



Rank-width of countable graphs

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Graph decompositions of finite graphs : useful for FPT algorithms and graph structure (Graph Minor Theorem, perfect graphs). **Width measures** are frequently associated with decompositions : tree-width, path-width, **rank-width**, **linear rank-width**, clique-width.

Graph decompositions and widths of countable graphs : Definitions ?

Compactness results : is $wd(G)$ the least upper bound of $wd(H)$, for all H , finite subgraphs of G ?

Yes for tree-width by Kriz & Thomas.

This talk : rank-width of countable graphs.

Two notions of linear rank-width based on two different linear orders : \mathbf{Q} and \mathbf{Z}

Two notions of rank-width based on two types of trees : quasi-trees (cf. \mathbf{Q}) and trees (cf. \mathbf{Z}).

Different compactness results :

Rank-width (quasi-trees)	Yes
Discrete rank-width (trees)	Yes with gap: $n \rightarrow 2.n$
Linear rank-width (\mathbf{Q})	Yes
Discrete linear rwd (\mathbf{Z})	No

Rank-width : Countable, simple, undirected graphs.

If X and Y are disjoint sets of vertices, $A[X, Y]$ is the **adjacency matrix** of G between vertices in X and in Y .

The **rank** over $GF(2)$ **of a countable matrix** $A[X, Y]$ is the *lub* (least upper bound) of ranks of its finite submatrices.

A **layout** of a graph G is a pair (T, f) :

T is a tree (without root) of degree at most 3,

f : injective mapping : $V(G) \rightarrow \text{Leaves}(T)$

Each edge e of T yields a bipartition (X_e, X_e^c) of $V(G)$
(X_e or X_e^c may be empty).

rk(e) := rank of $A[X_e, X_e^c]$ over $GF(2)$

rk(T, f) := *lub* {**rk**(e) / e , edge of tree T }.

rwd(G) := minimal **rk**(T, f) over all layouts (T, f) .

Property : (1) $\text{rwd}(H) \leq \text{rwd}(G)$ if $H \subseteq_i G$ (induced subgraph)

(2) If, between X and X^c , there are k pairwise disjoint edges, then $\text{rk}(A[X, X^c]) \geq k$.

Examples : Trees have rank-width 1,

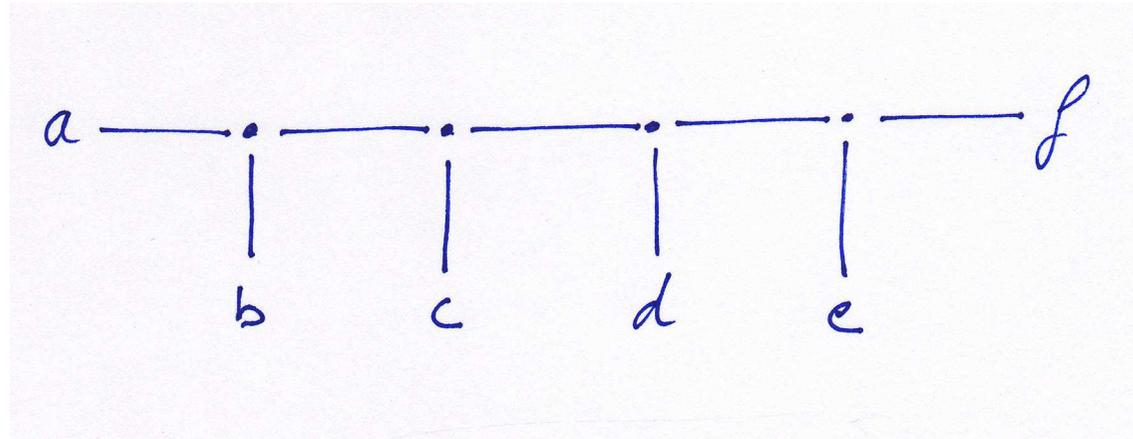
cycles have rank-width 2;

$n \times n$ square grids have rank-width $n-1$

(Jelinek, difficult proof).

Linear rank-width

Defined from layouts of the special form



Equivalent to $a < b < c < d < e < f$: a linear order on vertices. Edge bipartitions (X_e, X_e^c) are replaced by

Dedekind cuts : bipartitions (X, X^c) such that $X < X^c$

These definitions **work for countable graphs.**

Terminology : **discrete** rank-width, **drwd**(G); the good notion of “rank-width” will be different (for countable graphs).

Theorem : Discrete rank-width has compactness, but “only” with gap:

$$\mathbf{drwd}(G) \leq 2 \cdot \text{Sup}\{\mathbf{rwd}(H) \mid H \subseteq_i G, H \text{ finite}\}$$

Proof: Compactness with gap : will come later.

No compactness. Counter-example :

G has vertex set Q partitioned into two dense subsets A and B .

If $x < y$, then $x \text{ -- } y$ (edge) if and only if $y \in A$.

Every finite induced subgraph of G is a *threshold graph*, hence of rank-width 1. But $\text{drwd}(G) \geq 2$ (actually $\text{drwd}(G) = 2$ by the first fact).

Wanted: a definition of rank-width giving *exact compactness*.

Linear rank-width as warming up.

Same as rank-width with layout = linear order on $V(G)$.
Bipartitions are defined from Dedekind cuts (X, X^c) (such that $X < X^c$).

There are 3 types of layout :

N, **Z**, suborder of **Q** (arbitrary linear order).

Hence 3 notions of linear rank-width with:

$$\text{Lrwd}_{\mathbf{Q}}(G) \leq \text{Lrwd}_{\mathbf{Z}}(G) \leq \text{Lrwd}_{\mathbf{N}}(G) \leq 2 \cdot \text{Lrwd}_{\mathbf{Z}}(G)$$

$Lrwd_N(G)$ is called *discrete linear rank-width*, denoted by $dLrwd(G)$.

$Lrwd(G)$ called *linear rank-width* is based on arbitrary linear orders.

Theorem : (1) $Lrwd$ has the compactness property.

(2) $dLrwd$ has not, even with a gap function.

Proofs : (1) Using Koenig's lemma.

(2) Counter-example :

P = the infinite path isomorphic to \mathbf{N}

G = the union of ω disjoint copies of P

$\text{Lrwd}(G) = 1$ for order $\mathbf{N} + \mathbf{N} + \mathbf{N} + \dots$

$\text{dLrwd}(G) = \omega$.

Rank-width based on quasi-trees.

In order to apply Koenig's lemma, we need a notion of tree closed under countably many insertions of nodes on a path, hence has least upper bounds for topological ordering.

We will define **quasi-trees** whose "paths" between two nodes can have countably many nodes.

Betweenness : If T is a tree, let $B_T(x,y,z)$ mean that y is on the path that links x and z . The following properties hold:

$$A1 : B(x, y, z) \Rightarrow x \neq y \neq z \neq x.$$

$$A2 : B(x, y, z) \Rightarrow B(z, y, x).$$

$$A3 : B(x, y, z) \Rightarrow \neg B(x, z, y).$$

$$A4 : B(x, y, z) \wedge B(y, z, u) \Rightarrow B(x, y, u) \wedge B(x, z, u).$$

$$A5 : B(x, y, z) \wedge B(x, u, y) \Rightarrow B(x, u, z) \wedge B(u, y, z).$$

$$A6 : B(x, y, z) \wedge B(x, u, z) \Rightarrow \\ y = u \vee (B(x, u, y) \wedge B(u, y, z)) \vee (B(x, y, u) \wedge B(y, u, z)).$$

$$A7 : x \neq y \neq z \neq x \Rightarrow \\ B(x, y, z) \vee B(x, z, y) \vee B(y, x, z) \vee (\exists u. B(x, u, y) \wedge B(y, u, z) \wedge B(x, u, z)).$$

Definition: A *quasi-tree* is a pair (N, B) where N is a set (its *nodes*) and B a ternary relation that satisfies Properties A1-A7. It is *discrete* if for each x, z , the set of nodes y *between* x *and* z (i.e., such that $B(x, y, z)$ holds) is finite. Then $B = B_T$ for some tree T .

An increasing sequence of finite quasi-trees has a *lub* that is a quasi-tree.

A *cut* is a partition (X, X^c) such that X and X^c are convex; X is *convex* if $B(x, y, z) \wedge x \in X \wedge z \in X \Rightarrow y \in X$.

Degree. Nodes y and z are in the *same direction* relative to node x if :

$$y = z \vee B(y,z,x) \vee B(z,y,x) \vee$$

$$\exists u (B(y,u,x) \wedge B(z,u,x) \wedge B(y,u,z)).$$

This is an equivalence relation. Its classes are the *directions relative* to x .

The *degree of* x is the number of directions relative to it. A *leaf* has degree 1 (it is not between two nodes).

A *layout of* G is a quasi-tree $S = (N, B)$ whose nodes have degree ≤ 3 and such that $V(G) \subseteq \text{Leaves}(S)$.

If (X, X^c) is a cut of S :

$$\text{rk}(X, X^c) := \text{rk}(A[X \cap V(G), X^c \cap V(G)])$$

$$\text{rk}(S) := \text{Sup}\{ \text{rk}(X, X^c) / (X, X^c) \text{ is a cut } \}$$

$$\text{rwd}(G) := \text{Min}\{ \text{rk}(S) / S \text{ is a layout } \}$$

$$\text{drwd}(G) := \text{Min}\{ \text{rk}(S) / S \text{ is a discrete layout } \}$$

$$= \text{Min}\{ \text{rk}(T, f) / (T, f) \text{ is a layout, } T \text{ tree } \}$$

Theorem : (1) **rwd** has the compactness property.

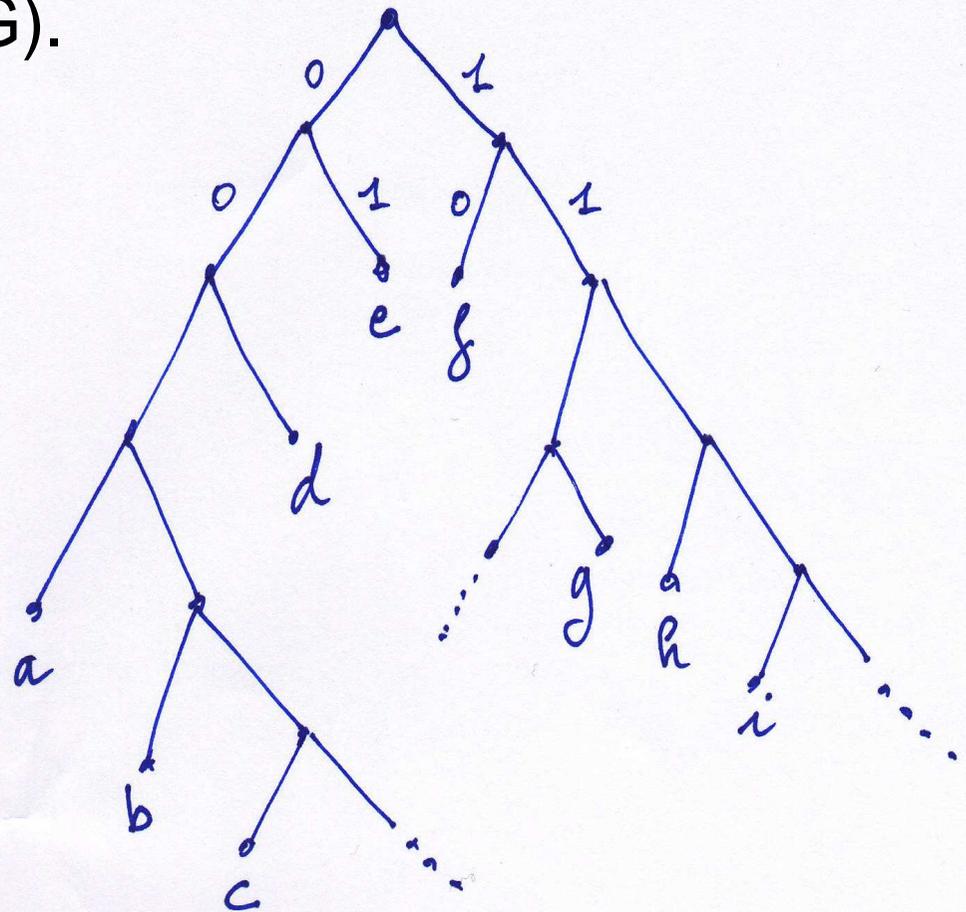
$$(2) \text{ drwd}(G) \leq 2 \cdot \text{rwd}(G).$$

Proof : (1) With Koenig's lemma, as for **Lrwd**.

$$(2) \text{ drwd}(G) \leq 2 \cdot \text{rwd}(G).$$

First: $\text{drwd}(G) \leq 2 \cdot \text{Lrwd}(G)$.

A countable linear order is *described* by a rooted binary tree.



$abc \dots def \dots ghi \dots$

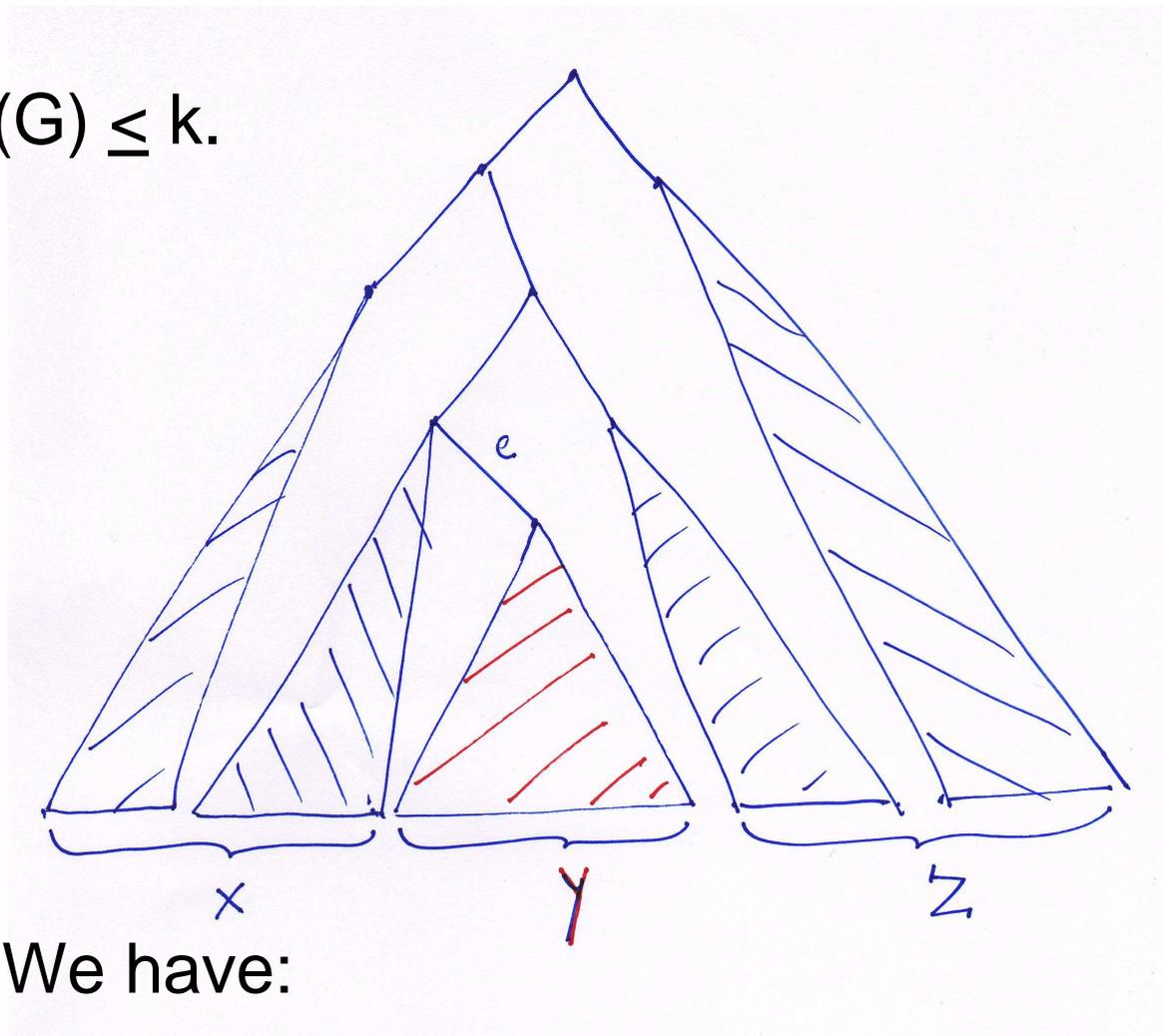
Let \leq witness that $\text{Lrwd}(G) \leq k$.

We describe \leq
by a binary tree.

We get as layout of G
a tree of degree ≤ 3 .

An edge e yields a
bipartition of $V(G)$

of the form $(X \cup Z, Y)$. We have:



$$\text{rk}(A[X \cup Z, Y]) \leq \text{rk}(A[X, Y]) + \text{rk}(A[Z, Y])$$

$$\leq \text{rk}(A[X, Y \cup Z]) + \text{rk}(A[X \cup Z, Y]) \leq 2k.$$

For proving that $\text{drwd}(G) \leq 2 \cdot \text{rwd}(G)$, we use essentially this idea: we encode a layout S by a binary rooted tree.

Structuring of a quasi tree S .

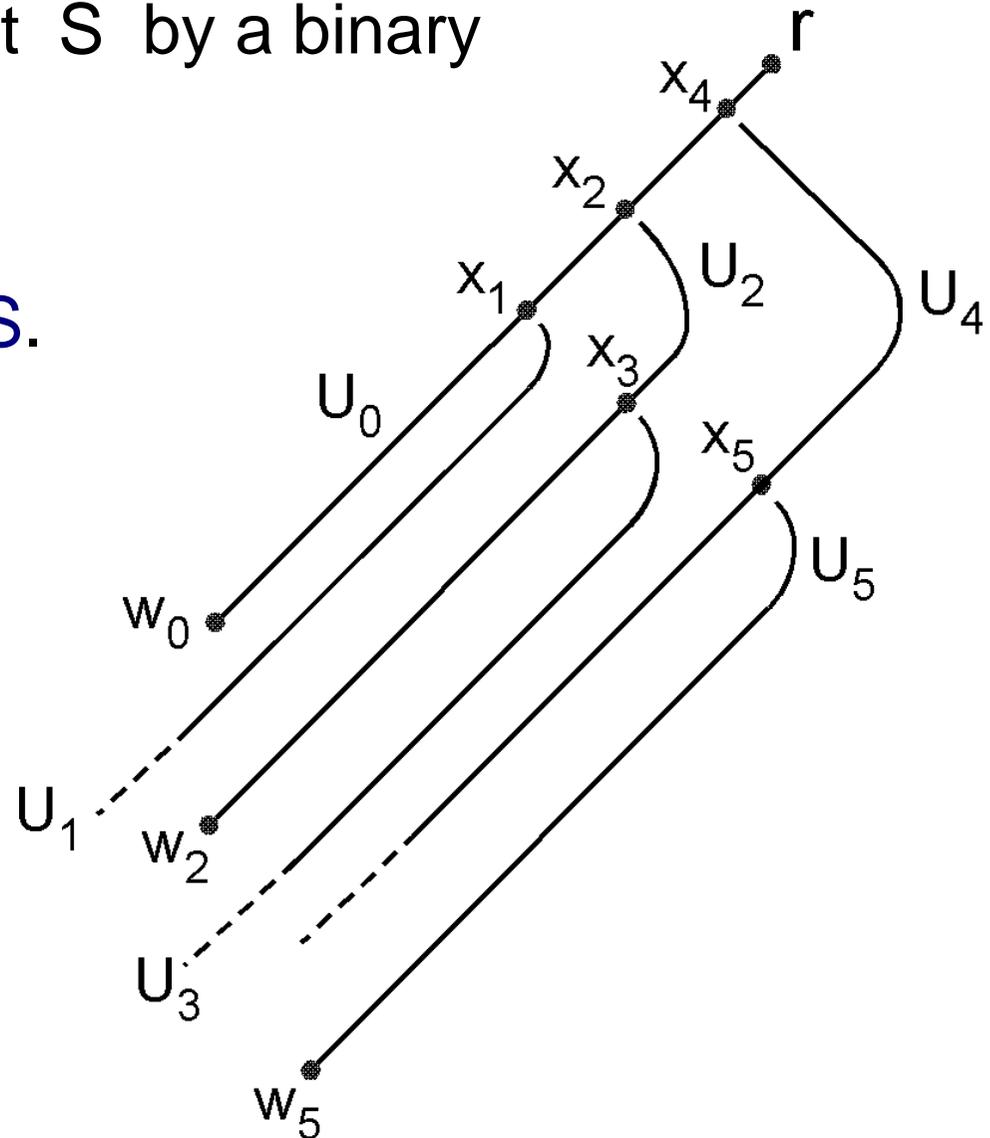
Choose a leaf r .

Maximal *lines* N_0, N_1, \dots containing r .

$U_0 := N_0,$

$U_1 := N_1 - U_0,$

$U_2 := N_2 - (U_0 \cup U_1),$ etc.



Describe each line U_i by a binary tree T_i

Connect trees T_0, T_1, T_2, \dots into a single tree.

This tree is a layout of G of rank $\leq 2k$ where $k = \text{rk}(S)$.

