

# Computing the rank of configurations on Complete Graphs

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The paper by M. Baker and S. Norine [1] in 2007 introduced a new parameter in Graph Theory it was called the rank of configurations on graphs. A configuration consists of giving an integer value to each vertex of the graph. It was proved 10 years after that the computation of the rank is an NP-complete problem for general graphs (see [6]). In a paper written with Yvan Le Borgne [3] we presented a linear time algorithm computing the rank of configurations in the complete graph. I give here a brief presentation of the main results of this paper, following the way in which I presented it in VIASM at Hanoi during october 2016.

## 1 Definitions and notation

### Graphs and configurations

Let  $G = (X, E)$  be a connected graph, where  $X = \{x_1, x_2, \dots, x_n\}$  is the vertex set and  $E$  the set of edges. We consider also  $E$  as a symmetric matrix such that  $e_{i,j} = 1$  if there is an edge between the vertices  $x_i$  and  $x_j$ , and  $e_{i,j} = 0$  if not. We assume that  $G$  is connected and has no loops, such that  $e_{i,i} = 0$  for all  $i$ .

We will consider *configurations* on graphs, these are elements of the discrete lattice  $\mathcal{Z}^n$ , where  $\mathcal{Z}$  is the set of integers (positive or negative). For a configuration  $u = (u_1, u_2, \dots, u_n)$  we will refer to the  $u_i$ 's as the *entries* of  $u$ . The symbol  $\varepsilon^{(i)}$  will denote the configuration in which the value 1 is assigned to vertex  $x_i$  and the value 0 is assigned to all other vertices.

The *degree* of the configuration  $u$  is the sum of the  $u_i$ 's it is denoted by  $\deg(u)$ .

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To any vertex is associated a *Laplacian configuration*  $\Delta^{(i)}$ , it is given by:  $\Delta^{(i)} = d_i \varepsilon^{(i)} - \sum_{j=1}^n e_{i,j} \varepsilon^{(j)}$ , where  $d_i = \sum_{j=1}^n e_{i,j}$  is the degree of the vertex  $x_i$ .

### $\mathcal{L}(G)$ -equivalence

We denote by  $\mathcal{L}(G)$  the subgroup of  $\mathcal{Z}^n$  generated by the  $\Delta^{(i)}$ , and two configurations  $u$  and  $v$  will be said *toppling equivalent* if  $u - v \in \mathcal{L}(G)$ , which will also be written as  $u \sim_{\mathcal{L}(G)} v$ .

In the sandpile model and the chip-firing game, the transition from configuration  $u$  to the configuration  $u - \Delta^{(i)}$  is allowed only if  $u_i \geq d_i$  and is called a toppling. Here we omit this condition and perform topplings even if  $u_i < d_i$ .

### Effectiveness

The notion of  $\mathcal{L}(G)$ -effectiveness introduced in [1] is central in this presentation.

**Definition 1.1.** A configuration  $u$  is effective if  $u_i \geq 0$  for all  $i$ . A configuration  $u$  is  $\mathcal{L}(G)$ -effective if there exists an effective configuration  $v$  toppling equivalent to  $u$  (recall that this means  $u - v \in \mathcal{L}(G)$ ).

Since two  $\mathcal{L}(G)$ -equivalent configurations have the same degree, it is clear that a configuration with negative degree is not  $\mathcal{L}(G)$ -effective. From now on it will be convenient to denote effective configurations using greek letters  $\lambda, \mu$  and configurations with no particular assumptions on them by letters  $u, v, w$ .

### Rank

The notion of rank indicates how far is a configuration  $\mathcal{L}(G)$ -effective. It is by subtracting effective configurations to it, and checking if it is still  $\mathcal{L}(G)$ -effective.

**Definition 1.2.** The rank  $\rho(u)$  of a configuration  $u$  is given by:

- if  $u$  is not  $\mathcal{L}(G)$ -effective then  $\rho(u) = -1$ ,
- if  $u$  is  $\mathcal{L}(G)$ -effective, then  $\rho(u)$  is the largest integer  $r$  such that for any effective configuration  $\lambda$  of degree  $r$  the configuration  $u - \lambda$  is  $\mathcal{L}(G)$ -effective.

Denoting  $\mathbb{P}$  the set of effective configurations and  $\mathbb{E}$  the set of  $\mathcal{L}(G)$ -effective configurations this definition can be given by the following compact formula which is valid in both cases:

$$\rho(u) + 1 = \min_{\lambda \in \mathbb{P}, u - \lambda \notin \mathbb{E}} \deg(\lambda)$$

An immediate consequence of this definition is that if two configurations  $u$  and  $v$  satisfy  $u_i \leq v_i$  for all  $i$  then  $\rho(u) \leq \rho(v)$ . Moreover if  $u_i = v - \varepsilon^{(i)}$  then  $\rho(v) - 1 \leq \rho(u) \leq \rho(v)$

The following notion will be useful to prove properties of the rank:

**Definition 1.3.** An effective configuration  $\mu$  is a proof for the rank  $\rho(u)$  of an  $\mathcal{L}(G)$ -effective configuration  $u$  if  $u - \mu$  is not  $\mathcal{L}(G)$ -effective and  $u - \lambda$  is  $\mathcal{L}(G)$ -effective for any effective configuration  $\lambda$  such that  $\deg(\lambda) < \deg(\mu)$ .

Notice that if  $\lambda$  is a proof for  $\rho(u)$  then  $\rho(u) = \deg(\lambda) - 1$ .

## 2 Laplacian configurations in the complete graph

In the complete graph  $K_n$  each vertex has degree  $n - 1$  and for any two vertices there is an edge between them. Hence the Laplacian group is generated by the  $n$  configurations:

$$\Delta^{(i)} = (-1, \dots, -1, n - 1, \dots, -1)$$

For each  $i = 1, 2, \dots, n$  we have:

$$\Delta_j^{(i)} = \begin{cases} n - 1 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$$

We notice that all the entries in these configuration are equal ( $\text{mod } n$ ), hence this is also true for any configuration in the Laplacian subgroup since its elements are linear combinations of the  $\Delta^{(i)}$ . Notice that the converse statement is also true giving the following:

**Proposition 2.1.** A configuration  $u = (u_1, u_2, \dots, u_n)$  in  $K_n$  belongs to the Laplacian subgroup  $\mathcal{L}(K_n)$  of the complete graph  $K_n$  if and only if the two conditions below are satisfied:

$$\deg(u) = 0, \tag{1}$$

$$u_i \equiv u_j \pmod{n} \quad \text{for all } 1 \leq i, j \leq n \tag{2}$$

*Proof.* We only have to prove that if conditions (1) and (2) are satisfied, then  $u \in \mathcal{L}(K_n)$ . We proceed by induction on  $\sum_{i=1}^n |u_i|$ . It is trivial to notice that if this sum is 0 then  $u = 0$  which is in  $\mathcal{L}$ . Suppose that the result is proved when this sum is less than  $p$  and let  $u$  be such a configuration for which the sum is equal to  $p + 1$ . Let  $j$  be an index such that  $|u_j|$  is maximal among the  $|u_i|$ . We may suppose that  $u_i > 0$  since  $u$  and  $-u$  both satisfy the same conditions.

Let  $v = u - \Delta^{(j)}$ , and consider the case when  $u_j \geq n - 1$ . Then we have  $|v_j| = |u_j| - (n - 1)$ , since the degree of  $u$  is 0, there exists at least one  $k$  such that  $u_k < 0$ . This gives  $v_k = u_k + 1$  and  $|v_k| = |u_k| - 1$ , for the  $i \neq j, k$ , the difference between  $|v_i|$  and  $|u_i|$  is at most 1. Hence:

$$\sum_{i=1}^n |v_i| \leq \sum_{i=1}^n |u_i| - (n - 1) - 1 + (n - 2) = \sum_{i=1}^n |u_i| - 2 < p$$

Applying the induction hypothesis we have that  $v \in \mathcal{L}(K_n)$  and  $u = v + \Delta^{(j)} \in \mathcal{L}(K_n)$ .

If  $u_j < n - 1$ , then  $u$  is such that  $k$  among the entries are equal to  $k - n$  and  $n - k$  are equal to  $k$ , giving

$$u = \sum_{i \in I} \Delta^{(i)}$$

where  $I$  is the subset of  $\{1, 2, \dots, n\}$  consisting of those  $i$  such that  $f_i = k$ . Clearly  $u \in \mathcal{L}(K_n)$   $\square$

This Proposition shows that to check a property on a configuration  $u$  which depends only on the class of  $u$  with respect to the Laplacian subgroup, one can consider a configuration  $v$  equivalent to  $u$  and satisfying  $v_i = u_i \pmod{n}$  for  $i \neq n$  and  $v_n = \deg(u) - \sum_{i=1}^{n-1} v_i$ . Then we have  $u \sim_{\mathcal{L}(K_n)} v$  and since  $0 \leq v_i < n$  for  $i \neq n$  it may be more convenient to consider  $v$  instead of  $u$ .

In the sequel we will say that a configuration  $u$  such that  $0 \leq u_i \leq n$  for  $i \neq n$  is *reduced*.

What we have seen gives an algorithm allowing to build a reduced configuration  $\mathcal{L}(K_n)$ -equivalent to a given configuration in  $K_n$ . It is described below, the variable  $\deg$  contains the value to add to  $u_n$  in order that  $u$  and  $v$  have the same degree.

**Input:** A configuration  $u$  on  $K_n$

**Output:** A configuration  $v$   $\mathcal{L}(K_n)$ -equivalent to  $u$  such that

$0 \leq v_i < n$  for any  $i < n$

$deg \leftarrow 0;$

**for**  $(i = 1, \dots, n-1)$  **do**;

- $v_i \leftarrow u_i \pmod{n}$
- $deg \leftarrow deg + u_i - v_i$

$v_n \leftarrow u_n + deg$

### 3 Superstable configurations

An *almost effective* configuration is a configuration  $u$  such that  $u_i \geq 0$  for  $i < n$ . It is easy to see that any configuration is equivalent by  $\mathcal{L}(G)$  to an almost effective configuration. This is very easy to prove for the complete graph (add sufficiently many times  $\Delta^{(n)}$  when  $u_i < 0$  for some  $i < n$ ). Notice that this is also true for any graph.

**Definition 3.1.** A configuration  $u$  is superstable if it is almost effective and if for any non empty  $I \subset \{1, 2, \dots, n-1\}$  the configuration  $u - \sum_{i \in I} \Delta^{(i)}$  is not almost effective.

To check if a configuration  $u$  of the graph  $K_n$  is superstable one we have the following

**Proposition 3.1.** *Let  $u$  be an almost effective configuration in  $K_n$  and let  $v$  be the configuration obtained from  $u$  by sorting in increasing order the elements  $u_1, u_2, \dots, u_{n-1}$ , then  $u$  is superstable if and only if  $v_i < i$  for all  $i < n$ .*

*Proof.* Since the graph  $K_n$  has  $\mathcal{S}_n$  as automorphism group, the property of being superstable is invariant by permuting the entries  $u_i$ . Hence  $v$  is superstable if and only if  $u$  is.

We first prove that if  $v_i \geq i$  for some  $i$ , then  $v$  is not superstable. Indeed consider  $I = \{i, i+1, \dots, n-1\}$  and let  $w = v - \sum_{i \in I} \Delta^{(i)}$ . The values of the entries of  $w$  satisfy:  $w_j = v_j - (n-1) + n - i - 1 = v_j - i$ , and for  $j < i$ ,  $w_j > v_j$ . Since  $v_i \geq i$  and  $v_j \geq v_i$  for  $j \geq i$ ,  $w$  is almost effective. Hence  $v$  is not superstable.

Conversely we show that if  $v_i < i$  for all  $i < n$  then for any  $I \subset \{1, 2, \dots, n-1\}$ ,

$w = v - \sum_{i \in I} \Delta^{(i)}$  is not almost effective. Let  $j$  be the smallest element in  $I$ . We will prove that  $w_j < 0$ . Let  $k$  be the number of elements in  $I$ , since  $I$  is a subset of  $\{j, j+1, \dots, n-2, n-1\}$  a set having  $n-j$  elements we have  $k \leq n-j$ . Let us compute  $w_j$  from  $v_j$ , we have to subtract  $n-1$  and add  $k-1$  giving:

$$w_j = v_j - n + k \leq v_j - n + n - j = v_j - j$$

Since  $v_j < j$  we get the result. □

We use the following result which may be considered as a straightforward consequence of an old result by D. Dhar proving that any class of Laplacian equivalence contains a unique "recurrent" configuration. Different proofs can be found in: [2, 4, 5]. Since one can show that a configuration  $u$  is recurrent if and only if the configuration  $u'$  such that  $u'_i = d_i - u_i$  is superstable (where  $d_i$  is the degree of vertex  $i$ ) we have:

**Theorem 3.2.** *Any configuration is  $\mathcal{L}(G)$ -equivalent to a unique superstable configuration.*

## 4 Words and Dyck words

We consider the alphabet  $A = \{a, b\}$  and the set  $A^*$  of words on  $A$ . A prefix of a word  $f$  is a word  $g$  such that there exists  $h$  such that  $f = gh$ , the prefix  $g$  is strict if  $g \neq f$ . For any word  $f$  the number of occurrences of the letter  $x$  in  $f$  will be denoted by  $|f|_x$  and the length of  $f$  by  $|f|$ , hence  $|f| = |f|_a + |f|_b$ . We denote by  $A_n$  the set of words of length  $2n-1$  which have one more occurrence of  $b$  than that of  $a$ , hence we may write:

$$A_n = \{f \in A^* \mid |f|_a = n-1, |f|_b = n\}$$

The usual notation on words will be used, two words  $f$  and  $f'$  are *conjugate* if there exists  $g, h$  such that  $f = gh$  and  $f' = hg$ . The mirror image of a word  $f = f_1 f_2 \dots f_m$  will be denoted  $\tilde{f}$  it is equal to  $f = f_m f_{m-1} \dots f_2 f_1$ .

In this paper we call Dyck word a word  $f$  satisfying  $|f|_a = |f|_b - 1$  and such that for any strict prefix  $g$  of  $f$  the number of occurrences of  $a$  is not less than the number of occurrences of  $b$ . It is useful to define the function  $\delta$  assigning to any word  $f$  of  $A^*$  the integer  $\delta(f) = |f|_a - |f|_b$  then  $f$  is Dyck

word if  $\delta(f) = -1$  and  $\delta(g) \geq 0$  for any strict prefix of  $f$ . Notice that the Dyck words are very often considered with the last  $b$  omitted. We decided to add an additional  $b$  in order to have a simple statement for the following simple Cyclic Lemma which we will use constantly. The subset consisting of Dyck words of length  $2n - 1$  will be denoted  $D_n$ .

One of the appealing results in combinatorics of words is:

**Lemma 4.1.** *Any word  $f$  in  $A_n$  can be decomposed in a unique way as  $f = gh$  such that  $hg$  is a Dyck word.*

From this lemma one has that the number of elements in  $D_n$  is the Catalan number:

$$\frac{1}{2n-1} \binom{2n-1}{n} = \frac{(2n-2)!}{n!(n-1)!}$$

**Definition 4.1.** The sequence of heights  $\eta = (\eta_1, \eta_2, \dots, \eta_{n-1})$  of  $f \in D_n$  is given by  $\eta_i = \delta(f^{(i)})$ , where  $f^{(i)}$  the prefix of  $f$  preceding the  $i$ -th occurrence of  $a$  in  $f$ .

Notice that for a sorted and superstable configuration  $u$  the heights for the Dyck word  $f$  such that  $\phi(u) = (w, k)$  satisfy  $\eta_i = i - 1 - u_i$ . Indeed for a prefix  $g^{(i)}$  of  $f$  preceding the  $i$ -th occurrence of the letter  $a$ , we have  $|g^{(i)}|_a = i - 1$  and  $|g^{(i)}|_b = u_i$ .

**Lemma 4.2.** *Any  $f$  in  $D_n$  has a unique decomposition  $f = agh$  such that  $g, h \in D_n$  and  $g$  has no proper prefix in  $D_n$ . And another decomposition may be written:*

$$f = ag^{(1)}ag^{(2)} \dots ag^{(p)}b,$$

where  $g^{(i)} \in D_n$ , for  $i = 1 \dots, p$ .

This first one is called *first return to the origin decomposition* since  $ag$  is the shortest not empty prefix  $f'$  of  $f$  such that  $\delta(f') = 0$ . We will call the other one the *full decomposition* of  $f$ ,  $p$  is the number of prefixes  $g$  of  $f$  which satisfy  $\delta(g) = 0$ . Moreover each such prefix is obtained from the previous one by concatenating  $ag^{(i+1)}$  to it.

## 5 Configurations and words

**Definition 5.1.** To any word  $f$  in  $A_n$  we build the sequence  $u = \psi(f)$  of length  $n - 1$ , such that  $u_i$  is the number of occurrences of the letter  $b$  of the prefix of  $f$  ending with the  $i$ -th occurrence of the letter  $a$ .

Notice that we have  $u_1 \leq u_2 \leq \dots \leq u_{n-2} \leq u_{n-1} \leq n$

Conversely let  $u$  be a reduced configuration of  $K_n$  ( meaning  $0 \leq u_i \leq n$ ) such that the first  $n - 1$  entries appear in increasing order, we associate to  $u$  the word  $\psi^{-1}(u)$  given by:

$$\psi^{-1}(u) = b^{u_1} a b^{u_2 - u_1} a b^{u_3 - u_2} a \dots b^{u_{n-1} - u_{n-2}} a b^{n - u_{n-1}}$$

**Definition 5.2.** Given any configuration  $u$  of  $K_n$  we denote  $\phi(u) = (f, k)$  the pair consisting of a Dyck word  $f$  and an integer  $k$  which are obtained from the superstable configuration  $v$ ,  $\mathcal{L}(K_n)$ -equivalent to  $u$  by:

- $f = \psi(v')$ , where  $v'$  is obtained by sorting in increasing order the first  $n - 1$ -entries of  $v$ ,
- $p = v_n$ .

We also define the operation  $\psi'$  associating to any pair consisting of a word  $f \in A_n$  and an integer  $k$  the configuration  $u$  of  $K_n$  given by  $u = (\psi(f), k)$ .

Notice that this implies that  $\psi(\phi(u))$  is a configuration for which there exists a configuration  $v$   $\mathcal{L}(K_n)$ -equivalent to a configuration obtained from  $u$  by permuting the first  $n - 1$  entries.

Notice that if  $f$  is a Dyck word then  $u = \psi(f, p)$  is superstable, since  $\delta(f^{(i)}) \geq 0$  is equivalent to  $u_i < i$  for  $i < n$ .

We now consider the relationship between two configurations obtained from two conjugate words  $f g$  and  $g f$ .

**Lemma 5.1.** *Let  $f, g$  be two words such that  $f g \in A_n$  and let  $k, k'$  be two integers, then there exists a configuration  $w$  obtained by permuting the entries of  $u = \psi'(f g, k)$  which is  $\mathcal{L}(K_n)$ -equivalent to  $v = \psi'(g f, k')$  if and only if the following holds:*

$$k' = k + |f|_b |g|_a - |f|_a |g|_b = k + n(|f|_b - |f|_a) - |f|_b \quad (3)$$

*Proof.* We first examine the entries of  $u$  and  $v$ . Denote  $p = |f|_a$  then  $u$  may be written

$$u = (u_1, u_2, \dots, u_p, u_{p+1}, \dots, u_{n-1}, k)$$

Since we have taken off the sequence  $u'_{p+1}, \dots, u'_{n-1}$  of to put in front of the word we have:

$$v = (u'_{p+1}, \dots, u'_{n-1}, u'_1, u'_2, \dots, u'_p, k')$$

where  $u'_j = u_j - |f|_b$  if  $j > p$ , similarly  $u'_j = u_j + |g|_b$  if  $j \leq p$ .



Now we prove that the condition is necessary by computing the degree of the two configurations and expressing the fact that the equality of degrees is a necessary condition to be equivalent. Notice also that permuting the entries of a configuration does not affect the value of its degree.

The degree of  $v$  is given by

$$\deg(v) = \sum_{i=1}^{n-1} u_i - |g|_a |f|_b + p |g|_b + k'$$

and that of  $u$  by:

$$\deg(v) = \sum_{i=1}^{n-1} u_i + k$$

Equating these two values gives equation (2). The last part of the equality is obtained by using the following relations:  $|g|_b = n - |f|_b$  and  $|g|_a = n - 1 - |f|_a$ .

We now suppose  $k' = k + |f|_b |g|_a - |f|_a |g|_b$  and consider the configuration  $u' = (u'_1, u'_2, \dots, u'_p, u'_{p+1}, \dots, u'_{n-1}, k')$  it has the same degree as  $u$  and:

$$u'_i - u_i = \begin{cases} |g|_b & \text{if } i > p \\ -|f|_b & \text{if } i \leq p \end{cases}$$

Since  $|g|_b + |f|_b = n$  all these values are equal (*mod*  $n$ ) showing that  $u' \sim_{\mathcal{L}(K_n)} u$ . Moreover  $v$  is obtained from  $u'$  by permuting circularly its elements, proving the Proposition.  $\square$

We have the following propositions which are not difficult to deduce from the Lemma above:

**Proposition 5.2.** *For two configurations  $u$  and  $u'$ ,  $\phi(u) = \phi(u')$  if and only if  $u'$  is  $\mathcal{L}(K_n)$ -equivalent to a configuration obtained by a permutation of the first  $n - 1$  entries of  $u$ .*

**Proposition 5.3.** *Let  $u = \psi'(f, k)$  where  $f \in A_n$  then  $\phi(u) = (g, k')$  where  $g$  is the conjugate of  $f$  which belongs to  $D_n$ , and :*

$$k' = \deg(u) - \deg(\psi(g)). \quad (4)$$

## 6 Interpretation of the involution $u \longleftrightarrow (\kappa - u)$ on Dyck words

The configuration  $\kappa$  on any graph, was defined in the paper of Baker and Norine paper setting  $\kappa_i = d_i - 2$ , where  $d_i$  is the degree of the vertex  $x_i$ . For the complete graph  $K_n$  we have  $\kappa = (n-3, n-3, \dots, n-3)$ . The Baker and Norine theorem of graphs gives a formula allowing to determine the rank of  $u$  when one knows that of  $\kappa - u$ .

It seems interesting to express  $\phi(\kappa - u)$  in terms of  $\phi(u)$ . In fact we get the following:

**Proposition 6.1.** *Let  $\phi(u) = (f, k)$  and  $\phi(\kappa - u) = (f', k')$  then  $f'$  is the element of  $D_n$  which is the unique conjugate belonging to  $D_n$  of  $\tilde{f}$ , the mirror image of  $f$ .*

*Proof.* The mirror image of the word  $f = f_1 f_2 \cdots f_{m-1} f_m$ , of length  $m$  (where the  $f_i$  are letters) is the word

$$f_m f_{m-1} \cdots f_3 f_2 f_1$$

We remark that  $\psi(f) = u$  and  $\psi(\tilde{f}) = v$  satisfy  $v_i = n - u_i$ . Since we have defined  $\psi'(f, k)$  for any word in  $A_n$  we can calculate  $v = \psi'(\tilde{f}, n - u_n)$ , it is clearly equal to :

$$(n - u_{n-1}, \dots, n - u_2, n - u_1, n - u_n)$$

Reordering the entries of  $v$  we get

$$v' = (n - u_1, \dots, n - u_2, \dots, n - u_{n-1}, n - u_n)$$

Since

$$\kappa - u = (n - 3 - u_1, \dots, n - 3 - u_2, \dots, n - 3 - u_{n-1}, n - 3 - u_n)$$

applying Proposition 2.1 we have the result. □

## 7 Main fact for computing the rank

In order to determine the rank of a configuration  $u$  one may use the exhaustive search method, it consists of determining all compositions  $\lambda$  of the integer  $k$  into  $n$  parts for  $k = 1, 2, 3, \dots$ . Then to subtract each of those  $\lambda$

from  $u$  and determine if the configuration  $u - \lambda$  obtained is  $\mathcal{L}(G)$ -effective. The algorithm stops when one non  $\mathcal{L}(G)$ -effective is found, the rank being one less than the value of  $k$  at that point. This is not an efficient algorithm. Another technique is to guess a configuration  $\lambda$  which we will call *a proof for the rank of  $u$* , satisfying the following property. First it is effective ( $\lambda_i \geq 0$  for all  $i$ , and  $u - \lambda$  is not  $\mathcal{L}(G)$ -effective). Then try to show that any effective configuration  $\mu$  with degree strictly less than that of  $\lambda$  is such that  $u - \mu$  is  $\mathcal{L}(G)$ -effective. When such a configuration is obtained not only the rank of  $u$  is determined but also all the ranks of the configurations  $u - \mu$  where  $\mu$  is any configuration such that  $0 \leq \mu_i \leq \lambda_i$  for all  $i$ . Of course such a configuration has rank  $\deg(\lambda) - \deg(\mu) - 1$  because of the following:

**Proposition 7.1.** *If  $\lambda$  is a proof for the rank of  $u$ , and  $\mu$  is an effective configuration such that  $\mu_i \leq \lambda_i$  for  $i = 1, 2, \dots, n$  then  $\lambda - \mu$  is a proof for the rank of  $u - \mu$ .*

*Proof.* The condition  $\mu_i \leq \lambda_i$  implies that  $\lambda - \mu$  is an effective configuration, moreover we have that  $(u - \mu) - (\lambda - \mu) = (u - \lambda)$  hence it is not  $\mathcal{L}(G)$ -effective. Now let  $\nu$  be an effective configuration such that  $\deg(\nu) < \deg(\lambda - \mu)$  then  $\mu + \nu$  is such that  $\deg(\mu + \nu) < \deg(\mu) + \deg(\lambda - \mu) = \deg(\lambda)$ , hence  $u - \mu - \nu$  is  $\mathcal{L}(G)$ -effective which writes  $(u - \mu) - \nu$  showing the result.  $\square$

The main result allowing to compute the rank of a configuration in  $K_n$ , using a polynomial time algorithm is the following:

**Theorem 7.2.** *Let  $u$  be a superstable configuration of  $K_n$  such that the first  $n - 1$  entries are sorted in increasing order. Then the rank of  $u$  and the rank of  $u - \varepsilon^{(1)}$  are related by the formula:*

$$\rho(u) = 1 + \rho(u - \varepsilon^{(1)}) \quad (5)$$

*Proof.* Let  $u$  be a configuration satisfying the conditions above and let  $\lambda$  be a proof for  $\rho(u)$ , we will build a proof for  $\rho(u - \varepsilon^{(1)})$ . If  $\lambda_1 > 0$  then by the remark above  $\lambda - \varepsilon^{(1)}$  is a proof for  $\rho(u - \varepsilon^{(1)}) = \rho(u) - 1$ . If not, since  $u - \lambda$  is non  $\mathcal{L}(G)$ -effective, there is at least one  $j$  such that  $\lambda_j > u_j$ . We will check that

$$\lambda' = (\lambda_j - u_j, \lambda_2, \dots, \lambda_{j-1}, u_j, \lambda_{j+1}, \lambda_n)$$

is also a proof for  $\rho(u)$  satisfying  $\lambda'_1 > 0$ . Since  $\deg(\lambda') = \deg(\lambda)$  we have that for any effective configuration  $\mu$  with degree less than  $\deg(\lambda')$ , the

configuration  $u - \mu$  is  $\mathcal{L}(G)$ -effective. We just have to prove that  $u - \lambda'$  is not  $\mathcal{L}(G)$ -effective. We consider a configuration obtained from  $u$  by subtracting a part of the  $\lambda_j$  to  $u$ , that is we let  $v$  be:

$$v = u - (\lambda_1, \lambda_2, \lambda_{j-1}, u_j, \lambda_{j+1} \dots, \lambda_n).$$

We check that this configuration is such that  $v_1 = v_j = 0$  and we notice that subtracting the configuration  $(\lambda_j - u_j)\varepsilon^{(j)}$  from it we obtain  $u - \lambda$  which is not  $\mathcal{L}(G)$ -effective. Using the property of symmetry of  $K_n$  this implies that subtracting  $(\lambda_j - u_j)\varepsilon^{(1)}$  to it we obtain also a non  $\mathcal{L}(G)$ -effective configuration. Since  $v - (\lambda_j - u_j)\varepsilon^{(1)} = u - \lambda'$  this shows that  $u - \lambda'$  is not  $\mathcal{L}(G)$ -effective, ending the proof.  $\square$

This result allows to obtain the rank of  $u$  by using the following algorithm. This algorithm uses the functions *superstable*( $u$ ) which computes the superstable configuration  $\mathcal{L}(K_n)$ -equivalent to a given configuration  $u$  and *sort*( $u$ ) which sorts in weakly increasing order the first  $n - 1$  entries of the configuration  $u$ .

**Input:** A superstable configuration  $u$  on  $K_n$  such that  $u_n \geq 0$ .

**Output:** The rank of  $u$

*rank*  $\leftarrow 0$  ;

**While** ( $u_n \geq 0$ )

- $u_1 \leftarrow -1$
- $u \leftarrow \text{superstable}(u)$
- $u \leftarrow \text{sort}(u)$
- **if**  $u_n < 0$  **return** *rank*
- **else** *rank*  $\leftarrow \text{rank} + 1$

## 8 Translating the subtraction $u \rightarrow u - \varepsilon^{(1)}$ in terms of Dyck words

We have seen that a superstable configuration  $u$  such that the first  $n - 1$  entries are in increasing order may be represented by a pair  $\phi(u) = (f, k)$  where  $f$  is a Dyck word and  $k = u_n$ . It is interesting to describe how to

obtain  $\phi(u - \varepsilon^{(1)})$  knowing  $\phi(u) = (f, k)$ . This will lead us to a more efficient algorithm computing the rank. We will also obtain a formula for this rank using some parameters on the Dyck word  $f$ . To do that we need, the classical decomposition of a Dyck word:

**Proposition 8.1.** *Let  $u$  be a superstable configuration where the  $n - 1$  first entries are in increasing order. Let  $\phi(u) = (f, k)$  and let  $f = agh$  be the first return to the origin decomposition of  $f$  then:*

$$\phi(u - \varepsilon^{(1)}) = (h'abg, k - |ag|_a) \quad (6)$$

where  $h'$  is given by  $h = h'b$ .

*Proof.* Let  $u = (u_1, u_2, \dots, u_{n-1}, k)$ , where  $u_1 = 0$ , we have to determine the superstable configuration  $\mathcal{L}(K_n)$ -equivalent to

$$u' = u - \varepsilon^{(1)} = (-1, u_2, \dots, u_{n-1}, k)$$

First we subtract  $\Delta^{(n)}$  from  $u'$ , giving:

$$u' \sim_{\mathcal{L}(K_n)} (0, u_2 + 1, \dots, u_{n-1} + 1, k - (n - 1))$$

This configuration is such that  $u' = \psi(abgh', k - (n - 1))$ , where  $h = h'b$ . In order to find a superstable configuration which is a rearrangement of the superstable configuration  $\mathcal{L}(K_n)$ -equivalent to  $u'$  we must take the conjugate of  $abgh'$  which belongs to  $D_n$ . This conjugate is:  $h'abg$ . Moreover by Proposition 5.1, we have to prove that:

$$k - |abg|_a = k - n + 1 + n(|abg|_b - |abg|_a) - |abg|_b$$

Since  $abg \in D_n$ , we have:  $|abg|_b - |abg|_a = 1$ , giving the result.  $\square$

**Remark 8.1.** The first observation we can make is that we are able to check that, for a Dyck word  $f$ , the rank of the configuration  $\psi(f, k)$  is 0. This happens if and only if  $0 \leq k < |ag|_a$ . Since the value of  $k - |ag|_a$  is negative.

### Small rank determination

We suppose here that the configuration  $u$  with  $\phi(u) = (f, k)$  is such that  $|ag|_a \leq k < n - 1$  then we may apply a few times the operation above in order to determine the rank of  $u$ . Indeed we have:

**Corollary 8.2.** *Let,  $u, f, k$  be as above, write*

$$f = ag^{(1)}ag^{(2)} \dots ag^{(p)}b$$

*for the full decomposition of  $f$ . Then the values  $|ag^{(1)}|, |ag^{(2)}|_a \dots |g^{(p)}|_a$ , satisfy that the rank  $r$  is such that:*

$$\sum_{i=1}^r |ag^{(i)}|_a \leq k \quad \text{and} \quad \sum_{i=1}^{r+1} |ag^{(i)}|_a > k$$

*Proof.* Let  $g^{(i)} = h^{(i)}b$  then we have  $\rho(\psi(f, k)) = 1 + \rho(\psi(f'), k - |ag^{(1)}|_a)$  Where  $f' = ag^{(2)} \dots ag^{(p)}ab h^{(1)}b$ . Since  $k \geq |ag^{(1)}|_a$ , we may repeat this operation getting:  $\rho(\psi(f, k)) = 2 + \rho(\psi(f''), k - |ag^{(1)}|_a - |ag^{(2)}|_a)$  where  $f'' = ag^{(3)} \dots ag^{(p)}ab h^{(1)}ab h^{(2)}b$ .

We may repeat this transformation until we get a negative value for  $k - \sum_{j=1}^i |ag^{(j)}|_a$ .

□

**Remark 8.2.** The value of  $r$  may also be determined using  $\eta_i$  the sequence of heights of  $f$ . Indeed,  $|ag^{(i)}|_a$  is the length of the sub-sequence of  $\eta$  beginning with the  $i$ -th occurrence of 0 and ending just before the  $(i+1)$ -th (or ending with  $\eta_{n-1}$  if there is  $i$  occurrences of 0 in  $\eta$ ). This shows that in order to compute  $r$  one may count the number of occurrences of 0 in the sequence  $\eta_1, \eta_2, \dots, \eta_{k+1}$  and subtract 1 to this value.

The calculation of the rank in that case becomes easy, it needs only the computation of the sequence of heights.

**Input:** A Dyck word  $f$  and an integer  $k$ ; such that  $\phi(u) = (f, k)$ , where  $u$  is a configuration on  $K_n$  satisfying  $k < n - 1$

**Output:** The rank of  $u$

Procedure **smallrank**

Compute  $\eta$  the sequence of heights of  $f$

$rank \leftarrow -1$  ;

**for**  $(i = 1, \dots, k + 1)$  **do**

    • **if**  $(\eta_i = 0)$  **then**  $rank \leftarrow rank + 1$

**return** rank

## The determination of the rank in the general case

Applying  $p$  times Proposition 8.1, (where  $p$  is the number of factors in the full decomposition of  $f$ ) we obtain a more interesting result:

**Proposition 8.3.** *Let  $u$  be a superstable configuration where the  $n - 1$  first entries are in increasing order and let  $\phi(u) = (f, k)$ . Consider  $f = ag^{(1)}ag^{(2)} \dots ag^{(p)}b$  the full decomposition of  $f$  if  $k \geq n - 1$ . Then*

$$\rho(u) = \rho(\psi(f', k - (n - 1)) + p,$$

where  $f' = h^{(1)}ab h^{(2)}ab \dots h^{(p)}ab b$ , and the words  $h^{(i)}$  are given by  $g^{(i)} = h^{(i)}b$ .

*Proof.* Performing  $p$  times the previous transformation one has to subtract  $|ag_1|_a$  from  $k$  then  $|ag_2|_a$  from it and so on until  $|ag_p|_a$  is subtracted. Notice that in total  $n - 1$  that is subtracted from  $k$ . Moreover during the procedure we did  $p$  times the subtraction of 1  $p$  from the rank and replaced  $ag$ , by  $h'abg$ .  $\square$

This procedure will allow to obtain in a very efficient way the rank of  $u$  if we are able to determine at each step the number of words in the full decomposition of the word  $f$ . Notice that this word is modified at each step. However this can be obtained by using the sequence of heights of  $f$ , since we have:

**Remark 8.3.** The number of elements in the full decomposition of  $f$  is equal to the number of times 0 appears in its sequence of heights. Moreover the sequence of heights  $\eta'$  of  $f' = h^{(1)}ab h^{(2)}ab \dots h^{(p)}ab b$  is obtained from the sequence of heights of  $f$  by:

$$\eta'_i = \begin{cases} 0 & \text{if } \eta_i = 0 \\ \eta_i - 1 & \text{if } \eta_i \neq 0 \end{cases}$$

We thus have a procedure which computes the rank by applying as many times as possible the full decomposition of the Dyck word  $f$ . This subtracts  $n - 1$  from  $k$ , adds  $p$  (the number of words in the full decomposition of  $f$ ) to the rank, and transforms  $f$  by replacing  $ah^{(i)}b$  by  $h^{(i)}ab$ . When the value of  $k$  gets less than  $n - 1$ , the small rank procedure is used. Notice that it is not necessary to compute the words  $f'$  at each step of the algorithm since it suffices to know their height sequences. This gives a procedure which performs the calculus of the rank using the parameters  $\eta$  and  $k$  when

$k \geq n - 1$ , which is described below. To obtain the rank of the configuration it is necessary to determine the total number of  $\eta'_i$ 's equal to 0 during the execution of the algorithm below. This includes the situation in the **for** loop and in the call of the procedure **smallrank**.

#### Procedure **allranks**

**Input:** The sequence  $\eta$  of heights of a Dyck word  $f$  and an integer  $k$  such that  $\phi(u) = (f, k)$ , where  $u$  is a configuration on  $K_n$

**Output:** The rank of  $u$

$rank \leftarrow 0$  ;

**while** ( $k \geq n - 1$ ) **do**

- **for** ( $i = 1, 2, \dots, n - 1$ ) **do**
- **if**  $\eta_i = 0$   $rank \leftarrow rank + 1$
- **else**  $\eta_i \leftarrow \eta_i - 1$
- $k \leftarrow k - (n - 1)$

**return**  $rank + \text{smallrank}(\eta, k)$

#### A direct formula

The expression of the algorithm as above allows to give a formula for the rank, which in turn will give an algorithm of linear complexity.

**Theorem 8.4.** *Let  $u$  be a configuration in  $K_n$ ,  $\phi(u) = (f, k)$  and  $\eta$  the sequence of heights of  $f$ . Denote  $q$  the quotient in the Euclidean division of  $k + 1$  by  $n - 1$  and  $r$  the remainder in this division so that:*

$$k + 1 = (n - 1)q + r, \quad r < n - 1$$

*Then*

$$\rho(u) + 1 = \sum_{i=1}^r \text{Max}(0, q - \eta_i + 1) + \sum_{i=r+1}^{n-1} \text{Max}(0, q - \eta_i) \quad (7)$$

*Proof.* To prove 7 we go back to the algorithm **allranks**. We remark that the quotient  $q$  of  $k + 1$  by  $n - 1$  is the number of iterations in the **for** loop which determines successive values of the sequence of heights  $\eta$ . In order to



determine the number of occurrences of 0 in total one can consider the initial values, since they are decremented by 1 each time they are non zero. Hence the number of steps in which their value is equal to 0 is  $q - \eta_i$  when  $i > r$  and  $q - \eta_i + 1$  if  $i \leq r$  (since they will be considered in the call of **small rank**). However this value is negative when  $\eta_i$  never reaches the value 0 during the execution of the algorithm, so that it has not to be taken into account. This explains the reason why the function  $Max(0, q - \eta_i)$  is used.  $\square$

**Remark 8.4.** Formula (7) may also be written:

$$\rho(u)+1 = \deg(u) - \frac{(n-1)(n-2)}{2} + \sum_{i=1}^r Max(0, \eta_i - q - 1) + \sum_{i=r+1}^{n-1} Max(0, \eta_i - q) \quad (8)$$

*Proof.* Consider the sum

$$s = \sum_{i=1}^r (q - \eta_i + 1) + \sum_{i=r+1}^{n-1} (q - \eta_i)$$

Since  $\eta_i = i - 1 - u_i$  we have

$$s = \sum_{i=1}^{n-1} (q + u_i - (i - 1)) + r$$

giving

$$s = q(n-1) + r + \sum_{i=1}^{n-1} u_i - \frac{(n-1)(n-2)}{2}$$

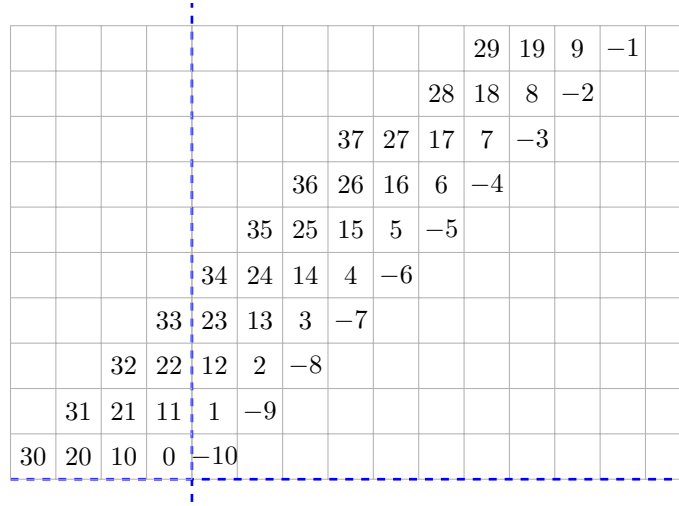
Notice that  $q(n-1) + r$  is  $u_n + 1$  giving

$$s = 1 + \deg(u) - \frac{(n-1)(n-2)}{2}$$

We observe that replacing  $Max(0, \eta_i - q - 1)$  by  $\eta_i - q - 1$  has the effect to have subtracted the value  $q - \eta_i - 1$ , when it is positive. Hence we have to add it to  $s$  in order to obtain the rank of  $u$ .  $\square$

## 9 Computing the rank by reading a graphical representation of a Dyck word

We can illustrate the above formula by a representation of the Dyck word in the plane starting from the origin, with two steps: an up step from  $(i, j)$  to  $(i, j + 1)$  representing the letter  $a$  and a right step from  $(i, j)$  to  $(i + 1, j)$  representing the letter  $b$ . We first consider a strip of squares in the plane of height  $n - 1$  and we number the squares determined by the lines  $x = i$  and  $y = j$ , a square with south east corner of coordinates  $(i, j)$  will be numbered  $nj - (n - 1)i$ .



Then we draw the path corresponding to the Dyck word, for that we use the rule above. Notice that the square numbered 0 is always on the left of the path and all the squares with negative numbers are on the right of the path.



- [6] V. Kiss and L. Tóthmész. Chip-firing games on eulerian digraphs and np-hardness of computing the rank of a divisor on a graph. *Discrete Applied Mathematics* 193: 48—56, 2015.