

Graph structure and Monadic second-order logic : Exercises

Bruno Courcelle
courcell@labri.fr

September 2007

1 About tree-width

1-1) Let T be a binary tree where each node either is a leaf or has a left and a right son. Let \hat{T} be the graph made of T with an edge between each leaf and the next one in the left-right order. What is the tree-width of \hat{T} ? Build an HR-equational system defining the set of all graphs \hat{T} .

1-2) Let us consider T as above with m leaves numbered from 1 to m in the left-right order. The i -th leaf is made into the i -source. What is the tree-width of $fg_{Au}(T//T)$? Build an HR-equational system defining the set of graphs $fg_{Au}(T//T)$ for all trees T . (fg_{Au} forgets all sources).

1-3) Prove that the set of graphs $fg_{Au}(S//T)$ for all trees S and T with same number of leaves has unbounded tree-width.

1-4) The graph $G(f)$ of a partial function $f : V \rightarrow V$ where V is finite has vertex set V and arcs $x \rightarrow f(x)$ for all $x \in V$. What is the maximum tree-width of such graphs? Prove that the set of all graphs $G(f)$ is HR- and also VR-equational.

1-5) For f a bijection $V \rightarrow V$ where V is of the form $\{1, \dots, k\}$, we let $H(f)$ consist of $G(f)$ augmented with edges between i and $i+1$ for every $i = 1, \dots, k-1$. Prove that the set of all graphs $H(f)$ has unbounded tree-width.

2 Monadic second-order logic

2-1) For H a finite simple loop-free undirected graph, show that the property " G has a minor isomorphic to H " is MS expressible without edge set quantifications. Conclude that planarity is MS expressible without edge set quantifications.

2-2) Let C be an MS-definable class of graphs. Let $Apex(C)$ be the class of graphs such that, by deleting one vertex, one obtains a graph in C . Show that this class is MS-definable.

2-3) Prove that the class $Apex(PLANAR)$ is minor-closed and that the corresponding set K of minimal excluded minors is MS-definable. Do we obtain from this fact an algorithm that enumerates the set K which is finite up to isomorphism by the Graph Minor Theorem (that is can one get one concrete graph for each element of K) ?

2-4) The *rank* of an undirected graph is defined as the rank over $GF(2)$ of its adjacency matrix. Show that this number is the common cardinality of the sets X of vertices that satisfy an MS-formula with free variable X and written with the even cardinality set predicate $Even(U)$.

2-5) Assume we augment MS logic with a set predicate $Eq(X, Y)$ expressing that two sets X and Y have the same cardinality. Try to establish the Fefermann-Vaught theorem and explain what does not work.

2-6) How to obtain disjoint union in the Fefermann-Vaught theorem?

2-7) Find classes of graphs on which a linear order of the set of vertices can be defined by monadic second-order formulas. (That is, for each such class, there exists a formula $\varphi(x, y, X, \dots, Z)$ that defines a linear order $x < y$ on vertices, with the help of auxiliary sets X, \dots, Z that have to satisfy another MS formula $\theta(X, \dots, Z)$. For each graph in the class, there must exist X, \dots, Z satisfying $\theta(X, \dots, Z)$.)

There are two cases, depending on whether one uses edge set quantifications.