)-vertex adjacent to a vertex \( v \) with \( d^{k^+}(v) \leq 2k - 1 \). Moreover, vertex \( v \) has at most \( k - 1 \) neighbors with \( d^{k^+} \geq 2k^2 \).

(C3) A \((2k - 1)^-\)-vertex adjacent to a \( k \)-quasilight vertex.

Proof. Let \( G \) be a graph with \( \text{Mad}(G) < 2k \) that does not contain configuration (C1), (C2) or (C3). Let \( G' \) be the graph obtained from \( G \) by deleting the \( k^- \)-vertices. For every vertex \( v \) of \( G' \), the degree of \( v \) in \( G' \) is equal to \( d^{k^+}(v) \) in \( G \). Every \( k \)-light vertex in \( G' \) is a \( k \)-quasilight vertex in \( G \).

If \( G \) has no configuration (C1), then \( \delta(G') > k \). By Lemma 19, then \( G' \) has a \((2k - 1)^-\)-vertex \( v \) adjacent to \( k - 1 \) or less \((2k^2)^+\)-vertices and to at least one \( k \)-light vertex. If \( v \) has a \( k^-\)-neighbor in \( G \), then \( G \) has configuration (C2), otherwise, in \( G \) we have \( d(v) \leq 2k - 1 \) and \( G \) has configuration (C3).

**Drawing conventions:** In Figures 8 and 9, we represent the configurations C1, C2 and C3. In these figures, the color we use for each vertex depends on its degree: \( k^- \)-vertices are in white, vertices with a degree bounded, but larger than \( k \), are in grey, and vertices with unbounded degree (i.e. only bounded by \( \Delta \)) are in black.

### 3.3 Coloring

We now prove Theorem 11. More precisely, we prove the following result:

**Theorem 22.** Let \( k \geq 2 \) be an integer.

Every graph \( G \) with \( \text{Mad}(G) < 2k \) has

\[
\chi(G^2) \leq \max \left\{ \frac{(2k - 1)\Delta + \frac{k^4 - 2k^3 - 4k^2 + 4k}{k + 2} + 1}{(2k - 2)\Delta + 2k^3 - 2k^2 - 2k + 2}, \frac{k^5 - k^4 - k^3 + k^2 + 4k + 2}{3k + 1} + 1 \right\}
\]
Proof. Let $G$ be a minimal counterexample (with the lesser number of edges and vertices) of Theorem 11, and let $\Delta(G) = \Delta$. There is an integer $k$ such that we have $\text{Mad}(G) < 2k$ and $\chi(G^2) > \Phi$, with

$$\Phi = \max \left\{ \frac{(2k-1)\Delta + \frac{k^4-2k^3-4k^2+4k}{k+2} + 1}{(2k-1)\Delta + \frac{k^5-k^4-k^3+6k+2}{3k+1} + 1}, \frac{(2k-2)\Delta + 2k^3-2k^2-2k+2}{(2k-2)\Delta + 2k^3-2k^2-2k+2} \right\}$$

Easily, $G$ is connected. By Corollary 21, graph $G$ contains one of the configurations (C1) to (C3). We consider three cases:

- If $G$ contains configuration (C1), then let $u$ be a $k^-$-vertex of $G$ and $v$ a neighbor of $u$ with $d^+(v) \leq k$. Let $G' = G - uv$. By minimality, $G'$ has a 2-distance coloring with less than $\Phi$ colors. We construct a good coloring for $G$ by discoloring $v$ and every $k^-$-vertex of $G$ (including $u$). We then recolor $v$, then the $k^-$-vertices of $G$.

  When we recolor $v$, we have $F(v) \leq (k-1)(\Delta - k) + k \times k < (2k-1)\Delta$.

  When we recolor any $k^-$-vertex $w$ of $G$, we have $F(w) \leq k\Delta(G)$.

So we can obtain a 2-distance coloring of $G$ with $\Phi$ colors, a contradiction.

- If $G$ contains configuration (C2), then let $u$ be a $k^-$-vertex of $G$ and $v$ a neighbor of $u$ with $d^+(v) \leq 2k - 1$, adjacent to at most $k - 1$ neighbors with $d^+ \geq 2k^2$. Let $G' = G - uv$. By minimality, $G'$ has a 2-distance coloring with less than $\Phi$ colors. We construct a good coloring for $G$ by discoloring $v$ and every $k^-$-vertex of $G$ (including $u$). We then recolor $v$, then the $k^-$-vertices of $G$.

  When we recolor $v$, we have $F(v) \leq (k-1)(\Delta - 2k-1) + k(2k^2-1) + (k-1)\Delta = (2k-2)\Delta + 2k^3-2k^2-2k+1$.
When we recolor any $k^-$-vertex $w$ of $G$, we have $F(w) \leq k\Delta(G)$.
So we can obtain a 2-distance coloring of $G$ with $\Phi$ colors, a contradiction.

- If $G$ contains configuration (C3), then let $u$ be a $(2k - 1)^-$-vertex of $G$ and $v$ a $k$-quasilight neighbor of $u$. Let $G' = G - uv$. By minimality, $G'$ has a 2-distance coloring with less than $\Phi$ colors, we then discolor $u$, $v$ and every $k^-$-vertex of $G$. We construct a 2-distance coloring of $G$ by recoloring $u$, then $v$, then the $k^-$-vertices of $G$. When we recolor $u$, we have $F(u) \leq (2k - 1)\Delta(G)$.

When we recolor $v$, we have three cases, depending on if $v$ corresponds to description (i), (ii) or (iii) in Definition 20:

- Case (i): As we observe in Figure 9.(a),

\[
F(v) \leq (2k - 1) + k\Delta + (k - 1) \times (\Delta - (k + 1)) \\
F(v) \leq (2k - 1)\Delta + (2k - 1) - (k - 1)(k + 1) \\
F(v) \leq (2k - 1)\Delta - k^2 + 2k \leq (2k - 1)\Delta
\]

- Case (ii): As we observe in Figure 9.(b),

\[
F(v) \leq (2k - 1) + (k - 1) \times (\Delta - (2k + 1)) + (k - 2) \times \frac{k(k + 1)^2}{k + 2} + k\Delta \\
F(v) \leq (2k - 1)\Delta + \frac{k(k - 2)(k + 1)^2}{k + 2} - (2k^2 - 3k) \\
F(v) \leq (2k - 1)\Delta + \frac{(k^4 - 3k^2 - 2k) - (2k^3 + k^2 - 6k)}{k + 2} \\
F(v) \leq (2k - 1)\Delta + \frac{k^4 - 2k^3 - 4k^2 + 4k}{k + 2}
\]

- Case (iii): As we observe in Figure 9.(c),

\[
F(v) \leq (2k - 1) + (k - 1) \times (\Delta - (k + 1) + 1) + (k^2 - 2) \times \left(\frac{2k^2(k + 1)}{3k + 1} - 1\right) + k\Delta \\
F(v) \leq (2k - 1)\Delta + \frac{k^2(k^2 - 2)(k + 1)}{3k + 1} - (k^2 - 2) + (2k - 1) - (k^3 - 1) \\
F(v) \leq (2k - 1)\Delta + \frac{k^2(k^2 - 2)(k + 1)}{3k + 1} - (k^3 + k^2 - 2k - 2) \\
F(v) \leq (2k - 1)\Delta + \frac{(k^5 + 2k^4 + k^3 - 4k^2 - 2k) - (3k^4 + 2k^3 - 5k^2 - 8k - 2)}{3k + 1} \\
F(v) \leq (2k - 1)\Delta + \frac{k^5 - k^4 - k^3 + k^2 + 6k + 2}{3k + 1}
\]

Finally, when we recolor any $k^-$-vertex $w$ of $G$, we have $F(w) \leq k\Delta(G)$.

So we can obtain a 2-distance coloring of $G$ with $\Phi$ colors, a contradiction. This completes the proof of our main theorem.
References


