

Computing the rank of configurations on Complete Graphs

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The paper by M. Baker and S. Norine [1] in 2007 introduced a new parameter in Graph Theory it was called the rank of configurations on graphs. A configuration consists of giving an integer value to each vertex of the graph. It was proved 10 years after that the computation of the rank is an NP-complete problem for general graphs (see [6]). In a paper written with Yvan Le Borgne [3] we presented a linear time algorithm computing the rank of configurations in the complete graph. I give here a brief presentation of the main results of this paper, following the way in which I presented it in VIASM at Hanoi during october 2016.

1 Definitions and notation

Graphs and configurations

Let $G = (X, E)$ be a connected graph, where $X = \{x_1, x_2, \dots, x_n\}$ is the vertex set and E the set of edges. We consider also E as a symmetric matrix such that $e_{i,j} = 1$ if there is an edge between the vertices x_i and x_j , and $e_{i,j} = 0$ if not. We assume that G is connected and has no loops, such that $e_{i,i} = 0$ for all i .

We will consider *configurations* on graphs, these are elements of the discrete lattice \mathcal{Z}^n , where \mathcal{Z} is the set of integers (positive or negative). For a configuration $u = (u_1, u_2, \dots, u_n)$ we will refer to the u_i 's as the *entries* of u . The symbol $\varepsilon^{(i)}$ will denote the configuration in which the value 1 is assigned to vertex x_i and the value 0 is assigned to all other vertices.

The *degree* of the configuration u is the sum of the u_i 's it is denoted by $\deg(u)$.

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To any vertex is associated a *Laplacian configuration* $\Delta^{(i)}$, it is given by: $\Delta^{(i)} = d_i \varepsilon^{(i)} - \sum_{j=1}^n e_{i,j} \varepsilon^{(j)}$, where $d_i = \sum_{j=1}^n e_{i,j}$ is the degree of the vertex x_i .

$\mathcal{L}(G)$ -equivalence

We denote by $\mathcal{L}(G)$ the subgroup of \mathbb{Z}^n generated by the $\Delta^{(i)}$, and two configurations u and v will be said *toppling equivalent* if $u - v \in \mathcal{L}(G)$, which will also be written as $u \sim_{\mathcal{L}(G)} v$.

In the sandpile model and the chip-firing game, the transition from configuration u to the configuration $u - \Delta^{(i)}$ is allowed only if $u_i \geq d_i$ and is called a toppling. Here we omit this condition and perform topplings even if $u_i < d_i$.

Effectiveness

The notion of $\mathcal{L}(G)$ -effectiveness introduced in [1] is central in this presentation.

Definition 1.1. A configuration u is effective if $u_i \geq 0$ for all i . A configuration u is $\mathcal{L}(G)$ -effective if there exists an effective configuration v toppling equivalent to u (recall that this means $u - v \in \mathcal{L}(G)$).

Since two $\mathcal{L}(G)$ -equivalent configurations have the same degree, it is clear that a configuration with negative degree is not $\mathcal{L}(G)$ -effective. From now on it will be convenient to denote effective configurations using greek letters λ, μ and configurations with no particular assumptions on them by letters u, v, w .

Rank

The notion of rank indicates how far is a configuration $\mathcal{L}(G)$ -effective. It is by subtracting effective configurations to it, and checking if it is still $\mathcal{L}(G)$ -effective.

Definition 1.2. The rank $\rho(u)$ of a configuration u is given by:

- if u is not $\mathcal{L}(G)$ -effective then $\rho(u) = -1$,
- if u is $\mathcal{L}(G)$ -effective, then $\rho(u)$ is the largest integer r such that for any effective configuration λ of degree r the configuration $u - \lambda$ is $\mathcal{L}(G)$ -effective.

Denoting \mathbb{P} the set of effective configurations and \mathbb{E} the set of $\mathcal{L}(G)$ -effective configurations this definition can be given by the following compact formula which is valid in both cases:

$$\rho(u) + 1 = \min_{\lambda \in \mathbb{P}, u - \lambda \notin \mathbb{E}} \deg(\lambda)$$

An immediate consequence of this definition is that if two configurations u and v satisfy $u_i \leq v_i$ for all i then $\rho(u) \leq \rho(v)$. Moreover if $u_i = v - \varepsilon^{(i)}$ then $\rho(v) - 1 \leq \rho(u) \leq \rho(v)$

The following notion will be useful to prove properties of the rank:

Definition 1.3. An effective configuration μ is a proof for the rank $\rho(u)$ of an $\mathcal{L}(G)$ -effective configuration u if $u - \mu$ is not $\mathcal{L}(G)$ -effective and $u - \lambda$ is $\mathcal{L}(G)$ -effective for any effective configuration λ such that $\deg(\lambda) < \deg(\mu)$.

Notice that if λ is a proof for $\rho(u)$ then $\rho(u) = \deg(\lambda) - 1$.

2 Laplacian configurations in the complete graph

In the complete graph K_n each vertex has degree $n - 1$ and for any two vertices there is an edge between them. Hence the Laplacian group is generated by the n configurations:

$$\Delta^{(i)} = (-1, \dots, -1, n - 1, \dots, -1)$$

For each $i = 1, 2, \dots, n$ we have:

$$\Delta_j^{(i)} = \begin{cases} n - 1 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$$

We notice that all the entries in these configuration are equal ($\text{mod } n$), hence this is also true for any configuration in the Laplacian subgroup since its elements are linear combinations of the $\Delta^{(i)}$. Notice that the converse statement is also true giving the following:

Proposition 2.1. *A configuration $u = (u_1, u_2, \dots, u_n)$ in K_n belongs to the Laplacian subgroup $\mathcal{L}(K_n)$ of the complete graph K_n if and only if the two conditions below are satisfied:*

$$\deg(u) = 0, \tag{1}$$

$$u_i \equiv u_j \pmod{n} \quad \text{for all } 1 \leq i, j \leq n \tag{2}$$

Proof. We only have to prove that if conditions (1) and (2) are satisfied, then $u \in \mathcal{L}(K_n)$. We proceed by induction on $\sum_{i=1}^n |u_i|$. It is trivial to notice that if this sum is 0 then $u = 0$ which is in \mathcal{L} . Suppose that the result is proved when this sum is less than p and let u be such a configuration for which the sum is equal to $p+1$. Let j be an index such that $|u_j|$ is maximal among the $|u_i|$. We may suppose that $u_i > 0$ since u and $-u$ both satisfy the same conditions.

Let $v = u - \Delta^{(j)}$, and consider the case when $u_j \geq n-1$. Then we have $|v_j| = |u_j| - (n-1)$, since the degree of u is 0, there exists at least one k such that $u_k < 0$. This gives $v_k = u_k + 1$ and $|v_k| = |u_k| - 1$, for the $i \neq j, k$, the difference between $|v_j|$ and $|u_j|$ is at most 1. Hence:

$$\sum_{i=1}^n |v_i| \leq \sum_{i=1}^n |u_i| - (n-1) - 1 + (n-2) = \sum_{i=1}^n |u_i| - 2 < p$$

Applying the induction hypothesis we have that $v \in \mathcal{L}(K_n)$ and $u = v + \Delta^{(j)} \in \mathcal{L}(K_n)$.

If $y_j < n-1$, then u is such that k among the entries are equal to $k-n$ and $n-k$ are equal to k , giving

$$u = \sum_{i \in I} \Delta^{(i)}$$

where I is the subset of $\{1, 2, \dots, n\}$ consisting of those i such that $f_i = k$. Clearly $u \in \mathcal{L}(K_n)$ \square

This Proposition shows that to check a property on a configuration u which depends only on the class of u with respect to the Laplacian subgroup, one can consider a configuration v equivalent to u and satisfying $v_i = u_i \pmod{n}$ for $i \neq n$ and $v_n = \deg(u) - \sum_{i=1}^{n-1} v_i$. Then we have $u \sim_{\mathcal{L}(K_n)} v$ and since $0 \leq v_i < n$ for $i \neq n$ it may be more convenient to consider v instead of u .

In the sequel we will say that a configuration u such that $0 \leq u_i \leq n$ for $i \neq n$ is *reduced*.

What we have seen gives an algorithm allowing to build a reduced configuration $\mathcal{L}(K_n)$ -equivalent to a given configuration in K_n . It is described below, the variable \deg contains the value to add to u_n in order that u and v have the same degree.

Input: A configuration u on K_n

Output: A configuration v $\mathcal{L}(K_n)$ -equivalent to u such that $0 \leq v_i < n$ for any $i < n$

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 $deg \leftarrow 0;$ 
for  $(i = 1, \dots, n - 1)$  do;
  •  $v_i \leftarrow u_i \pmod n$ 
  •  $deg \leftarrow deg + u_i - v_i$ 
 $v_n \leftarrow u_n + deg$ 

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3 Superstable configurations

An *almost effective* configuration is a configuration u such that $u_i \geq 0$ for $i < n$. It is easy to see that any configuration is equivalent by $\mathcal{L}(G)$ to an almost effective configuration. This is very easy to prove for the complete graph (add sufficiently many times $\Delta^{(n)}$ when $u_i < 0$ for some $i < n$). Notice that this is also true for any graph.

Definition 3.1. A configuration u is *superstable* if it is almost effective and if for any non empty $I \subset \{1, 2, \dots, n - 1\}$ the configuration $u - \sum_{i \in I} \Delta^{(i)}$ is not almost effective.

To check if a configuration u of the graph K_n is superstable one we have the following

Proposition 3.1. *Let u be an almost effective configuration in K_n and let v be the configuration obtained from u by sorting in increasing order the elements u_1, u_2, \dots, u_{n-1} , then u is superstable if and only if $v_i < i$ for all $i < n$.*

Proof. Since the graph K_n has \mathcal{S}_n as automorphism group, the property of being superstable is invariant by permuting the entries u_i . Hence v is superstable if and only if u is.

We first prove that if $v_i \geq i$ for some i , then v is not superstable. Indeed consider $I = \{i, i + 1, \dots, n - 1\}$ and let $w = v - \sum_{i \in I} \Delta^{(i)}$. The values of the entries of w satisfy: $w_j = v_j - (n - 1) + n - i - 1 = v_j - i$, and for $j < i$, $w_j > v_j$. Since $v_i \geq i$ and $v_j \geq v_i$ for $j \geq i$, w is almost effective. Hence v is not superstable.

Conversely we show that if $v_i < i$ for all $i < n$ then for any $I \subset \{1, 2, \dots, n-1\}$,

$w = v - \sum_{i \in I} \Delta^{(i)}$ is not almost effective. Let j be the smallest element in I . We will prove that $w_j < 0$. Let k be the number of elements in I , since I is a subset of $\{j, j+1, \dots, n-2, n-1\}$ a set having $n-j$ elements we have $k \leq n-j$. Let us compute w_j from v_j , we have to subtract $n-1$ and add $k-1$ giving:

$$w_j = v_j - n + k \leq u_j - n + n - j = v_j - j$$

Since $v_j < j$ we get the result. □

We use the following result which may be considered as a straightforward consequence of an old result by D. Dhar proving that any class of Laplacian equivalence contains a unique "recurrent" configuration. Different proofs can be found in: [2, 4, 5]. Since one can show that a configuration u is recurrent if and only if the configuration u' such that $u'_i = d_i - u_i$ is superstable (where d_i is the degree of vertex i) we have:

Theorem 3.2. *Any configuration is $\mathcal{L}(G)$ -equivalent to a unique superstable configuration.*

4 Words and Dyck words

We consider the alphabet $A = \{a, b\}$ and the set A^* of words on A . A prefix of a word f is a word g such that there exists h such that $f = gh$, the prefix g is strict if $g \neq f$. For any word f the number of occurrences of the letter x in f will be denoted by $|f|_x$ and the length of f by $|f|$, hence $|f| = |f|_a + |f|_b$. We denote by A_n the set of words of length $2n-1$ which have one more occurrence of b than that of a , hence we may write:

$$A_n = \{f \in A^* \mid |f|_a = n-1, |f|_b = n\}$$

The usual notation on words will be used, two words f and f' are *conjugate* if there exists g, h such that $f = gh$ and $f' = hg$. The mirror image of a word $f = f_1 f_2 \dots f_m$ will be denoted \tilde{f} it is equal to $f = f_m f_{m-1} \dots f_2 f_1$.

In this paper we call Dyck word a word f satisfying $|f|_a = |f|_b - 1$ and such that for any strict prefix g of f the number of occurrences of a is not less than the number of occurrences of b . It is useful to define the function δ assigning to any word f of A^* the integer $\delta(f) = |f|_a - |f|_b$ then f is Dyck

word if $\delta(f) = -1$ and $\delta(g) \geq 0$ for any strict prefix of f . Notice that the Dyck words are very often considered with the last b omitted. We decided to add an additional b in order to have a simple statement for the following simple Cyclic Lemma which we will use constantly. The subset consisting of Dyck words of length $2n - 1$ will be denoted D_n .

One of the appealing results in combinatorics of words is:

Lemma 4.1. *Any word f in A_n can be decomposed in a unique way as $f = gh$ such that hg is a Dyck word.*

From this lemma one has that the number of elements in D_n is the Catalan number:

$$\frac{1}{2n-1} \binom{2n-1}{n} = \frac{(2n-2)!}{n!(n-1)!}$$

Definition 4.1. The sequence of heights $\eta = (\eta_1, \eta_2, \dots, \eta_{n-1})$ of $f \in D_n$ is given by $\eta_i = \delta(f^{(i)})$, where $f^{(i)}$ the prefix of f preceding the i -th occurrence of a in f .

Notice that for a sorted and superstable configuration u the heights for the Dyck word f such that $\phi(u) = (w, k)$ satisfy $\eta_i = i - 1 - u_i$. Indeed for a prefix $g^{(i)}$ of f preceding the i -th occurrence of the letter a , we have $|g^{(i)}|_a = i - 1$ and $|g^{(i)}|_b = u_i$.

Lemma 4.2. *Any f in D_n has a unique decomposition $f = agh$ such that $g, h \in D_n$ and g has no proper prefix in D_n . And another decomposition may be written:*

$$f = a g^{(1)} a g^{(2)} \cdots a g^{(b)} b,$$

where $g^{(i)} \in D_n$, for $i = 1 \dots, p$.

This first one is called *first return to the origin decomposition* since ag is the shortest not empty prefix f' of f such that $\delta(f') = 0$. We will call the other one the *full decomposition* of f , p is the number of prefixes g of f which satisfy $\delta(g) = 0$. Moreover each such prefix is obtained from the previous one by concatenating $a g^{(i+1)}$ to it.

5 Configurations and words

Definition 5.1. To any word f in A_n we build the sequence $u = \psi(f)$ of length $n - 1$, such that u_i is the number of occurrences of the letter b of the prefix of f ending with the i -th occurrence of the letter a .

Notice that we have $u_1 \leq u_2 \leq \cdots \leq u_{n-2} \leq u_{n-1} \leq n$

Conversely let u be a reduced configuration of K_n (meaning $0 \leq u_i \leq n$) such that the first $n - 1$ entries appear in increasing order, we associate to u the word $\psi^{-1}(u)$ given by:

$$\psi^{-1}(u) = b^{u_1} a b^{u_2-u_1} a b^{u_3-u_2} a \dots b^{u_{n-1}-u_{n-2}} a b^{n-u_{n-1}}$$

Definition 5.2. Given any configuration u of K_n we denote $\phi(u) = (f, k)$ the pair consisting of a Dyck word f and an integer k which are obtained from the superstable configuration v , $\mathcal{L}(K_n)$ -equivalent to u by:

- $f = \psi(v')$, where v' is obtained by sorting in increasing order the first $n - 1$ -entries of v ,
- $p = v_n$.

We also define the operation ψ' associating to any pair consisting of a word $f \in A_n$ and an integer k the configuration u of K_n given by $u = (\psi(f), k)$.

Notice that this implies that $\psi(\phi(u))$ is a configuration for which there exists a configuration v $\mathcal{L}(K_n)$ -equivalent to a configuration obtained from u by permuting the first $n - 1$ entries.

Notice that if f is a Dyck word then $u = \psi(f, p)$ is superstable, since $\delta(f^{(i)}) \geq 0$ is equivalent to $u_i < i$ for $i < n$.

We now consider the relationship between two configurations obtained from two conjugate words words $f g$ and $g f$.

Lemma 5.1. *Let f, g be two words such that $fg \in A_n$ and let k, k' be two integers, then there exists a configuration w obtained by permuting the entries of $u = \psi'(f g, k)$ which is $\mathcal{L}(K_n)$ -equivalent to $v = \psi'(g f, k')$ if and only if the following holds:*

$$k' = k + |f|_b |g|_a - |f|_a |g|_b = k + n(|f|_b - |f|_a) - |f|_b \quad (3)$$

Proof. We first examine the entries of u and v . Denote $p = |f|_a$ then u may be written

$$u = (u_1, u_2, \dots, u_p, u_{p+1}, \dots, u_{n-1}, k)$$

Since we have taken off the sequence $u'_{p+1}, \dots, u'_{n-1}$ of to put in front of the word we have:

$$v = (u'_{p+1}, \dots, u'_{n-1}, u'_1, u'_2, \dots, u'_p, k')$$

where $u'_j = u_j - |f|_b$ if $j > p$, similarly $u'_j = u_j + |g|_b$ if $j \leq p$.

Now we prove that the condition is necessary by computing the degree of the two configurations and expressing the fact that the equality of degrees is a necessary condition to be equivalent. Notice also that permuting the entries of a configuration does not affect the value of its degree.

The degree of v is given by

$$\deg(v) = \sum_{i=1}^{n-1} u_i - |g|_a |f|_b + p|g|_b + k'$$

and that of u by:

$$\deg(v) = \sum_{i=1}^{n-1} u_i + k$$

Equating these two values gives equation (2). The last part of the equality is obtained by using the following relations: $|g|_b = n - |f|_b$ and $|g|_a = n - 1 - |f|_a$.

We now suppose $k' = k + |f|_b |g|_a - |f|_a |g|_b$ and consider the configuration $u' = (u'_1, u'_2, \dots, u'_p, u'_{p+1}, \dots, u'_{n-1}, k')$ it has the same degree as u and:

$$u'_i - u_i = \begin{cases} |g|_b & \text{if } i > p \\ -|f|_b & \text{if } i \leq p \end{cases}$$

Since $|g|_b + |f|_b = n$ all these values are equal ($\text{mod } n$) showing that $u' \sim_{\mathcal{L}(K_n)} u$. Moreover v is obtained from u' by permuting circularly its elements, proving the Proposition. \square

We have the following propositions which are not difficult to deduce from the Lemma above:

Proposition 5.2. *For two configurations u and u' , $\phi(u) = \phi(u')$ if and only if u' is $\mathcal{L}(K_n)$ -equivalent to a configuration obtained by a permutation of the first $n - 1$ entries of u .*

Proposition 5.3. *Let $u = \psi'(f, k)$ where $f \in A_n$ then $\phi(u) = (g, k')$ where g is the conjugate of f which belongs to D_n , and :*

$$k' = \deg(u) - \deg(\psi(g)). \quad (4)$$

6 Interpretation of the involution $u \longleftrightarrow (\kappa - u)$ on Dyck words

The configuration κ on any graph, was defined in the paper of Baker and Norine paper setting $\kappa_i = d_i - 2$, where d_i is the degree of the vertex x_i . For the complete graph K_n we have $\kappa = (n-3, n-3, \dots, n-3)$. The Baker and Norine theorem of graphs gives a formula allowing to determine the rank of u when one knows that of $\kappa - u$.

It seems interesting to express $\phi(\kappa - u)$ in terms of $\phi(u)$. In fact we get the following:

Proposition 6.1. *Let $\phi(u) = (f, k)$ and $\phi(\kappa - u) = (f', k')$ then f' is the element of D_n which is the unique conjugate belonging to D_n of \tilde{f} , the mirror image of f .*

Proof. The mirror image of the word $f = f_1 f_2 \dots f_{m-1} f_m$, of length m (where the f_i are letters) is the word

$$f_m f_{m-1} \dots f_3 f_2 f_1$$

We remark that $\psi(f) = u$ and $\psi(\tilde{f}) = v$ satisfy $v_i = n - u_i$. Since we have defined $\psi'(f, k)$ for any word in A_n we can calculate $v = \psi'(\tilde{f}, n - u_n)$, it is clearly equal to :

$$(n - u_{n-1}, \dots, n - u_2, n - u_1, n - u_n)$$

Reordering the entries of v we get

$$v' = (n - u_1, \dots, n - u_2, \dots, n - u_{n-1}, n - u_n)$$

Since

$$\kappa - u = (n - 3 - u_1, \dots, n - 3 - u_2, \dots, n - 3 - u_{n-1}, n - 3 - u_n)$$

applying Proposition 2.1 we have the result. □

7 Main fact for computing the rank

In order to determine the rank of a configuration u one may use the exhaustive search method, it consists of determining all compositions λ of the integer k into n parts for $k = 1, 2, 3 \dots$. Then to subtract each of those λ

from u and determine if the configuration $u - \lambda$ obtained is $\mathcal{L}(G)$ -effective. The algorithm stops when one non $\mathcal{L}(G)$ -effective is found, the rank being one less than the value of k at that point. This is not an efficient algorithm. Another technique is to guess a configuration λ which we will call *a proof for the rank of u* , satisfying the following property. First it is effective ($\lambda_i \geq 0$ for all i , and $u - \lambda$ is not $\mathcal{L}(G)$ -effective). Then try to show that any effective configuration μ with degree strictly less than that of λ is such that $u - \mu$ is $\mathcal{L}(G)$ -effective. When such a configuration is obtained not only the rank of u is determined but also all the ranks of the configurations $u - \mu$ where μ is any configuration such that $0 \leq \mu_i \leq \lambda_i$ for all i . Of course such a configuration has rank $\deg(\lambda) - \deg(\mu) - 1$ because of the following:

Proposition 7.1. *If λ is a proof for the rank of u , and μ is an effective configuration such that $\mu_i \leq \lambda_i$ for $i = 1, 2, \dots, n$ then $\lambda - \mu$ is a proof for the rank of $u - \mu$.*

Proof. The condition $\mu_i \leq \lambda_i$ implies that $\lambda - \mu$ is an effective configuration, moreover we have that $(u - \mu) - (\lambda - \mu) = (u - \lambda)$ hence it is not $\mathcal{L}(G)$ -effective. Now let ν be an effective configuration such that $\deg(\nu) < \deg(\lambda - \mu)$ then $\mu + \nu$ is such that $\deg(\mu + \nu) < \deg(\mu) + \deg(\lambda - \mu) = \deg(\lambda)$, hence $u - \mu - \nu$ is $\mathcal{L}(G)$ -effective which writes $(u - \mu) - \nu$ showing the result. \square

The main result allowing to compute the rank of a configuration in K_n , using a polynomial time algorithm is the following:

Theorem 7.2. *Let u be a superstable configuration of K_n such that the first $n - 1$ entries are sorted in increasing order. Then the rank of u and the rank of $u - \varepsilon^{(1)}$ are related by the formula:*

$$\rho(u) = 1 + \rho(u - \varepsilon^{(1)}) \quad (5)$$

Proof. Let u be a configuration satisfying the conditions above and let λ be a proof for $\rho(u)$, we will build a proof for $\rho(u - \varepsilon^{(1)})$. If $\lambda_1 > 0$ then by the remark above $\lambda - \varepsilon^{(1)}$ is a proof for $\rho(u - \varepsilon^{(1)}) = \rho(u) - 1$. If not, since $f - \lambda$ is non $\mathcal{L}(G)$ -effective, there is at least one j such that $\lambda_j > u_j$. We will check that

$$\lambda' = (\lambda_j - u_j, \lambda_2, \dots, \lambda_{j-1}, u_j, \lambda_{j+1}, \lambda_n)$$

is also a proof for $\rho(u)$ satisfying $\lambda'_1 > 0$. Since $\deg(\lambda') = \deg(\lambda)$ we have that for any effective configuration μ with degree less than $\deg(\lambda')$, the

configuration $u - \mu$ is $\mathcal{L}(G)$ -effective. We just have to prove that $u - \lambda'$ is not $\mathcal{L}(G)$ -effective. We consider a configuration obtained from u by subtracting a part of the λ_j to u , that is we let v be:

$$v = u - (\lambda_1, \lambda_2, \lambda_{j-1}, u_j, \lambda_{j+1}, \dots, \lambda_n).$$

We check that this configuration is such that $v_1 = v_j = 0$ and we notice that subtracting the configuration $(\lambda_j - u_j)\varepsilon^{(j)}$ from it we obtain $u - \lambda$ which is not $\mathcal{L}(G)$ -effective. Using the property of symmetry of K_n this implies that subtracting $(\lambda_j - u_j)\varepsilon^{(1)}$ to it we obtain also a non $\mathcal{L}(G)$ -effective configuration. Since $v - (\lambda_j - u_j)\varepsilon^{(1)} = u - \lambda'$ this shows that $u - \lambda'$ is not $\mathcal{L}(G)$ -effective, ending the proof. \square

This result allows to obtain the rank of u by using the following algorithm. This algorithm uses the functions $superstable(u)$ which computes the superstable configuration $\mathcal{L}(K_n)$ -equivalent to a given configuration u and $sort(u)$ which sorts in weakly increasing order the first $n - 1$ entries of the configuration u .

Input: A superstable configuration u on K_n such that $u_n \geq 0$.

Output: The rank of u

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rank ← 0 ;
While ( $u_n \geq 0$ )
    •  $u_1 \leftarrow -1$ 
    •  $u \leftarrow superstable(u)$ 
    •  $u \leftarrow sort(u)$ 
    • if  $u_n < 0$  return rank
    • else  $rank \leftarrow rank + 1$ 

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8 Translating the subtraction $u \rightarrow u - \varepsilon^{(1)}$ in terms of Dyck words

We have seen that a superstable configuration u such that the first $n - 1$ entries are in increasing order may be represented by a pair $\phi(u) = (f, k)$ where f is a Dyck word and $k = u_n$. It is interesting to describe how to

obtain $\phi(u - \varepsilon^{(1)})$ knowing $\phi(u) = (f, k)$. This will lead us to a more efficient algorithm computing the rank. We will also obtain a formula for this rank using some parameters on the Dyck word f . To do that we need, the classical decomposition of a Dyck word:

Proposition 8.1. *Let u be a superstable configuration where the $n - 1$ first entries are in increasing order. Let $\phi(u) = (f, k)$ and let $f = agh$ be the first return to the origin decomposition of f then:*

$$\phi(u - \varepsilon^{(1)}) = (h'abg, k - |a|_a) \quad (6)$$

where h' is given by $h = h'b$.

Proof. Let $u = (u_1, u_2, \dots, u_{n-1}, k)$, where $u_1 = 0$, we have to determine the superstable configuration $\mathcal{L}(K_n)$ -equivalent to

$$u' = u - \varepsilon^{(1)} = (-1, u_2, \dots, u_{n-1}, k)$$

First we subtract $\Delta^{(n)}$ from u' , giving:

$$u' \sim_{\mathcal{L}(K_n)} (0, u_2 + 1, \dots, u_{n-1} + 1, k - (n - 1))$$

This configuration is such that $u' = \psi(abgh', k - (n - 1))$, where $h = h'b$. In order to find a superstable configuration which is a rearrangement of the superstable configuration $\mathcal{L}(K_n)$ -equivalent to u' we must take the conjugate of $abgh'$ which belongs to D_n . This conjugate is: $h'abg$. Moreover by Proposition 5.1, we have to prove that:

$$k - |a|_a = k - n + 1 + n(|a|_b - |a|_a) - |a|_b$$

Since $a|b \in D_n$, we have: $|a|_b - |a|_a = 1$, giving the result. \square

Remark 8.1. The first observation we can make is that we are able to check that, for a Dyck word f , the rank of the configuration $\psi(f, k)$ is 0. This happens if and only if $0 \leq k < |a|_a$. Since the value of $k - |a|_a$ is negative.

Small rank determination

We suppose here that the configuration u with $\phi(u) = (f, k)$ is such that $|a|_a \leq k < n - 1$ then we may apply a few times the operation above in order to determine the rank of u . Indeed we have:

Corollary 8.2. Let, u, f, k be as above, write

$$f = ag^{(1)}ag^{(2)} \dots ag^{(p)}b$$

for the full decomposition of f . Then the values $|ag^{(1)}|, |ag^{(2)}|_a \dots a|g^{(p)}|_a$, satisfy that the rank r is such that:

$$\sum_{i=1}^r |a g^{(i)}|_a \leq k \quad \text{and} \quad \sum_{i=1}^{r+1} |a g^{(i)}|_a > k$$

Proof. Let $g^{(i)} = h^{(i)} b$ then we have $\rho(\psi(f, k)) = 1 + \rho(\psi(f'), k - |a g^{(1)}|_a)$ Where $f' = ag^{(2)} \dots ag^{(p)} a b h^{(1)} b$. Since $k \geq |a g^{(1)}|_a$, we may repeat this operation getting: $\rho(\psi(f, k)) = 2 + \rho(\psi(f''), k - |a g^{(1)}|_a - |a g^{(2)}|_a)$ where $f'' = a g^{(3)} \dots ag^{(p)} a b h^{(1)} a b h^{(2)} b$.

We may repeat this transformation until we get a negative value for $k - \sum_{j=1}^i |a g^{(j)}|_a$.

□

Remark 8.2. The value of r may also be determined using η_i the sequence of heights of f . Indeed, $|a g^{(i)}|_a$ is the length of the sub-sequence of η beginning with the i -th occurrence of 0 and ending just before the $(i+1)$ -th (or ending with η_{n-1} if there is i occurrences of 0 in η). This shows that in order to compute r one may count the number of occurrences of 0 in the sequence $\eta_1, \eta_2, \dots, \eta_{k+1}$ and subtract 1 to this value.

The calculation of the rank in that case becomes easy, it needs only the computation of the sequence of heights.

Input: A Dyck word f and an integer k ; such that $\phi(u) = (f, k)$, where u is a configuration on K_n satisfying $k < n - 1$

Output: The rank of u

Procedure **smallrank**

Compute η the sequence of heights of f

$rank \leftarrow -1$;

for $(i = 1, \dots, k + 1)$ do

• **if** ($\eta_i = 0$) **then** $rank \leftarrow rank + 1$

return $rank$

The determination of the rank in the general case

Applying p times Proposition 8.1, (where p is the number of factors in the full decomposition of f) we obtain a more interesting result:

Proposition 8.3. *Let u be a superstable configuration where the $n - 1$ first entries are in increasing order and let $\phi(u) = (f, k)$. Consider $f = ag^{(1)}ag^{(2)}\dots ag^{(p)}b$ the full decomposition of f if $k \geq n - 1$. Then*

$$\rho(u) = \rho(\psi(f', k - (n - 1)) + p,$$

where $f' = h^{(1)}abh^{(2)}ab\dots h^{(p)}abb$, and the words $h^{(i)}$ are given by $g^{(i)} = h^{(i)}b$.

Proof. Performing p times the previous transformation one has to subtract $|ag_1|_a$ from k then $|ag_2|_a$ from it and so on until $|ag_p|_a$ is subtracted. Notice that in total $n - 1$ that is subtracted from k . Moreover during the procedure we did p times the subtraction of 1 p from the rank and replaced ag , by $h'abg$. \square

This procedure will allow to obtain in a very efficient way the rank of u if we are able to determine at each step the number of words in the full decomposition of the word f . Notice that this word is modified at each step. However this can be obtained by using the sequence of heights of f , since we have:

Remark 8.3. The number of elements in the full decomposition of f is equal to the number of times 0 appears in its sequence of heights. Moreover the sequence of heights η' of $f' = h^{(1)}abh^{(2)}ab\dots h^{(p)}abb$ is obtained from the sequence of heights of f by:

$$\eta'_i = \begin{cases} 0 & \text{if } \eta_i = 0 \\ \eta_i - 1 & \text{if } \eta_i \neq 0 \end{cases}$$

We thus have a procedure which computes the rank by applying as many times as possible the full decomposition of the Dyck word f . This subtracts $n - 1$ from k , adds p (the number of words in the full decomposition of f) to the rank, and transforms f by replacing $ah^{(i)}b$ by $h^{(i)}ab$. When the value of η gets less than $n - 1$, the small rank procedure is used. Notice that it is not necessary to compute the words f' at each step of the algorithm since it suffices to know their height sequences. This gives a procedure which performs the calculus of the rank using the parameters η and k when

$k \geq n - 1$, which is described below. To obtain the rank of the configuration it is necessary to determine the total number of η'_i 's equal to 0 during the execution of the algorithm below. This includes the situation in the **for** loop and in the call of the procedure **smallrank**.

Procedure **allranks**

Input: The sequence η of heights of a Dyck word f and an integer k such that $\phi(u) = (f, k)$, where u is a configuration on K_n

Output: The rank of u

```

rank ← 0 ;
while (k ≥ n - 1) do
    • for (i = 1, 2, ..., n - 1) do
        •   – if  $\eta_i = 0$  rank ← rank + 1
            – else  $\eta_i \leftarrow \eta_i - 1$ 
    • k ← k - (n - 1)
return rank + smallrank( $\eta$ , k)

```

A direct formula

The expression of the algorithm as above allows to give a formula for the rank, which in turn will give an algorithm of linear complexity.

Theorem 8.4. *Let u be a configuration in K_n , $\phi(u) = (f, k)$ and η the sequence of heights of f . Denote q the quotient in the Euclidean division of $k + 1$ by $n - 1$ and r the remainder in this division so that:*

$$k + 1 = (n - 1)q + r, \quad r < n - 1$$

Then

$$\rho(u) + 1 = \sum_{i=1}^r \text{Max}(0, q - \eta_i + 1) + \sum_{i=r+1}^{n-1} \text{Max}(0, q - \eta_i) \quad (7)$$

Proof. To prove 7 we go back to the algorithm **allranks**. We remark that the quotient q of $k + 1$ by $n - 1$ is the number of iterations in the **for** loop which determines successive values of the sequence of heights η . In order to

determine the number of occurrences of 0 in total one can consider the initial values, since they are decremented by 1 each time they are non zero. Hence the number of steps in which their value is equal to 0 is $q - \eta_i$ when $i > r$ and $q - \eta_i + 1$ if $i \leq r$ (since they will be considered in the call of **small rank**). However this value is negative when η_i never reaches the value 0 during the execution of the algorithm, so that it has not to be taken into account. This explains the reason why the function $\text{Max}(0, q - \eta_i)$ is used. \square

Remark 8.4. Formula (7) may also be written:

$$\rho(u) + 1 = \deg(u) - \frac{(n-1)(n-2)}{2} + \sum_{i=1}^r \text{Max}(0, \eta_i - q - 1) + \sum_{i=r+1}^{n-1} \text{Max}(0, \eta_i - q) \quad (8)$$

Proof. Consider the sum

$$s = \sum_{i=1}^r (q - \eta_i + 1) + \sum_{i=r+1}^{n-1} (q - \eta_i)$$

Since $\eta_i = i - 1 - u_i$ we have

$$s = \sum_{i=1}^{n-1} (q + u_i - (i - 1)) + r$$

giving

$$s = q(n-1) + r + \sum_{i=1}^{n-1} u_i - \frac{(n-1)(n-2)}{2}$$

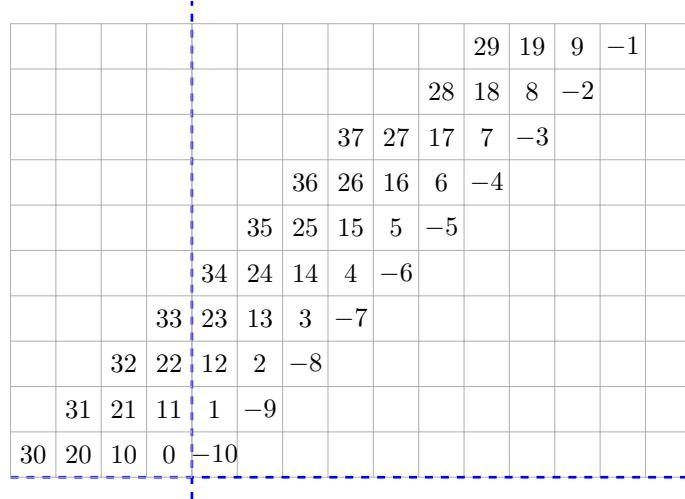
Notice that $q(n-1) + r$ is $u_n + 1$ giving

$$s = 1 + \deg(u) - \frac{(n-1)(n-2)}{2}$$

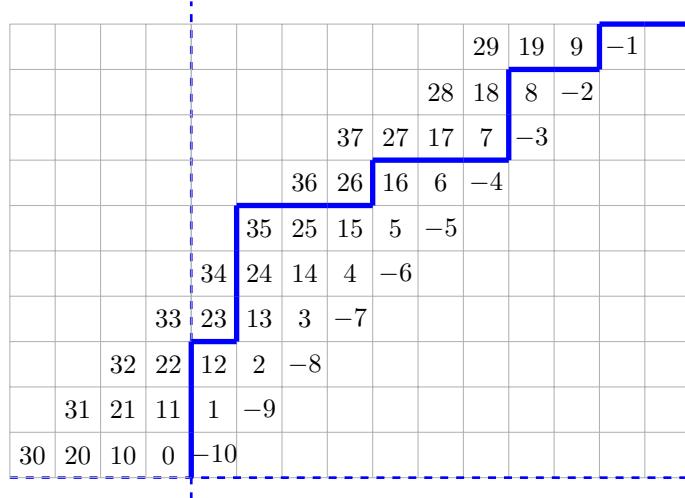
We observe that replacing $\text{Max}(0, \eta_i - q - 1)$ by $\eta_i - q - 1$ has the effect to have subtracted the value $q - \eta_i - 1$, when it is positive. Hence we have to add it to s in order to obtain the rank of u . \square

9 Computing the rank by reading a graphical representation of a Dyck word

We can illustrate the above formula by a representation of the Dyck word in the plane starting from the origin, with two steps: an up step from (i, j) to $(i, j+1)$ representing the letter a and a right step from (i, j) to $(i+1, j)$ representing the letter b . We first consider a strip of squares in the plane of height $n-1$ and we number the squares determined by the lines $x = i$ and $y = j$, a square with south east corner of coordinates (i, j) will be numbered $nj - (n-1)i$.



Then we draw the path corresponding to the Dyck word, for that we use the rule above. Notice that the square numbered 0 is always on the left of the path and all the squares with negative numbers are on the right of the path.



Path representing the Dyck word $a a a b a a a b b b a b b b a a b b a b b$

To determine the rank of u such that $\phi(u) = (f, k)$ one has to count the number p of squares which numbers that are not greater than k and that appear on the left of the path. To obtain the rank one has to subtract 1 from p . For instance when $k = 6$ the rank is 0, and when $k = 27$ the rank is 13.

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