

# Some permutations on Dyck words

Marilena Barnabei<sup>a</sup>, Flavio Bonetti<sup>b</sup>, Niccolò Castronuovo<sup>c</sup>, Robert Cori<sup>d,\*</sup>

<sup>a</sup>Università di Bologna, Dipartimento di Matematica, Italy

<sup>b</sup>Università di Bologna, Dipartimento di Matematica, Italy

<sup>c</sup>Università di Ferrara, Dipartimento di Matematica, Italy

<sup>d</sup>LaBRI Université Bordeaux 1, France

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## Abstract

We examine three permutations on Dyck words. The first one,  $\alpha$ , is related to the Baker and Norine theorem on graphs, the second one,  $\beta$ , is the symmetry, and the third one is the composition of these two. The first two permutations are involutions and it is not difficult to compute the number of their fixed points, while the third one has cycles of different lengths. We show that the lengths of these cycles are odd numbers. This result allows us to give some information about the interplay between  $\alpha$  and  $\beta$ , and a characterization of the fixed points of  $\alpha \circ \beta$ .

*Keywords:* Dyck word, Dyck path, permutation

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*In memory of Marcel-Paul Schützenberger (1920-1996)*

## 1. Introduction

Dyck words are a central object in combinatorics and formal language theory. They are enumerated by Catalan numbers as many other objects, for instance Chapter 6 of [16] points out the role of Dyck words (or Dyck paths) in enumerative combinatorics.

From the point of view of formal languages they are generated by a context free grammar, whose important properties were described in [5].

In this paper we will consider Dyck words on the alphabet  $A = \{a, b\}$  to represent Dyck paths consisting of up steps, represented by the letter  $a$ , and down steps, represented by the letter  $b$ . Paths

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\*Corresponding author

Email addresses: [marilena.barnabei@unibo.it](mailto:marilena.barnabei@unibo.it) (Marilena Barnabei), [flavio.bonetti@unibo.it](mailto:flavio.bonetti@unibo.it) (Flavio Bonetti), [niccol.castronuovo@unife.it](mailto:niccol.castronuovo@unife.it) (Niccolò Castronuovo), [robert.cori@labri.fr](mailto:robert.cori@labri.fr) (Robert Cori)

are better for the visual intuition and words are better for writing proofs, but of course they represent in different ways the same combinatorial object. For this reason, almost all the examples of the paper will be given in the language of Dyck paths, while proofs are mostly written with Dyck words.

We are interested here in transformations that do not modify the length of a word. This kind of transformations was considered by many authors (see [3], [7], [8], [10], [12], [13], [14]). This paper studies two new ones. The first one is an involution which we denote by  $\alpha$ . It was recently introduced in [6] in the context of the sandpile (or chip-firing game) model in order to determine the rank of configurations, as defined by Baker and Norine [1], for the case of the complete graph. It relies on the so called *Cyclic Lemma* which was obtained in [9] long time before the name Dyck path was used to speak of these combinatorial objects. This first involution is added to a very classical one, that is, the symmetry of paths along a vertical line, denoted  $\beta$  here, to define a new transformation  $\gamma$ .

The main result of the paper shows that the composition  $\gamma$  of the two involutions  $\alpha$  and  $\beta$  has cycles of odd lengths. In order to prove this result, we remarked that it was sufficient to show that for each word  $w$  there exists an odd integer  $k$  such that  $\gamma^k(w) = w$ , since of course this implies that the length of the cycle of  $\gamma$  containing  $w$  is also odd. Then the first ingredient for the proof is to consider a subset of the set of Dyck words which we call *smooth* words, namely, Dyck words which do not have  $aba$  or  $bab$  as factors. We then associate to any Dyck word a smooth word that we call its *skeleton*. The last ingredient is to associate to each Dyck word a sequence of integers which allows to rebuild the word from its skeleton. A property of the action of  $\gamma$  on the skeleton and the sequence allows to end the proof.

The paper is organized as follows. In Section 2 we briefly recall the main facts about Dyck words. Section 3 contains the definition of the involutions  $\alpha$  and  $\beta$ , together with a characterization of their fixed elements. In Section 4 we describe the action of the map  $\gamma = \alpha \circ \beta$ , while in Section 5 we define the skeleton of a Dyck word and examine its interaction with the map  $\gamma$ . Section 6 contains the main theorem of the paper (Theorem 3) and its consequences about the cycles of  $\gamma$ . Finally, in Section 7 we give a characterization of the fixed points of  $\gamma$  and deduce an upper bound for their numbers.

## 2. Dyck words and their parameters

We consider words on the alphabet with two letters  $A = \{a, b\}$ . For a word  $w$ , we denote  $|w|_x$  (where  $x \in \{a, b\}$ ) the number of occurrences of the letter  $x$  in  $w$ . Hence the *length* of  $w$ , denoted  $|w|$ , is equal

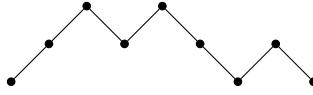


Figure 1: A Dyck path

to  $|w|_a + |w|_b$ . We use also the mapping  $\delta$  associating to any word  $w$  the integer  $\delta(w) = |w|_a - |w|_b$ .

The word  $u$  is a *prefix* of  $w$  if  $w = uv$ . This prefix is *strict* if  $u \neq w$ .

We will use the following notation. For any word  $w = w_1 w_2 \cdots w_m$  where  $w_i \in A$  we denote by  $\tilde{w}$  the mirror image of  $w$ , that is, the word obtained by reading  $w$  from right to left, hence  $\tilde{w} = w_m w_{m-1} \cdots w_1$ . Moreover, we denote by  $\overline{w}$  the word obtained from  $w$  by replacing any occurrence of  $b$  by an occurrence of  $a$  and vice versa, giving  $\overline{w} = \overline{w}_1 \overline{w}_2 \cdots \overline{w}_m$ .

A Dyck word is often defined as a word  $w$  such that  $\delta(w) = 0$  and  $\delta(u) \geq 0$  for any prefix  $u$  of  $w$ . We consider here a slight modification adding a letter  $b$  at the end of it. We define for each non negative integer  $n$  the set  $D_n$  of words of length  $2n + 1$  first by considering  $A_n$  to be the set of words on the alphabet  $A$  having  $n$  occurrences of the letter  $a$  and  $n + 1$  occurrences of the letter  $b$ . Then the set  $D_n$  is the subset of words  $w$  in  $A_n$  satisfying

$$\delta(w') \geq 0 \text{ for any strict prefix } w' \text{ of } w. \quad (1)$$

Notice that  $D_0 = \{b\}$  and  $D_1 = \{abb\}$ .

Dyck words in  $D_n$  correspond bijectively to Dyck paths of semilength  $n$ , namely, lattice paths consisting of  $2n$  steps  $a = (1, 1)$  (*up steps*) and  $b = (1, -1)$  (*down steps*), starting at  $(0, 0)$ , ending at  $(0, 2n)$ , and never going below the  $x$ -axis. Note that, when we see a Dyck word as a lattice path, we ignore its last letter  $b$ .

As an example, the Dyck path corresponding to the Dyck word  $aabbabbabb$  is drawn in Figure 1.

Many parameters are defined for Dyck words. We recall here the definition of those we use in the paper. A *peak* of a Dyck word is an occurrence of the letter  $a$  followed by an occurrence of  $b$  – or, in the Dyck path language, the vertex between an up step and a down step. Any word in  $D_n$  for  $n \neq 0$  has at least one peak and at most  $n$  peaks.

The *height* of a Dyck word  $w$  is the maximum value of  $\delta(u)$ , as  $u$  ranges over all prefixes of  $w$ .

The number of elements of  $D_n$  are given by the Catalan numbers which satisfy the formula

$$C_n = \frac{(2n)!}{n!(n+1)!}. \quad (2)$$

An elegant way to obtain the above formula is to use the so-called *Cyclic Lemma* (see. e.g. [9]) which will be used very often in this paper. This Lemma considers the set  $A_n$  and uses the operation of conjugation. Two words  $w$  and  $w'$  on an alphabet  $A$  are *conjugate* if there exist  $u$  and  $v$  such that  $w = uv$  and  $w' = vu$ . In other terms, if you write a word  $w$  in counterclockwise order, wrapped around a circle, the conjugates of  $w$  are then all words obtained by starting reading at any letter in counterclockwise order and making a full circle.

**Lemma 1. (Cyclic Lemma)** *Any word  $w$  of  $A_n$  has exactly one decomposition into two factors  $w = uv$  such that  $vu$  is an element of  $D_n$ . Moreover the decomposition into two factors  $w = uv$  is such that  $u$  is the smallest prefix of  $w$  attaining the minimal value for  $\delta(u)$ .*

It is well known that the number of different conjugates of a word  $w$  divides its length and that if a word  $uv$  is equal to one of its conjugates  $vu$ , where  $u, v \neq \emptyset$ , then there exist a word  $q$  and an integer  $k > 1$  such that  $uv = q^k$  (see [15, Paragraph 1.3]). Since the words in  $A_n$  contain  $n$  occurrences of the letter  $a$  and  $n+1$  occurrences of  $b$ , it follows that an element in  $A_n$  has  $2n+1$  different conjugates. Clearly the number of elements of  $A_n$  is  $\binom{2n+1}{n}$ . This proves Formula (2).

### 3. Involutions for words of $D_n$

#### 3.1. The involution related to the Baker Norine theorem on graphs

We define an involution  $\alpha$  on Dyck words in  $D_n$  by setting  $\alpha(w)$  to be the unique conjugate of  $\tilde{w}$  which belongs to  $D_n$ . Recall that  $\tilde{w}$  denotes the mirror image of  $w$ , that is, if  $w = w_1 w_2 \cdots w_{2n+1}$ , then  $\tilde{w} = w_{2n+1} w_{2n} \cdots w_2 w_1$ .

Observe that this map was introduced in [6] in order to compute the rank of configurations in complete graphs where it was proved that it keeps invariant the *dinv* parameter (extensively studied in [11]).

**Example 1.** Let  $w = aabaababbabb$  then  $\tilde{w} = bbabbbabaabaa$ . This decomposes as  $\tilde{w} = (bbabbb)(abaabaa)$  and the conjugate  $(abaabaa)(bbabbb)$  is in  $D_n$ , hence

$$\alpha(aabaababbabb) = abaabaabbabb.$$

**Proposition 1.** *The mapping  $\alpha$  is an involution.*

*Proof.* Let  $w$  be a word in  $D_n$  and set  $v = \alpha(w)$ . Consider the word  $\tilde{w}$  and decompose it as  $\tilde{w} = w'w''$ , where  $w'$  is the shortest prefix of  $w$  such that  $\delta(w')$  is minimal. Then we have  $v = w''w'$ . To compute  $\alpha(v)$  we consider  $\tilde{v}$  which is equal to  $\tilde{w}'\tilde{w}''$ . But since  $\tilde{w} = w'w''$  we have  $w = \tilde{w}''\tilde{w}'$  showing that  $w$  is a conjugate of  $\tilde{v}$  which is in  $D_n$ . The unicity of such a conjugate given by the cyclic lemma implies  $\alpha(v) = w$ .  $\square$

In order to determine the number of fixed points of  $\alpha$  we need the following characterization.

**Proposition 2.** *The Dyck word  $w$  satisfies  $\alpha(w) = w$  if and only if  $w$  is the concatenation of two palindromes, namely,*

$$w = v u, \quad \text{with} \quad v = \tilde{v}, \quad u = \tilde{u}.$$

*Moreover each word in  $D_n$  has at most one decomposition as a concatenation of two palindromes.*

*Proof.* Let  $w \in D_n$  such that  $w = \alpha(w)$ . Consider its mirror image  $\tilde{w}$  and write it as  $\tilde{w} = u v$  such that  $vu \in D_n$ ; then  $\alpha(w) = v u$ . On the other hand, using the relation  $\tilde{u}\tilde{v} = \tilde{v}\tilde{u}$ , which is true for any pair of words  $u, v$ , we have:  $w = \tilde{v}\tilde{u}$ . Since  $w = \alpha(w)$  and  $|v| = |\tilde{v}|$ , we get

$$v = \tilde{v}, \quad u = \tilde{u}.$$

Conversely, assume that  $w = u v$  is such that  $v = \tilde{v}, u = \tilde{u}$ . Then  $\tilde{w} = \tilde{v}\tilde{u} = v u$ . This shows that  $w$  is the unique conjugate of  $\tilde{w}$  which lies in  $D_n$ , proving that  $\alpha(w) = w$ .

The unicity of the decomposition of  $w$  as a concatenation of two palindromes comes from the fact that if  $w = u'v'$ , where  $u'$  and  $v'$  are two palindromes different from  $u, v$ , then  $\tilde{w} = \tilde{v}'\tilde{u}' = v'u'$ , thus  $\tilde{w}$  would have two different decompositions  $vu$  and  $v'u'$  such that  $uv = u'v' \in D_n$ , contradicting the Cyclic Lemma.  $\square$

Then we have, as done in [2, Theorem 4]:

**Corollary 1.** *The Dyck word  $w$  satisfies  $\alpha(w) = w$  if and only if  $w$  is the conjugate of a palindrome.*

*Proof.* If  $w$  has a conjugate which is a palindrome then it can be written as  $w = w'w''$ , and  $w''w'$  is a palindrome. Since the length of  $w$  is odd we can write  $w''w' = ux\tilde{u}$  where  $x \in \{a, b\}$ . If  $|w''| \leq |u|$  we have  $u = w''v$  for some  $v$  and  $w' = vx\tilde{u}$ . Thus

$$w'w'' = vx\tilde{u}w''. \tag{3}$$

But since  $u = w''v$  we have  $\tilde{u} = \tilde{v}\tilde{w}''$ , whence

$$w = w'w'' = vx\tilde{v}\tilde{w}''w'', \quad (4)$$

that is,  $w$  is the concatenation of the palindromes  $vx\tilde{v}$  and  $\tilde{w}''w''$ .

If  $|w''| > |u|$  we have  $w'' = uxv$  and  $\tilde{u} = vw'$ , hence

$$w = w'w'' = w'uxv = w'\tilde{w}'\tilde{v}xv,$$

which is the concatenation of two palindromes.

Conversely, suppose that  $w$  is the concatenation of the two palindromes  $u$  and  $v$ . Since  $w$  is of odd length, then one of  $u, v$  is of odd length and the other one of even length. Suppose  $u$  is of odd length. Then  $u = u'x\tilde{u}'$  where  $x \in \{a, b\}$  and  $v = v'\tilde{v}'$ . We have

$$w = uv = u'x\tilde{u}'v'\tilde{v}'. \quad (5)$$

This shows that the conjugate of  $w$  equal to  $\tilde{v}'u'x\tilde{u}'v'$  is a palindrome. A similar construction holds if  $v$  is of odd length, and  $u$  of even length.  $\square$

**Theorem 1.** *The number of fixed points of  $\alpha$  in  $D_n$  is equal to the central binomial coefficient  $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ .*

*Proof.* We give a bijection between words of length  $n$  with  $\lfloor \frac{n}{2} \rfloor$  occurrences of  $a$  and words in  $D_n$  which are fixed by  $\alpha$ . Let  $w$  be any word of length  $n$  with  $\lfloor \frac{n}{2} \rfloor$  occurrences of  $a$  and consider the word  $wx\tilde{w}$  where  $x = a$  if  $n$  is odd and  $x = b$  if  $n$  is even. This word has  $n$  occurrences of  $a$  and  $n + 1$  occurrences of  $b$  in both cases. By the cyclic Lemma, it has a unique conjugate in  $D_n$ . All these words are obviously distinct and by Proposition 2 they are fixed points of  $\alpha$ .  $\square$

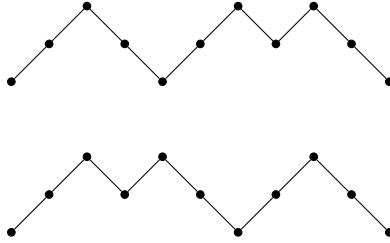
**Example 2.** Consider the 10 words of length 5 with 2 occurrences of  $a$ :

$$aabbb, ababb, abbab, abbba, baabb, babab, babba, bbaab, bbaba, bbbaa.$$

Building  $wa\tilde{w}$  for each of these words, we obtain:

$$aabbbabbbaa, ababbabbaba, abbabababba, abbbaaabbba, baabbabbaab$$

$$bababababab, babbaaabab, bbaababaabb, bbabaaaabb, bbbaaaaabb.$$

Figure 2: The paths  $w$  and  $\beta(w)$ 

Now computing the conjugate of each of these words which belongs to  $D_5$  we get:

$aaaabbabb, abaababbabb, aabbabababb, aaabbbabbb, aabbaabbabb$

$abababababb, aaabbabbabb, aababaabbbb, abaaababbbb, aaaaabbbbb.$

The words above can be decomposed as the concatenation of two palindromes as follows:

$(aaaa)(bbbabb), (abaaba)(bbabb), (aa)(bbabababb), (aaa)(bbbaabbb),$

$(aabbaa)(bbabb), (ababababa)(bb) (aaa)(bbabbabb),$

$(aababaa)(bbbb), (abaaaba)(bbbb), (aaaaa)(bbbbb).$

### 3.2. Symmetry

The easiest involution on Dyck paths is the symmetry, denoted  $\beta$  here. It consists in seeing the path from right to left and replacing each up step by a down step, and vice versa. We translate this definition to the words of  $D_n$  by setting for  $w = w_1 w_2 \dots w_m$ , where  $m = 2n + 1$ ,

$$\beta(w) = \overline{w}_{m-1} \overline{w}_{m-2} \dots \overline{w}_2 \overline{w}_1 w_m.$$

Recall that  $\overline{w}_i$  means changing  $a$  into  $b$  and  $b$  into  $a$ .

**Example 3.** If  $w = aabbaabbbb$ , then

$$\beta(w) = aababbaabbb.$$

The corresponding Dyck paths are given in Figure 2.

Notice that this symmetry is such that for  $w \in D_n$  we have  $\beta(w) \in D_n$ , and that  $w$  and  $\beta(w)$  have the same number of peaks and the same height.

**Proposition 3.** *The number of elements  $w$  of  $D_n$  such that  $w = \beta(w)$  is the central binomial coefficient  $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ .*

To prove this we consider the words  $u$  of length  $n$  which are prefixes of a Dyck word of length  $2n+1$ . It is well known (see, e.g., [4]) that the number of such words is  $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ . To each such word we associate the word  $\bar{u}$  obtained by reversing  $u$  and exchanging  $a$  into  $b$ . Then the word  $w$  is such that  $w = \beta(w)$  if and only if  $w = u\bar{u}b$ .

#### 4. The permutation $\gamma$

The permutation  $\gamma$  is defined on  $D_n$  as the composition of  $\alpha$  and  $\beta$ :

$$\gamma(w) = \alpha(\beta(w)).$$

In order to give a direct determination of  $\gamma(w)$  we need to introduce the following:

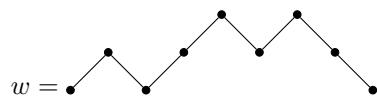
**Definition 1.** *The principal prefix of a word  $w \in D_n$  is the shortest prefix  $u$  of  $w$  such that  $\delta(u)$  is maximum.*

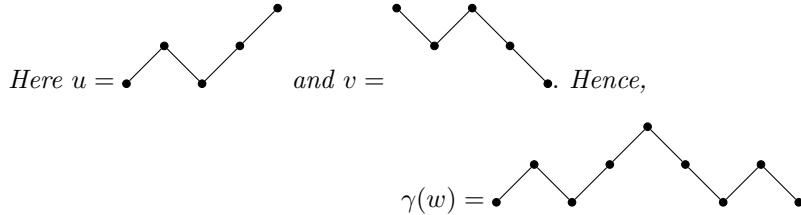
**Proposition 4.** *For  $w = w'b \in D_n$ , the word  $\gamma(w)$  is the unique conjugate of the word  $\bar{w}'b$  belonging to  $D_n$ . Moreover, if  $w = uvb$ , where  $u$  is the principal prefix of  $w$ , we have:*

$$\gamma(w) = \bar{v}b\bar{u}.$$

*Proof.* By definition of  $\alpha$ ,  $\alpha(\beta(w))$  is the unique conjugate of the mirror image of  $\beta(w)$  which is in  $D_n$ . Denoting  $w = w'b$  we have that the mirror image of  $\beta(w'b)$  is equal to  $\bar{b}\bar{w}'$  which has the same conjugates as  $\bar{w}'b$ . The shortest prefix  $v'$  of  $\bar{w}'b$  such that  $\delta(v')$  is minimal is equal to  $\bar{u}$ , where  $u$  is the principal prefix of  $w$ .  $\square$

**Example 4.** Consider the Dyck path





The following result is an immediate consequence of Proposition 4.

**Corollary 2.** *Let  $w \in D_n$ . The word  $\gamma(w)$  has the same height and the same number of peaks as  $w$ .*

We point out that the definition of the map  $\gamma$  and the fact that  $\alpha$  and  $\beta$  are involutions imply immediately the following commutation rules:

$$\beta \circ \gamma = \gamma^{-1} \circ \beta, \quad \alpha \circ \gamma = \gamma^{-1} \circ \alpha.$$

Hence,  $\alpha$  and  $\gamma$  generate a dihedral group.

## 5. Smooth words and skeleton for words in $D_n$

Consider a Dyck word  $w \in D_n$ ,  $w = w_1 w_2 \dots w_m$ , where  $w_i \in \{a, b\}$  and  $m = 2n + 1$ .

**Definition 2.** *An integer  $i$ ,  $i = 1, 2, \dots, 2n + 1$ , is said to be an active site of  $w$  if both  $w_{i-1} w_i$  and  $w_i w_{i+1}$  are different from  $ab$ .*

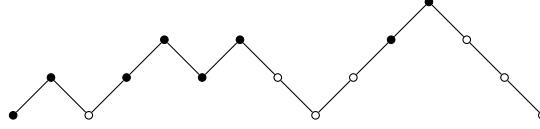
In other words, if we see  $w$  as a lattice path, the integer  $i$  is an active site for  $w$  if the  $i$ -th vertex of  $w$  is neither a peak nor the vertex preceding a peak.

The number of active sites of a Dyck word  $w$  depends only on the length and the number of peaks of  $w$ . In fact, we have the following result, whose proof is straightforward.

**Proposition 5.** *If  $w$  is a Dyck word of length  $2n + 1$ , then the number of active sites of  $w$  is  $2n + 1 - 2\text{peak}(w)$ , where  $\text{peak}(w)$  denotes the number of peaks of  $w$ . In particular the number of active sites of  $w$  is always odd.*

Recalling that the map  $\gamma$  preserves the number of peaks of a word (Corollary 2), the previous Proposition implies that  $\gamma$  preserves also the number of active sites.

**Example 5.** In the following Dyck path the white vertices are the active sites.



**Definition 3.** A Dyck word  $w$  is said to be smooth if  $w$  may be written as

$$w = a^{i_1}b^{j_1}a^{i_2}b^{j_2} \cdots a^{i_k}b^{j_k},$$

where  $k$  is the number of peaks of  $w$  and  $i_s > 1$ ,  $j_t > 1$  for all  $s, t$ .

Roughly speaking,  $w$  is smooth whenever it can not be obtained from a shorter word by adding one peak in any active site.

Definition 3 implies immediately that a Dyck word  $w$  is smooth if and only if it does not contain any sequence  $bab$  or  $aba$ . As a consequence, if  $w$  is smooth so is  $\gamma(w)$ .

Given a smooth word

$$w = a^{i_1}b^{j_1}a^{i_2}b^{j_2} \cdots a^{i_k}b^{j_k},$$

consider the operation  $\theta$  which decreases by 1 all the exponents, giving

$$\theta(w) = a^{i_1-1}b^{j_1-1}a^{i_2-1}b^{j_2-1} \cdots a^{i_k-1}b^{j_k-1}.$$

As a consequence, if at least one of the exponents is equal to 2, then  $\theta(w)$  is not smooth. However we have:

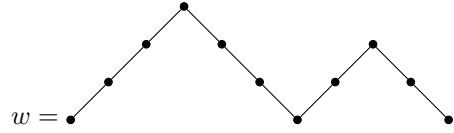
**Lemma 2.** Let  $w$  be a smooth word in  $D_n$ . Then,

- i)  $\gamma(\theta(w)) = \theta(\gamma(w))$ .
- ii)  $\theta(w) = \theta(w')$  implies  $w = w'$ .
- iii) The words  $w$  and  $\theta(w)$  are in cycles of  $\gamma$  of the same length.

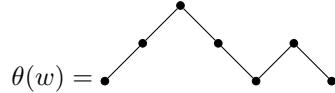
*Proof.* The first assertion follows from the fact that if  $a^{i_1}b^{j_1}a^{i_2}b^{j_2} \cdots a^{i_n}$  is the principal prefix of  $w$  then  $a^{i_1-1}b^{j_1-1}a^{i_2-1}b^{j_2-1} \cdots a^{i_n-1}$  is the principal prefix of  $\theta(w)$ . The second assertion follows immediately from the definition of  $\theta$ , and the third one is a consequence of the first two.  $\square$

Notice that here we are considering  $\gamma$  as an operator acting on all Dyck words of any length at the same time. Observe that  $\gamma$  preserves the length of the Dyck word, whereas  $\theta$  changes the length of a smooth Dyck word.

**Example 6.** Consider the Dyck path



Then



We now associate to any element  $w \in D_n$  a smooth word  $\text{Sk}(w)$ , given by:

**Definition 4.** The skeleton  $\text{Sk}(w)$  of a word  $w$  obtained by deleting iteratively all the factors  $ab$  of  $w$  which are followed by  $a$  or preceded by  $b$ .

Note that the above definition is well posed, since the end result is obviously independent of the order in which the factors  $ab$  are removed.

The following result, whose proof is straightforward, will be useful in the sequel.

**Proposition 6.** The words  $w$  and  $\text{Sk}(w)$  have the same height and the same number of active sites.

**Example 7.** Let

$$w = a \ b \ a \ a \ b \ a \ b \ b \ a \ a \ b \ b \ a \ b \ a \ a \ a \ b \ b \ b \ b.$$

We get:

$$\text{Sk}(w) = a \ a \ b \ b \ a \ a \ b \ b \ a \ a \ a \ b \ b \ b,$$

since we delete the factors  $ab$  which are delimited in the equation below by parenthesis:

$$w = (a \ b) \ a \ a \ b \ (a \ b) \ b \ a \ a \ b \ b \ (a \ b) \ (a \ b) \ a \ a \ a \ b \ b \ b \ b.$$

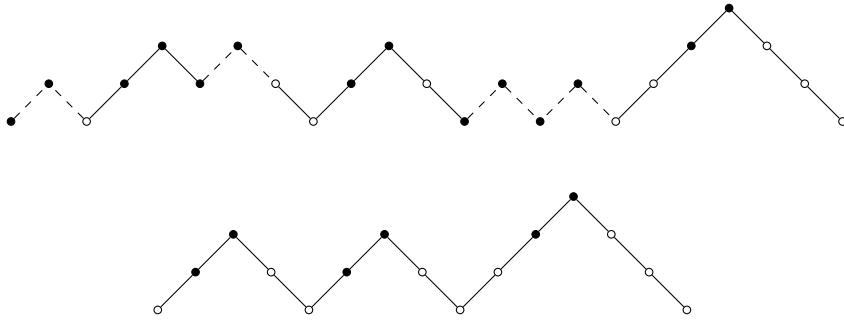


Figure 3: A Dyck path  $w$  and its skeleton. The nine white vertices in each path denote the active sites and the dashed peaks are removed from  $w$  to get  $\text{Sk}(w)$ .

The Dyck words  $w$  and  $\text{Sk}(w)$  correspond to the Dyck paths in Figure 3.

The word  $w$  can be obtained from  $\text{Sk}(w)$  as soon as the positions of the active sites where the factors  $ab$  were deleted are known. This can be done by a sequence  $(p_1, p_2, \dots, p_\ell)$  that gives for each active site the number of factors  $ab$  that were deleted. For our example the number of active sites in this sequence is 9, and the values are given by:

$$(1, 1, 0, 0, 2, 0, 0, 0).$$

In the sequel we will denote this sequence by  $\text{ins}(w)$ . Conversely, a smooth word  $\hat{w}$  and a sequence  $I = (i_1, i_2, \dots, i_\ell)$  of integers, where  $\ell$  is the number of active sites in  $\hat{w}$ , determine a unique word  $w$  such that  $\text{Sk}(w) = \hat{w}$  and  $\text{ins}(w) = I$ .

We have a first trivial lemma showing the effect of deleting a factor  $ab$  in a word when this factor is followed by an occurrence of  $a$  or preceded by an occurrence of  $b$ .

**Lemma 3.** *Let  $w = uabv \in D_n$  be such that  $u$  ends with  $b$  or  $v$  begins with  $a$ . Let  $\text{ins}(w) = (i_1, i_2, \dots, i_\ell)$  and set  $w' = uv$ . Then we have*

$$\text{Sk}(w) = \text{Sk}(w') \quad \text{and} \quad \text{ins}(w') = (i_1, i_2, \dots, i_{k+1} - 1, \dots, i_\ell), \quad (6)$$

where  $k$  is the number of active sites of  $w$  appearing in  $u$ .

The action of  $\gamma$  in relation with the deletion of a factor  $ab$  obeys to the rules expressed in the following Lemma.

**Lemma 4.** *If  $w'$  is obtained from  $w$  by deleting a factor  $ab$  followed by an occurrence of  $a$  or preceded by an occurrence of  $b$ , then  $\gamma(w')$  is obtained from  $\gamma(w)$  in the same way. Moreover, let  $\ell$  be the number of active sites of  $w$  and  $\ell_0$  the number of those situated in its principal prefix. Assume that the factor  $ab$  deleted from  $w$  is situated in the  $i$ -th active site of  $w$ . Then:*

- If  $i \leq \ell_0$  then  $\gamma(w')$  is obtained from  $\gamma(w)$  by deleting  $ab$  in its  $(\ell - \ell_0 + i)$ -th active site.
- If  $i > \ell_0$  then  $\gamma(w')$  is obtained from  $\gamma(w)$  by deleting  $ab$  in its  $(i - \ell_0)$ -th active site.

*Proof.* Assume  $w = w^{(1)}abaw^{(2)}b$  and  $w' = w^{(1)}aw^{(2)}b$ . The case  $w = w^{(1)}bab^{(2)}b$  and  $w' = w^{(1)}bw^{(2)}b$  can be treated similarly.

Let  $u$  be the principal prefix of  $w$ . Then we have  $w = uvb$  and  $\gamma(w) = \bar{v}b\bar{u}$ .

- If  $|w^{(1)}a| < |u|$  then we have  $u = w^{(1)}abaw^{(3)}$ , with  $w^{(3)} \neq \emptyset$  since the principal prefix cannot end with  $aba$ , and

$$\gamma(w) = \bar{v}b\bar{w}^{(1)}bab\bar{w}^{(3)}.$$

We notice that the number of active sites in  $\bar{v}$  is  $\ell - \ell_0$  and their number in  $\bar{w}^{(1)}$  is  $i$ , giving the first part of the Lemma, since  $\gamma(w') = \bar{v}b\bar{w}^{(1)}b\bar{w}^{(3)}$ .

- If  $w^{(1)}a = u$ , then we have  $i = \ell_0 + 1$  and

$$\gamma(w) = ab\bar{w}^{(2)}b\bar{w}^{(1)}b, \quad \gamma(w') = \bar{w}^{(2)}b\bar{w}^{(1)}b,$$

thus  $\gamma(w')$  is obtained from  $\gamma(w)$  by deleting  $ab$  in the first position in accordance with  $i - \ell_0 = 1$ .

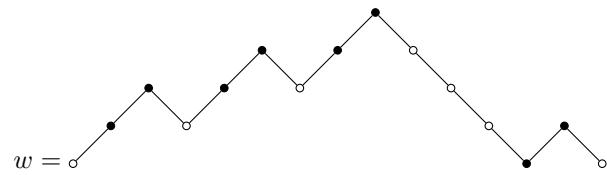
- If  $|w^{(1)}a| > |u|$  then we have  $w^{(1)}a = uw^{(3)}$  where  $w^{(3)}$  ends with an occurrence of  $a$ , thus  $w = uw^{(3)}baw^{(2)}b$  and

$$\gamma(w) = \bar{w}^{(3)}ab\bar{w}^{(2)}b\bar{u}$$

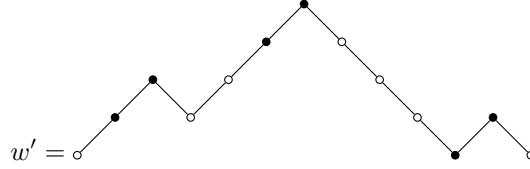
The number of active sites in  $\bar{w}^{(3)}$  is  $i - \ell_0$ , proving the second part of the Lemma.

□

**Example 8.** Consider the following path

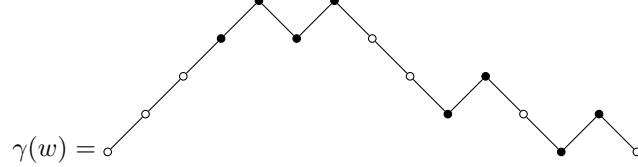


where the white vertices are the active sites. Here  $\ell = 7$  and  $\ell_0 = 3$ .  $w$  is obtained from the path

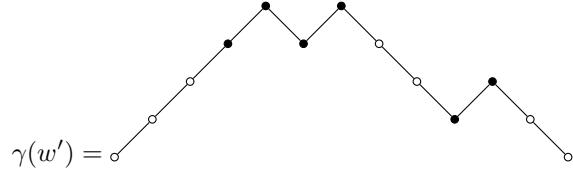


by adding a peak in the third active site. Hence  $i = 3 \leq \ell_0$ .

We have



and



$\gamma(w)$  is obtained from  $\gamma(w')$  by adding a peak in the active site number  $\ell - \ell_0 + i = 7 - 3 + 3 = 7$ .

We now compare  $(\text{Sk}(w), \text{ins}(w))$  and  $(\text{Sk}(\gamma(w)), \text{ins}(\gamma(w)))$ .

We denote by  $\text{cyc}$  the cyclic shift of a finite sequence. It is given by:

$$\text{cyc}((i_1, i_2, \dots, i_n)) = (i_2, i_3, \dots, i_n, i_1).$$

**Lemma 5.** *Let  $w \in D_n$ . Then we have*

$$\text{Sk}(\gamma(w)) = \gamma(\text{Sk}(w)),$$

and

$$\text{ins}(\gamma(w)) = \text{cyc}^{\ell_0}(\text{ins}(w)),$$

where  $\ell_0$  is the number of active sites contained in the principal prefix of  $w$ .

*Proof.* This Lemma is obtained by applying the two Lemmas 3 and 4 sufficiently many times.  $\square$

## 6. Orbits of $\gamma$

We are now in the position to prove our main result.

**Theorem 2.** *Let  $w$  be a word in  $D_n$ . Then there exists an odd integer  $k$  such that  $\gamma^k(w) = w$ .*

*Proof.* The proof proceeds by induction on  $n$ . The assertion is trivially true for  $n = 1$ . Assume that for any  $n' < n$  and for each word  $w' \in D_{n'}$  there exists an odd integer  $k'$  such that  $\gamma^{k'}(w') = w'$ . Let  $w \in D_n$ . We distinguish two cases:

- If  $w$  is smooth then  $w' = \theta(w)$  is an element of  $D_{n'}$  with  $n' < n$ . By the inductive hypothesis there exists an odd integer  $k'$  such that  $\gamma^{k'}(w') = w'$ . But by Lemma 2  $\gamma(\theta(w)) = \theta(\gamma(w))$ , hence  $\gamma^{k'}(w) = w$ .
- If  $w$  is not smooth then its skeleton  $\hat{w}$  belongs to  $D_{n'}$  with  $n' < n$ . By the inductive hypothesis we have that there exists an odd integer  $k'$  such that  $\gamma^{k'}(\hat{w}) = \hat{w}$ .

Set  $\text{ins}(w) = (i_1, i_2, \dots, i_\ell)$ . Denote by  $\ell_i$ ,  $0 \leq i \leq k' - 1$ , the number of active sites in the principal prefixes of  $w$ ,  $\gamma(w)$ ,  $\dots$ ,  $\gamma^{k'-1}(w)$ , respectively, and set  $L = \sum_{i=0}^{k'-1} \ell_i$ .

By Lemma 5, we have

$$\begin{aligned} \text{Sk}(\gamma(w)) &= \gamma(\hat{w}), & \text{ins}(\gamma(w)) &= \text{cyc}^{\ell_0}(\text{ins}(w)), \\ \text{Sk}(\gamma^2(w)) &= \gamma^2(\hat{w}), & \text{ins}(\gamma^2(w)) &= \text{cyc}^{\ell_0 + \ell_1}(\text{ins}(w)), \\ &\dots & & \\ \text{Sk}(\gamma^{k'}(w)) &= \gamma^{k'}(\hat{w}) = \hat{w}, & \text{ins}(\gamma^{k'}(w)) &= \text{cyc}^L(\text{ins}(w)), \\ &\dots & & \\ \text{Sk}(\gamma^{k'\ell}(w)) &= \gamma^{k'\ell}(\hat{w}) = \hat{w}, & \text{ins}(\gamma^{k'\ell}(w)) &= \text{cyc}^{\ell L}(\text{ins}(w)). \end{aligned}$$

Now,  $\ell L$  is a multiple of the length of the sequence  $\text{ins}(w)$ , so we have that  $\text{cyc}^{\ell L}(\text{ins}(w)) = \text{ins}(w)$ , giving  $w = \gamma^{k'\ell}(w)$ . Since the two integers  $\ell$  and  $k'$  are odd we get the assertion.

□

In the following we denote by  $\text{Orb}(w)$  the orbit of the word  $w$  under the action of the map  $\gamma$ , namely, the set  $\{\gamma(w), \gamma^2(w), \dots, \gamma^{k-1}(w)\}$ , where  $k$  is the minimum positive integer such that  $\gamma^k(w) = w$ .

**Theorem 3.** *For every  $w \in D_n$  the orbit of  $w$  has odd cardinality.*

*Proof.* The assertion is an immediate consequence of the preceding result, since the cardinality of  $\text{Orb}(w)$  must divide any  $k$  such that  $\gamma^k(w) = w$ .  $\square$

In the table below we give for each  $n$  the number of cycles of lengths  $1, 3, \dots, 2n-3$  of the permutation  $\gamma$  on  $D_n$ , and the list of lengths of the cycles which are greater than  $2n-3$ . This table was obtained by a computer program that generates all Dyck words in  $D_n$  with  $2 \leq n \leq 10$  and determines the cycles of the permutation  $\gamma$ .

	1	3	5	7	9	11	13	15	17	other values
n=2	2									
n=3	2	1								
n=4	3	2	1							
n=5	3	4	4	1						
n=6	4	5	11	4	1					21
n=7	2	10	21	14	6	1				21,45
n=8	6	10	44	34	24	6	1			21,27,33,45,77
n=9	5	17	67	83	74	27	8	2		21,27,33,45,55 65,77,117
n=10	4	17	119	162	212	92	44	11	1	21,27,33,45,55 65,77,91,105 117,165,273

**Proposition 7.** For every odd number  $2n+1$  there exists a word  $w \in D_{n+2}$  such that  $\text{Orb}(w)$  has length  $2n+1$ .

*Proof.* Consider the word  $w = aba^{n+1}b^{n+2}$ . Notice that the principal prefix of  $w$  has  $n$  active sites. We have

$$\text{Sk}(w) = a^{n+1}b^{n+2}, \quad \text{ins}(w) = \underbrace{(1, 0, \dots, 0)}_{2n+1}.$$

The word  $\text{Sk}(w)$  is obviously fixed under  $\gamma$ , hence, by Lemma 5,

$$\text{Sk}(\gamma(w)) = \text{Sk}(w), \quad \text{ins}(\gamma(w)) = \text{cyc}^n(\text{ins}(w)).$$

Since  $\gcd(n, 2n+1) = 1$ , we deduce that  $|\text{Orb}(w)| = 2n+1$ .  $\square$

In order to give some additional properties of the orbits of  $\gamma$  we build a graph  $G_n$  whose vertices are the words in  $D_n$ . The edges of  $G_n$  are all the pairs  $(w, \beta(w))$  and  $(w, \alpha(w))$ , thus the fixed points of  $\beta$  and  $\alpha$  correspond to loops in  $G_n$ . Taking the convention that a loop contributes for 1 in the degree of the vertex adjacent to it, all vertices in  $G_n$  have degree 2.

The graph  $G_n$  has some elementary properties:

**Lemma 6.** *Each connected component  $C_i$  of  $G_n$  corresponds to an orbit of the group generated by  $\alpha$  and  $\beta$  and satisfies one of the conditions below:*

- $C_i$  has a unique vertex incident with two loops.
- $C_i$  is a path of odd length connecting two vertices each one being incident with a loop.
- $C_i$  is a cycle of length  $4p + 2$ , where  $p \geq 1$ .

*Proof.* One of the first trivial results in graph theory states that a graph with no loops whose vertices have degree 0, 1 or 2 is the union of isolated vertices, paths and cycles. Let  $G'_n$  be the graph obtained from  $G_n$  by deleting every loop. Notice that the connected components of the graph  $G'_n$  have the same blocks of vertices as the connected components of  $G_n$ . Thus the graph  $G'_n$  satisfies the condition stated above. Let  $C_i$  be a connected component of  $G'_n$ . Then:

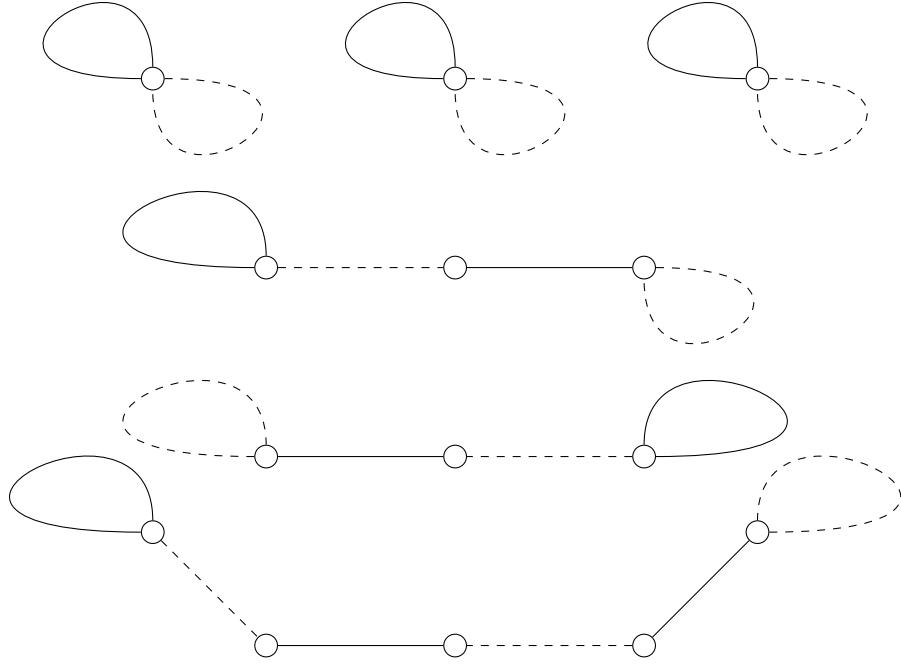
- if  $C_i$  is an isolated vertex, then it corresponds in  $G_n$  to a vertex incident with two loops.
- If  $C_i$  is a path, it corresponds in  $G_n$  to a path connecting two vertices each one being incident with a loop. By definition of  $G_n$  this path may be written  $w^{(1)}, w^{(2)}, \dots, w^{(p)}$ . Without loss of generality (by exchanging  $\beta$  and  $\alpha$ ), we may suppose that  $\beta(w^{(1)}) = w^{(1)}$ ,  $\beta(w^{(i)}) = w^{(i+1)}$  for  $i$  even and  $\alpha(w^{(i)}) = w^{(i+1)}$  for  $i \neq 1$  odd. If  $p$  is even,  $p = 2q$ , then the cycle of  $\gamma$  containing  $w^{(1)}$  is equal to:

$$(w^{(1)}, w^{(2)}, w^{(4)}, \dots, w^{(2q)}, w^{(2q-1)}, \dots, w^{(3)}),$$

hence it has even length contradicting Theorem 3.

- If  $C_i$  is a cycle, it is also a cycle in  $G_n$ , hence this cycle may be written  $C_i = w^{(1)}, w^{(2)}, \dots, w^{(p)}$ . The images under  $\beta$  and  $\alpha$  alternate in this cycle, hence,  $p$  is even. Let  $p = 2q$ . Computing the cycle  $\mathcal{C}$  of  $\gamma$  containing  $w^{(1)}$  we get

$$\mathcal{C} = (w^{(1)}, w^{(3)}, \dots, w^{(2q-1)}).$$

Figure 4: The graph  $G_4$ 

As a consequence of Theorem 3 we have that  $q$  is odd, completing the proof. □

As an immediate consequence we have:

**Proposition 8.** *Any orbit  $\mathcal{O}$  of  $\gamma$  satisfies one and only one of the two assertions below:*

1. *Any  $w$  in  $\mathcal{O}$  satisfies  $\beta(w) \notin \mathcal{O}$ , and  $\alpha(w) \notin \mathcal{O}$ , moreover there exists an orbit  $\mathcal{O}'$  different from  $\mathcal{O}$  such that  $w \in \mathcal{O} \Leftrightarrow \beta(w) \in \mathcal{O}'$  and  $w \in \mathcal{O} \Leftrightarrow \alpha(w) \in \mathcal{O}'$ .*
2. *There exists  $w'$  and  $w''$  in  $\mathcal{O}$  such that  $\beta(w') = w'$  and  $\alpha(w'') = w''$ . Moreover for any  $w \in \mathcal{O}$  we have  $\alpha(w) \in \mathcal{O}$  and  $\beta(w) \in \mathcal{O}$ .*

Propositions 8 and 2 imply immediately the following.

**Corollary 3.** *The pair of statistics (number of peaks, height) is equidistributed over the set of symmetric words in  $D_n$  and the set of words in  $D_n$  that are the concatenation of two palindromes.*

## 7. The fixed points of $\gamma$

We are now interested in the fixed points of  $\gamma$ .

**Proposition 9.** *Let  $w \in D_n$  be such that  $\gamma(w) = w$ , and let  $p$  be the length of the principal prefix  $u$  of  $w$ . Then we have*

$$w_{m-p} = w_m = b,$$

and for all  $i \neq m-p$

$$w_{p+i} = \overline{w_i}$$

where  $p+i$  is taken mod  $m$ , where  $m = 2n+1$  is the common length of the words in  $D_n$ .

*Proof.* Let  $u$  be the principal prefix of  $w$ . Then we have  $w = uvb$  and  $\gamma(w) = \overline{v}b\overline{u}$ , giving:

$$uvb = \overline{v}b\overline{u},$$

thus

$$w_1 w_2 \cdots w_p w_{p+1} \cdots w_{m-1} b = \overline{w}_{p+1} \cdots \overline{w}_{m-1} b \overline{w}_1 \cdots \overline{w}_p.$$

Hence

$$w_i = \overline{w}_{p+i} \text{ for } i = 1, 2, \dots, m-p-1,$$

$$w_{m-p} = w_m = b \quad \text{and} \quad w_i = \overline{w}_{p+i-m} \text{ for } m-p < i \leq m.$$

The result now follows from the fact that  $p+i-m \equiv p+i \pmod{m}$ .

□

**Corollary 4.** *Let  $w \in D_n$  be such that  $\gamma(w) = w$  and  $m = 2n+1$ . Let  $p$  be the length of the principal prefix of  $w$ . Then the permutation  $\tau \in S_m$  defined by  $\tau(i) \equiv i+p \pmod{m}$  has only one cycle. Hence*

$$\gcd(m, p) = 1.$$

*Proof.* It is well known that the permutation  $\tau$  has cycles of the same length, which is equal to  $\frac{m}{\gcd(m,p)}$ . Since  $m$  is odd this common length cannot be even. By Proposition 9 we have that the letters  $w_i$  and  $w_{\tau(i)}$  of the word  $w$  are different except when  $\tau(i) = m$ . Hence, if a cycle of  $\tau$  does not contain  $m$ , its elements correspond to occurrences of letters in  $w$  alternating between  $a$  and  $b$ . This implies that such a cycle must be of even length. We conclude that such a cycle does not exist and that  $\tau$  has only one cycle.  $\square$

**Theorem 4.** *For any  $p < m$  such that  $\gcd(m,p) = 1$  there is at most one word  $w$  in  $D_n$  satisfying the two conditions:*

- $\gamma(w) = w$ .
- *The principal prefix of  $w$  has length  $p$ .*

Moreover each such word satisfies

$$\alpha(w) = \beta(w) = w.$$

*Proof.* We have seen that if  $w$  satisfies the condition  $w = \gamma(w)$ , the permutation  $\tau$  has only one cycle which we can write  $(i_1 = m, i_2, \dots, i_m)$ . With this notation  $w_{i_j}$  is equal to  $a$  if  $j$  is even and equal to  $b$  if  $j$  is odd. We also have  $i_{j+1} \equiv i_j + p \pmod{m}$ . This gives a unique word.

In order to prove that  $\beta(w) = w$  we have to show that  $w_k = \bar{w}_{m-k}$  for any  $k < m$ , or equivalently  $w_{i_j} = \bar{w}_{m-i_j}$  for any  $j$  such that  $i_j < m$ . Note that  $i_j \equiv jp \pmod{m}$ . Hence we have to prove that if  $k$  is such that  $m - i_j = kp$ , then one of  $j, k$  is even and the other one is odd. This follows from the fact that  $m$  is odd and  $\gcd(m,p) = 1$ , since we have:

$$kp \equiv m - i_j \equiv m - pj \pmod{m},$$

giving

$$p(j+k) \equiv 0 \pmod{m}.$$

Since  $\gamma = \alpha \circ \beta$ , we get immediately also  $\alpha(w) = w$ .  $\square$

This result implies the following characterization of the fixed points of  $\gamma$ .

**Proposition 10.** *A word  $w$  in  $D_n$  is fixed under  $\gamma$  whenever the following two conditions are satisfied:*

- $\gamma(\text{Sk}(w)) = \text{Sk}(w)$ , and
- $\text{ins}(w) = (i, i, \dots, i)$  for some  $i \geq 0$ .

*Proof.* Let  $w$  be a Dyck word satisfying the two conditions above. Then by Lemma 5 we get  $\gamma(w) = w$ .

Conversely, take  $w \in D_n$  such that  $w = \gamma(w)$ . Lemma 5 implies immediately that also  $\text{Sk}(w)$  is fixed under  $\gamma$  and  $\text{ins}(w) = \text{cyc}^{\ell_0}(\text{ins}(w))$ , where  $\ell_0$  is number of active sites in the principal prefix of  $w$ . Set

$$\text{Sk}(w) = a^{i_1}b^{j_1}a^{i_2}b^{j_2} \cdots a^{i_k}b^{j_k},$$

where  $k$  is the number of peaks of  $w$  and  $i_s > 1$ ,  $j_t > 1$  for all  $s, t$ . Hence, by Propositions 5 and 6, the words  $w$  and  $\text{Sk}(w)$  have

$$\ell = \sum_{s=1}^k (i_s + j_s) - 2k$$

active sites, and both their principal prefixes have

$$\ell_0 = \sum_{s=1}^{h-1} (i_s + j_s) - 2h + 1$$

active sites, for some  $h \leq k - 1$ .

Consider now the word

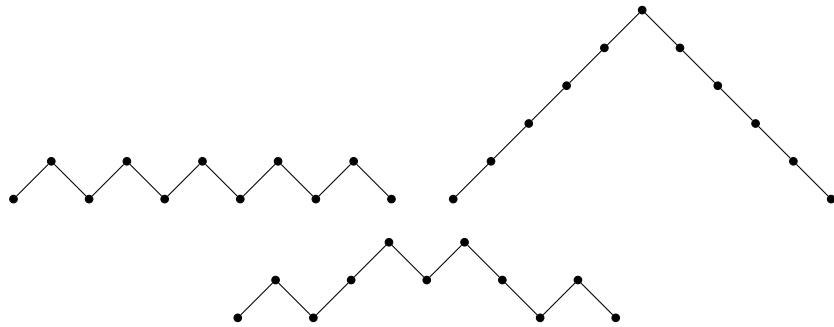
$$\hat{w} = \theta(\text{Sk}(w)) = a^{i_1-1}b^{j_1-1}a^{i_2-1}b^{j_2-1} \cdots a^{i_k-1}b^{j_k-1}.$$

Observe that by Proposition 2  $\hat{w}$  is fixed under  $\gamma$ .

It is immediately checked that  $\hat{w}$  has length  $\ell$  and its principal prefix has length  $\ell_0$ . Corollary 4 implies now that  $\text{gcd}(\ell, \ell_0) = 1$ . Since the sequence  $\text{ins}(w)$  has length  $\ell$  and  $\text{ins}(w) = \text{cyc}^{\ell_0}(\text{ins}(w))$ , we deduce that  $\text{ins}(w)$  must be a constant sequence.  $\square$

Theorem 4 gives a way to build all words  $w \in D_n$  such that  $\gamma(w) = w$ . We use the following algorithm:

Let  $m = 2n + 1$ . Take all integers  $p$  such that  $p \leq n$  and  $\text{gcd}(2n + 1, m) = 1$ . For each such  $p$  consider the permutation  $\tau$  on  $\{1, 2, \dots, m\}$  such that  $\tau(i) \equiv i + p \pmod{m}$ . This permutation has only one cycle, which may be written  $(i_1, i_2, \dots, i_m)$  with  $i_1 = p, i_m = m$ . Let  $w$  be the word given by  $w_i = a$  if  $i = i_j$  with  $j$  odd and  $i \neq m$ , and  $w_i = b$  if  $i = m$  or  $i = i_j$  with  $j$  even. If  $w$  is in  $D_n$ , then  $\gamma(w) = w$ .

Figure 5: The three Dyck paths of semilength 5 fixed by the action of  $\gamma$ .

**Example 9.** Consider  $n = 5$  and  $p = 4$ , then  $m = 11$  and the permutation  $\tau$  is the cycle

$$(4, 8, 1, 5, 9, 2, 6, 10, 3, 7, 11),$$

thus  $w_i = a$  for  $i = 4, 1, 9, 6, 3$  and  $w_i = b$  for  $i = 8, 5, 2, 10, 7, 11$  giving the word  $w = abaababbabb$  which is in  $D_n$  and is such that  $\gamma(w) = w$ .

Taking  $n = 5, p = 3$  gives

$$\tau = (3, 6, 9, 1, 4, 7, 10, 2, 5, 8, 11).$$

Since this implies  $w_1 = b$ , we have  $w \notin D_n$ .

Combining Theorem 4 and Proposition 10 we have:

**Corollary 5.** The number of words  $w$  fixed by  $\gamma$  in  $D_n$  is upper bounded by:

$$\min(n, \phi(2n + 1)),$$

where  $\phi$  is the Euler totient function.

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## 8. References

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