

On the closure of graphs under substitution

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Abstract

In this paper we investigate the closure \mathcal{F}^* under substitution-composition of a family of graphs \mathcal{F} , defined by a set \mathcal{L} of forbidden configurations. We first prove that \mathcal{F}^* can be defined by a set \mathcal{L}^* of forbidden subgraphs. Next, using a counterexample we show that \mathcal{L}^* is not necessarily a finite set, even when \mathcal{L} is finite. We then give a sufficient condition for \mathcal{L}^* to be finite and a simple algorithm for enumerating all the graphs of \mathcal{L}^* . As application, we obtain new classes of perfect graphs.

Keywords: Module; Substitution-composition; Closure; Perfect graphs

1. Motivation

The *substitution-composition* graph G of two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is obtained by first removing a vertex v from G_2 and then making every vertex in G_1 adjacent to all the neighbours of v in G_2 .

The wide utilization of the substitution-composition of graphs into theoretical and practical problems is certainly due to the fact that this operation preserves many of the properties of the composed graphs. Lovász, in [14], highlighted its importance: for establishing his fundamental result asserting that a graph is perfect if and only if its complement is perfect, he relied on the fact that substitution-composition preserves perfectness.

Let \mathcal{F} be a family of graphs defined by a set \mathcal{L} of forbidden subgraphs and let \mathcal{F}^* be the closure by substitution-composition of \mathcal{F} . Two natural questions concerning \mathcal{F}^* arise:

- (i) Is it possible to define \mathcal{F}^* by a set \mathcal{L}^* of forbidden subgraphs?
- (ii) If \mathcal{L}^* exists, is \mathcal{L}^* a finite set?

In this paper, we first show that \mathcal{L}^* exists but its cardinality is not necessarily finite. We then give a sufficient condition for establishing the finiteness of \mathcal{L}^* and we propose

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a simple algorithm for enumerating all the graphs of \mathcal{L}^* . As application, we obtain new classes of perfect graphs.

2. Terminology

For terms not defined in this paper the reader is referred to [5]. All graphs considered here are finite, without loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$ or V and the set of its edges by $E(G)$ or E , with cardinalities $|V(G)| = n$ and $|E(G)| = m$. For $X \subseteq V(G)$, $G[X]$ will denote the subgraph of G induced by X . $G[X]$ will be a *proper* induced subgraph of G , if X is strictly contained into $V(G)$.

The neighbourhood of a vertex v in G is $N(v) = \{w \mid vw \in E(G)\}$, while $N(X)$ for $X \subseteq V(G)$ is the set of vertices outside X which are adjacent to at least one vertex of X . A vertex x *distinguishes* the vertices of $X \subseteq V(G)$ if $x \notin X$ and x is adjacent to some, but not all the vertices of X .

Let Q be an induced subgraph of G , then we denote by $N_Q(X)$ the neighbourhood of X in Q , namely $N_Q(X) = N(X) \cap V(Q)$.

We shall say that G_2 contains G_1 , if G_1 is an induced subgraph of G_2 . Two graphs G_1 and G_2 will be called *incomparable* if none of them is included into the other.

A chordless path of k vertices will be denoted by P_k and a chordless cycle of k vertices will be denoted by C_k . A C_k cycle with $k \geq 5$ is called a *hole* while its complement is called an *antihole*.

A graph G will be called \mathcal{L} -free, where \mathcal{L} is a set of graphs, if no induced subgraph of G is isomorphic to a graph of \mathcal{L} . A set of graphs \mathcal{F} will be \mathcal{L} -free if every graph of \mathcal{F} is \mathcal{L} -free.

2.1. Modular decomposition of a graph G

A subset M of vertices of a graph G is said to be a *module* of G if every vertex outside M is either adjacent to all vertices of M or to none of them. Obviously, M is a module in G iff M is a module in \bar{G} . The empty set, the singletons and $V(G)$ are the *trivial* modules of G , and whenever G has only trivial modules, G will be called *prime* or *indecomposable*. Let G be a prime graph; if $n > 2$ then $n \geq 4$ and G and \bar{G} are connected. A nontrivial module M (i.e. $2 \leq |M| < n$) is also called a *homogeneous set*.

Lemma 2.1. *Let M be a nontrivial module of graph G and W be a prime induced subgraph of G , then either $V(W)$ is included into M or $|V(W) \cap M| \leq 1$.*

Proof. Otherwise W would contain $V(W) \cap M$ as a nontrivial module, a contradiction. \square

An induced prime subgraph G' of a graph G will be called a *maximal prime subgraph* of G , if G' is not strictly contained into any prime proper subgraph of G . In other words, any induced proper subgraph of G strictly containing G' is a decomposable graph.

Whenever a graph G has a nontrivial module M , in order to get some of its structural properties, it is useful to decompose G into two subgraphs $G[M]$ and G_v where G_v is defined as follows: $V(G_v) = V(G) - M \cup \{v\}$, where v is a new vertex called a *marker*. By definition the neighbourhood of v in G_v is the neighbourhood of M in G .

Hence, $E(G_v) = E(G \setminus M) \cup \{vy \mid y \in N(M)\}$. In other words, G_v is the graph obtained from G by contracting the module M to the marker v .

Obviously, G_v is isomorphic to an induced subgraph of G .

If $G[M]$ and/or G_v are not prime graphs, by applying recursively this process to $G[M]$ and/or to G_v , we can clearly associate with G a binary tree $\mathcal{A}(G)$ whose nodes correspond to the graphs obtained during this decomposition process. More precisely, let $G(f)$ be the graph corresponding to the node f of $\mathcal{A}(G)$, then if f is a leaf, $G(f)$ is isomorphic to a prime subgraph of G , otherwise $G(f)$ is isomorphic to a decomposable subgraph of G .

Notation. We shall denote henceforth by $G(f)$ the graph corresponding to the node f of a *binary modular decomposition tree* $\mathcal{A}(G)$.

Clearly, since there is no restriction on the method for choosing the modules of G , $\mathcal{A}(G)$ is not necessarily unique.

The *substitution-composition* graph G of two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ arises naturally as the inverse operation of the binary modular decomposition of G : G is obtained by first removing a vertex v from G_2 and then making every vertex in G_1 adjacent to all the neighbours of v in G_2 . We shall call also this operation the \mathcal{X} -join composition of G_1 and G_2 , and we shall note it henceforth by $\mathcal{X}(G_1, G_2; v)$.

The decomposition of a graph according to its modules has various names in the literature: substitution-decomposition [16], ordinal sum [12] and X -join [20] and it has been discovered independently by researchers in many different areas (see [16,17] for a summary of different applications). The *modular decomposition* is a form of decomposition of a graph G that associates with G a unique modular decomposition tree whose leaves are the vertices of G . The efficient construction of the modular decomposition tree has been extensively studied and two linear algorithms (on the number of edges of G) are proposed for it in [8,15].

The binary modular decomposition tree $\mathcal{A}(G)$ of a graph G will be our framework for here. This form of decomposition of graphs is a special case of the *split decomposition* of graphs. Cunningham, in [9,10] established many unique decomposition theorems for both directed and undirected variants of the split decomposition of a graph. In Theorem 2.1 below, we show that from $\mathcal{A}(G)$ we can associate with G a unique, up to isomorphism, set of prime graphs. This theorem is deduced from a result presented in [10]. We need for this the following notation:

Notation. $\pi(G)$ is the set of all graphs $G(l)$, where l is a leaf of $\mathcal{A}(G)$.

Theorem 2.1. $\pi(G)$ is unique up to isomorphism and each graph of $\pi(G)$ is isomorphic to a maximal prime subgraph of G .

Proof. Indeed, let us associate with G a binary decomposition tree $\mathcal{A}'(G)$, constructed in using the same process as for $\mathcal{A}(G)$, except that no complete or edgeless graph will be decomposed. In other words, a graph corresponding to a leaf of $\mathcal{A}'(G)$ is either prime or complete or edgeless. Let $\pi'(G)$ be the set of graphs corresponding to the set of leafs of $\mathcal{A}'(G)$, then Theorem 5 in [10] asserts that $\pi'(G)$ is unique up to isomorphism. But, the binary modular decomposition of a complete or edgeless graph is unique up to isomorphism and consequently $\pi(G)$ will be unique up to isomorphism.

Consider now a maximal prime subgraph W of G , then by Lemma 2.1, when G is decomposed into $G[M]$ and G_v , there will be a prime graph W' isomorphic to W in one of $G[M]$ or G_v . In using the fact that W is a maximal prime subgraph of any subgraph of G containing W , it is easy to see that W' will be a maximal prime subgraph of $G[M]$ or G_v . The result follows by observing that any graph G_i contains a maximal prime subgraph W_i (a vertex or an edge can be a maximal prime subgraph) and thus G_i will not be decomposable if and only if G_i is exactly W_i . \square

3. The closure of a family of graphs \mathcal{F} under substitution-composition

Let \mathcal{F} be a family of graphs defined by a set \mathcal{L} of forbidden subgraphs and let \mathcal{F}^* be the closure of \mathcal{F} under substitution-composition. The aim of this section is to show that \mathcal{F}^* can be defined by a set \mathcal{L}^* of forbidden subgraphs.

Theorem 3.1. Let W be a prime graph, then the graphs G_1 and G_2 are W -free if and only if $G = \mathcal{X}(G_1, G_2; v)$ is W -free.

Proof. The if part is obvious, since G_1 and G_2 are isomorphic to an induced subgraph of G . Assume now by contradiction that there is an induced subgraph W' of G that is isomorphic to W , while G_1 and G_2 are W -free graphs. Then, since W' is not entirely contained into G_1 or into $G_2 - \{v\}$, there must be a proper subset H of the vertices of W' belonging to G_1 . Since $V(G_1)$ is a nontrivial module of G and W' a prime graph, by Lemma 2.1 H must be a single vertex. Hence, since the neighbourhood of v in G_2 is the neighbourhood of H in G_2 , W' would be isomorphic to an induced subgraph of G_2 , a contradiction. \square

Theorem 3.2. Let W be a prime graph, then the graph G is W -free if and only if every graph of $\pi(G)$ is W -free.

Proof. The only if part is obvious since every graph of $\pi(G)$ is isomorphic to an induced subgraph of G . For the if part consider a binary modular decomposition tree $\mathcal{A}(G)$ associated with G , then G can be obtained by applying substitution-composition to the graphs corresponding to the nodes of $\mathcal{A}(G)$ following a post-order traversal of this tree. The result follows by applying Theorem 3.1. \square

We give now a necessary and sufficient condition for having $\mathcal{F} = \mathcal{F}^*$.

Theorem 3.3. $\mathcal{F} = \mathcal{F}^*$ if and only if every graph of \mathcal{L} is prime.

Proof. First, assume by contradiction that $\mathcal{F} = \mathcal{F}^*$ while there exists a graph W in \mathcal{L} having a homogeneous set H . Since the graphs of \mathcal{L} are mutually incomparable with respect to graph-inclusion, the graph $G[H]$ induced by H clearly belongs to \mathcal{F} . Consider the graph W' induced by $V(W) \cup \{v\} - H$, where v is a new vertex whose neighbourhood in W' is the neighbourhood of H in W . Then W' also belongs to \mathcal{F} since this graph is isomorphic to a proper induced subgraph of W . However, the graph $\mathcal{X} = (G[H], W'; v)$ that clearly belongs to \mathcal{F}^* , does not belong to \mathcal{F} since this graph is isomorphic to the forbidden graph W , a contradiction.

Assume now that every graph of \mathcal{L} is a prime graph, while $\mathcal{F} \neq \mathcal{F}^*$. By the definition of \mathcal{F}^* , \mathcal{F} is strictly contained into \mathcal{F}^* . Let G be a graph of $\mathcal{F}^* - \mathcal{F}$, then there must be an induced subgraph W of G isomorphic to a graph of \mathcal{L} . Consider the set of prime graphs $\pi(G)$ associated with G . Since no prime graph can be obtained by substitution-composition, by the definition of \mathcal{F}^* , every graph of $\pi(G)$ belongs to \mathcal{F} . Hence, $\pi(G)$ is \mathcal{L} -free and consequently, by Theorem 3.2, G is also \mathcal{L} -free, a contradiction. \square

The above theorem can also be stated as follows:

Theorem 3.4. \mathcal{F} is strictly contained into \mathcal{F}^* if and only if there exists a graph $W \in \mathcal{L}$ that is decomposable.

Definition. Let W be a decomposable graph, then a prime graph Q will be called W -minimal, if Q contains a graph isomorphic to W and Q is minimal with respect to this property and graph-inclusion, namely, there is no prime induced proper subgraph Q' of Q containing a graph isomorphic to W .

Remark. Two W -minimal graphs G_1 and G_2 are incomparable.

Notation. Let \mathcal{L}_1 be the set of prime graphs of \mathcal{L} and \mathcal{L}_2 its set of decomposable ones. Let also \mathcal{L}_2^* be the set of prime graphs: $\mathcal{L}_2^* = \{Z\text{-minimal} \mid Z \in \mathcal{L}_2\}$.

Finally, let \mathcal{L}^* be the set of graphs obtained as union of \mathcal{L}_1 with the set of all \mathcal{L}_1 -free graphs in \mathcal{L}_2^* .

Observation. Whenever \mathcal{L}_2 exists, \mathcal{L}_2^* also exists since we can always construct a prime graph Z' containing as induced subgraph a decomposable graph Z . Indeed, if Z

is connected, Z' can be obtained by adding a new ‘private’ neighbour to each vertex of Z . Then, either Z' is Z -minimal or there is a prime proper subgraph of Z' which is Z -minimal. If Z is not connected, we construct as previously a prime graph \bar{Z}' from \bar{Z} . Obviously, since the complementary graph of a prime graph is also prime, Z' is a prime graph containing Z .

Theorem 3.5 below shows that there exists a definition of \mathcal{F}^* in terms of forbidden configurations.

Theorem 3.5. $G \in \mathcal{F}^*$ if and only if G is \mathcal{L}^* -free.

Proof. First, assume by contradiction that a graph G of \mathcal{F}^* contains an induced subgraph W isomorphic to a graph of \mathcal{L}^* . Then since W is prime, by Theorem 3.2, there must be a graph G' of $\pi(G)$ containing W as induced subgraph. Observe that since no prime graph is obtained by substitution-composition, every graph of $\pi(G)$ belongs to \mathcal{F} . Consequently, $\pi(G)$ is \mathcal{L} -free and thus \mathcal{L}^* -free, a contradiction.

Assume now that a \mathcal{L}^* -free graph G does not belong to \mathcal{F}^* , then there must be a graph G' in $\pi(G)$ such that $G' \notin \mathcal{F}$. Hence, there must be an induced subgraph W of G' isomorphic to a graph of \mathcal{L} . This graph must be decomposable since G is assumed to be \mathcal{L}^* -free and consequently \mathcal{L}_1 -free. But, since G' is a prime graph, it contains an induced subgraph G'' that is W -minimal. Thus G'' will be isomorphic to a graph of \mathcal{L}_2^* , a contradiction. \square

Let $\overline{\mathcal{F}^*}$ and $\overline{\mathcal{L}^*}$ be the sets of the complementary graphs of \mathcal{F}^* and \mathcal{L}^* , respectively. Since a graph Z is included into a graph G if and only if \bar{Z} is included into \bar{G} , we can easily obtain the following result:

Theorem 3.6. $G \in \overline{\mathcal{F}^*}$ if and only if G is $\overline{\mathcal{L}^*}$ -free.

4. A sufficient condition for \mathcal{L}^* to be finite

Assume now that \mathcal{L} is a finite set, then it is natural to ask if the set \mathcal{L}^* is also a finite set. We shall first show in Theorem 4.1 below that this is not necessarily the case. Then, in Theorem 4.4 which is the main theorem of this paper, we shall prove that if every homogeneous set of any graph of \mathcal{L} has two vertices, then \mathcal{L}^* is finite.

Suppose that \mathcal{L} contains only the graph Z depicted in the Fig. 1 below. This graph is constructed by joining a hole C with the chordless path $abcd$, in such a way that the vertex a is adjacent to every vertex of C while there is no edge in Z between $\{b, c, d\}$ and $V(C)$. Thus, Z contains only one nontrivial module, the module formed by the set of vertices of C .

Let \mathcal{L}^* be the set of Z -minimal graphs.

Theorem 4.1. \mathcal{L}^* is not a finite set.

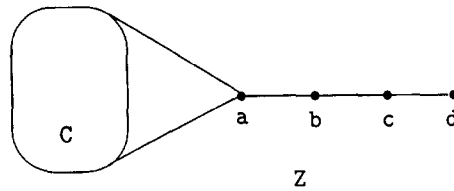


Fig. 1.

Proof. Let Z' be the graph obtained by adding a new vertex w to the graph Z , such that the neighbourhood of w in Z' is exactly one vertex of C . Clearly, Z' belongs to \mathcal{L}^* since it is prime, it contains Z and is a minimal graph with respect to these properties. Now, we construct from Z' a new graph Z'' , by first removing w from Z' and then replacing it by a chordless chain $l = x_1, \dots, x_k$, $k \geq 2$, such that the neighbourhood of l in Z'' is as follows: Every vertex of l but x_k , is adjacent to the vertex a and nonadjacent to the vertices b, c and d , the vertex x_1 is also adjacent to a vertex of the hole C while the vertex x_k is adjacent only to the vertex x_{k-1} of l .

We can easily verify that Z'' is a prime graph containing as induced subgraph the graph Z . Moreover, Z'' is Z -minimal. To justify the latter, first observe that there exists only one induced subgraph of Z'' isomorphic to Z , namely Z itself. Second, whenever we consider a proper induced subgraph G' of Z'' containing the graph Z as induced subgraph, G' contains a homogeneous set formed by the vertices of C and the vertices of l belonging to G' that are adjacent to vertex a . Hence, since the chain l can be arbitrarily long, we obtain the claimed result. \square

Consider now a connected decomposable graph W , every homogeneous set of which has exactly two vertices. Then we have the following:

Proposition 4.1. *Two nontrivial modules of W share a common vertex if and only if W is isomorphic to a C_3 .*

Proof. The ‘if’ part holds by observing that every pair of vertices of a C_3 forms a nontrivial module. Assume now that $|V(W)| > 3$ and consider two nontrivial modules H and H' of W such that $H \neq H'$ and $H \cap H' \neq \emptyset$. Then, by a property of modules, $H \cup H'$ will be a nontrivial module of W (see e.g. [8]), contradicting our assumption that every nontrivial module of W has two vertices. We can easily check now that since W is supposed to be connected W must be isomorphic to a C_3 . \square

Whenever W is isomorphic to a C_3 , we know that the set of W -minimal graphs is finite. Indeed, Olariu in [19] proved the following result:

Theorem 4.2 (Olariu [19]). *The closure of C_3 -free graphs under substitution is defined by the three forbidden configurations Q_1, Q_2 and Q_3 depicted in Fig. 2 below.*

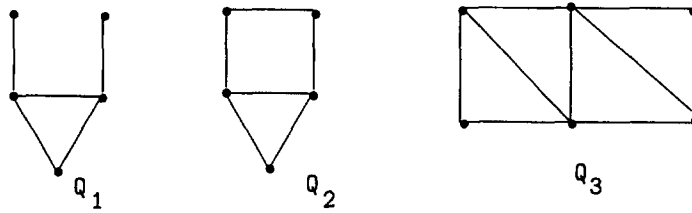


Fig. 2.

Hence, by Theorem 3.5, $\{Q_1, Q_2, Q_3\}$ is the set of C_3 -minimal graphs.

Assume then in what follows that W is not isomorphic to a C_3 . By Proposition 4.1 two homogeneous sets of W do not share a common vertex. In Theorem 4.3 below we shall show that the number of vertices of every W -minimal graph is at most equal to $|V(W)| + k$, where k is the number of the nontrivial modules of W .

We use as a prerequisite:

Proposition 4.2. *Let H be a proper subset of the vertices of a prime graph G , then if H does not induce a stable set in G (resp. a clique), then there exist two adjacent (resp. nonadjacent) vertices x, y in H and a vertex z outside H such that $xz \in E(G)$ and $yz \notin E(G)$.*

Proof. Consider the connected components of the subgraph $G[H]$ of G , then if H is not a stable set, there must be a connected component C in $G[H]$ having at least two vertices. Since C is not a homogeneous set of G (G is a prime graph), there exists a vertex z outside C adjacent to some but not all vertices of C . By connectedness of C , we find an edge xy in C such that $xz \in E(G)$ and $yz \notin E(G)$. Whenever $G[H]$ is not a clique, the result holds by considering the connected components of $\overline{G[H]}$. \square

Let H_1, \dots, H_k be the nontrivial modules of W and denote by $\{x_i, y_i\}$ the two vertices of H_i , $i = 1, \dots, k$. Since we assumed that W is connected and nonisomorphic to a C_3 , by Proposition 4.1 $H_i \cap H_j = \emptyset$, for $i \neq j$ and $i, j = 1, \dots, k$.

Theorem 4.3. *Let Q be a W -minimal graph then $|V(W)| < |V(Q)| \leq |V(W)| + k$.*

Proof. Instead of providing here just an existence proof, we choose to present a slightly more involved constructive one, which is used later in the algorithm below for enumerating all W -minimal graphs.

Let A_1 be the set of vertices of Q such that for any vertex u of A_1 and for any vertex v of $W - H_1$, $uv \in E(Q)$ if and only if $x_1v \in E(W)$. Clearly, A_1 is not empty since it contains H_1 . Since Q is prime, by Proposition 4.2 we can find two vertices a_1 and b_1 in A_1 and a vertex t outside A_1 such that t is adjacent to exactly one vertex of $\{a_1, b_1\}$. Moreover, we can choose a_1 and b_1 adjacent (resp. nonadjacent) whenever x_1

and y_1 are adjacent (resp. nonadjacent). Obviously $t \notin W$. Let W_1 be the graph induced by $V(W) - \{x_1, y_1\} \cup \{a_1, b_1\}$.

Claim 1. W_1 is a graph isomorphic to W whose nontrivial modules are $\{a_1, b_1\}$, H_2, \dots, H_k .

Proof. Indeed, since every homogeneous set of W has exactly two vertices, no vertex of $A_1 - H_1$ (if any) belongs to W . Moreover, since $H_i \cap H_j = \emptyset$, for $i \neq j$ and $i, j = 1, \dots, k$, $\{a_1, b_1\}$ is distinct from any H_i , $i \neq 1$. \square

Observe that Claim 1 is not necessarily true when W is isomorphic to a C_3 . Indeed, suppose that W is a C_3 with $V(W) = \{x_1, y_1, z_1\}$ and assume that a_1 is the vertex x_1 while b_1 is adjacent to a_1 and to z_1 and nonadjacent to y_1 . Then the graph W_1 induced by $\{a_1, b_1, z_1\}$ is isomorphic to W but its nontrivial modules are $\{a_1, b_1\}$, $\{a_1, z_1\}$ and $\{b_1, z_1\}$.

Let Q_1 the graph induced by $V(W_1) \cup T_1$ with T_1 reduced to $\{t\}$.

If $k > 1$, we construct a sequence of graphs Q_2, \dots, Q_k such that $V(Q_i) = V(W_i) \cup T_i$, $i = 2, \dots, k$ where W_i is a graph isomorphic to W whose nontrivial modules are $\{a_1, b_1\}, \dots, \{a_i, b_i\}, H_{i+1}, \dots, H_k$. W_i and T_i are constructed from Q_{i-1} as follows:

Let A_i be the set of vertices of Q such that the neighbourhood of any vertex of A_i in the graph induced by $V(W_{i-1}) - H_i$, is the neighbourhood of the nontrivial module $H_i = \{x_i, y_i\}$ in W_{i-1} .

Since $|H_i| = 2$, no vertex of $A_i - H_i$ (if any) belongs to W_{i-1} . Assume that x_i is adjacent (resp. nonadjacent) to y_i . First examine if there is a vertex of T_{i-1} (T_{i-1} is the set $V(Q_{i-1}) - V(W_{i-1})$), that distinguishes a pair $\{a_i, b_i\}$ of adjacent (resp. nonadjacent) vertices of A_i . If this is the case, W_i will be the graph induced by $V(W_{i-1}) - \{x_i, y_i\} \cup \{a_i, b_i\}$ and T_i will be the set T_{i-1} .

If no vertex of T_{i-1} distinguishes any pair of adjacent (resp. nonadjacent) vertices of A_i , then by Proposition 4.2 we can find a pair $\{a_i, b_i\}$ of adjacent (resp. nonadjacent) vertices of A_i and a vertex v of Q outside A_i that distinguishes a_i and b_i . Obviously, $v \notin Q_{i-1}$. Then W_i will be the graph induced by $V(W_{i-1}) - \{x_i, y_i\} \cup \{a_i, b_i\}$ and T_i will be the set $T_{i-1} \cup \{v\}$.

Using an analogous argumentation to the one used in Claim 1, we deduce that W_i is isomorphic to W having $\{a_1, b_1\}, \dots, \{a_i, b_i\}, H_{i+1}, \dots, H_k$ as nontrivial modules.

At the end of this process we obtain a graph Q_k having at most $|V(W)| + k$ vertices and strictly containing a graph W_k isomorphic to W . Moreover, since W is supposed to be connected, the construction of Q_k implies immediately that this graph is also connected.

We shall show now by contradiction that Q_k is a prime graph.

Let us denote by α_i the nontrivial module $\{a_i, b_i\}$ of W_k , $i = 1, \dots, k$. Then by the definition of T_k we have the following properties:

Fact 1. *There is no pair of vertices of T_k having the same neighbourhood in Q_k .*

Fact 2. For each α_i there exists a vertex of T_k that distinguishes the vertices of α_i , $i = 1, \dots, k$.

Denote henceforth by t_{α_i} a vertex of T_k that distinguishes the vertices of α_i , $i = 1, \dots, k$.

Fact 3. $N_{W_k}(x) \neq N_{W_k}(\alpha_i)$, $x \in T_k$ and $i = 1, \dots, k$.

Assume now that Q_k contains a nontrivial module M .

Claim 2. Let x and y be two vertices of W_k such that $x \in M$ and $y \notin M$, then there exists a vertex of W_k belonging to $N_{Q_k}(M)$.

Proof. Indeed, since W_k is connected, there must be a chain in W_k joining x to y , and this chain clearly contains at least one vertex of $N_{Q_k}(M)$. \square

Claim 3. At least one of the vertices of T_k belongs to M .

Proof. Indeed, assume that no vertex of T_k belongs to M . Then, Fact 2 implies that M contains at most one vertex of each α_i . Hence, Claim 2 implies that the set of vertices $N_{Q_k}(M)$ contains a vertex of W_k and consequently M would be a nontrivial module of W_k distinct from any α_i , a contradiction. \square

Let t_{α_r} be a vertex of T_k belonging to M .

Claim 4. One of a_r or b_r does not belong to M .

Proof. Indeed, assume to the contrary that both a_r and b_r belong to M . Then Fact 3 implies that there exists a vertex, say x , in W_k that distinguishes t_{α_r} and α_r . Since M is an homogeneous set of Q_k , clearly x belongs to M .

Observe now that $N_{Q_k}(M)$ cannot contain only vertices of T_k . Indeed, let t_{α_s} be a vertex of $N_{Q_k}(M)$, then since by definition t_{α_s} is not adjacent to both vertices of α_s , one at least of a_s or b_s belongs to $V(Q_k) - M$. Thus, since both, a_r and b_r belong to M , Claim 2 implies that $N_{Q_k}(M)$ contains a vertex of W_k .

It follows that $M - T_k$ is a nontrivial module of W_k of at least three vertices, namely a_r, b_r and x and this contradicts our assumption that any homogeneous set of W_k has exactly two vertices. \square

Claim 5. Exactly one vertex of α_r belongs to M .

Proof. Indeed, by Claim 4 one of a_r and b_r does not belong to M . Suppose now that none of these two vertices belong to M . Let a_r be the only vertex of α_r that is adjacent to t_r , then clearly a_r belongs to $N_{Q_k}(M)$. Observe that since by Fact 1 no pair of vertices of T_k has the same neighbourhood in W_k , M cannot contain only vertices of T_k . Let d be a vertex of W_k belonging to M , then since $\{a_r, b_r\}$ is a nontrivial module of W_k , d

is adjacent to both a_r and b_r and consequently $b_r \in N_{Q_k}(M)$. It follows that t_{α_r} is also adjacent to b_r , a contradiction. \square

We are in position now to conclude the proof of the theorem.

Let a_r be the only vertex of α_r belonging to M . Then by Fact 3 there exists a vertex x of W_k that distinguishes a_r and t_{α_r} . Thus, since M contains both a_r and t_{α_r} , clearly x must be a vertex of M . Since b_r does not belong to M , Claim 2 implies that $N_{Q_k}(M)$ contains at least one vertex of W_k . Consequently, the set of vertices $M - T_k$ is a nontrivial module of W' distinct from any α_i , $i = 1, \dots, k$, a contradiction.

Thus, Q_k is a prime graph containing a subgraph isomorphic to W . Clearly, since Q is supposed to be W -minimal, Q_k is exactly Q and W_k is exactly W . \square

Notation. The set of W -minimal graphs will be denoted henceforth by $\mathcal{P}(W)$.

We are in position now to state our main result.

Theorem 4.4 (Main result). *If every nontrivial module of any graph of \mathcal{L} has two vertices, then \mathcal{L}^* is a finite set.*

Proof. Let W be a decomposable graph of \mathcal{L} . If W is connected then by Theorems 4.2 and 4.3 we clearly have that $\mathcal{P}(W)$ is finite.

If on the contrary W is not connected, consider \overline{W} that is connected and observe that:

- (i) $\mathcal{H} = \{H_1, \dots, H_k\}$ is the set of the nontrivial modules of W if and only if \mathcal{H} is the set of the nontrivial modules of \overline{W} ;
- (ii) A graph Q is W -minimal if and only if \overline{Q} is \overline{W} -minimal.

Hence, the set of W -minimal graphs will be the set of the complementary graphs of $\mathcal{P}(\overline{W})$. To conclude the proof, we recall that \mathcal{L}^* is obtained as union of \mathcal{L}_1 (the set of prime graphs of \mathcal{L}) with the sets $\mathcal{P}(Z)$, $Z \in \mathcal{L}_2$ (\mathcal{L}_2 is the set of decomposable graphs of \mathcal{L}). From each such $\mathcal{P}(Z)$ we remove the graphs that are not \mathcal{L}_1 -free. \square

4.1. A simple algorithm for enumerating all graphs of \mathcal{L}^*

Step 1: Construct the modular decomposition tree $T(W)$ of any graph W of \mathcal{L} (e.g. using the linear algorithm in [8]), and determine the set \mathcal{L}_1 of prime graphs of \mathcal{L} and the set \mathcal{L}_2 of decomposable ones.

Step 2: Test on using $T(W)$ if each nontrivial module of any graph W of \mathcal{L}_2 contains exactly two vertices. If yes go to step 3, else exit.

Step 3: For each graph W of \mathcal{L}_2 , if W is connected construct the set of graphs $\mathcal{P}(W)$ and if W is nonconnected, construct the set of the complementary graphs of $\mathcal{P}(\overline{W})$.

Step 4: Define \mathcal{L}^* as the union of the graphs of \mathcal{L}_1 with the set of \mathcal{L}_1 -free graphs obtained on step 3.

In step 3, when a connected graph W of \mathcal{L}_2 is not isomorphic to a C_3 , by means of the construction used in the proof of Theorem 4.3, we can easily find all the graphs $\mathcal{P}(W)$ as follows:

- (i) Associate W with a set $T = \{t_1, \dots, t_k\}$ of new vertices.
- (ii) Construct a prime graph Q by making each vertex t_i of T adjacent to exactly one of the vertices of the nontrivial module H_i of W , $i = 1, \dots, k$.
- (iii) Construct from Q all possible prime graphs obtained by identification of the vertices of T and/or by adding to Q edges from vertices of T to vertices of W , or edges between vertices of T , with the following restriction: if a vertex t_i is adjacent to H_j , $i \neq j$, t_i must be adjacent to both vertices of H_j .

5. Applications

In this section we shall obtain new classes of perfect graphs by applying the algorithm of the previous section to some classes of perfect graphs.

We recall that the notion of perfect graph was first introduced by Berge in [2]. In that paper, a graph G is defined as *perfect* if for every induced subgraph H of G the chromatic number $\chi(H)$ of H equals the largest number $\omega(H)$ of pairwise adjacent vertices in H . A graph is *minimal imperfect* if G itself is imperfect but every induced subgraph of G is perfect. The only known minimal imperfect graphs are the odd long cycles (or holes) and their complements. Berge [3] conjectured that these are the only minimal imperfect graphs, conjecture that is still open (see also [4] for the history of perfect graphs). The above question, known also as the Strong Perfect Graph Conjecture (SPGC), stimulated over the years intensive research that established the perfection of many families of graphs. We could expect that improving our knowledge about perfection by increasing the list of known classes of perfect graphs takes us closer to the solution of SPGC. This part of our paper is an attempt in this direction.

5.1. New classes of perfect graphs

Each family of graphs \mathcal{F} presented below has been shown perfect and is defined by forbidden configurations that are not all prime graphs. Each nontrivial module of a decomposable forbidden subgraph Z contains exactly two vertices. Thus, we can apply the algorithm of the previous section for enumerating all the forbidden configurations of \mathcal{F}^* . \mathcal{F}^* is the closure by substitution-composition of \mathcal{F} and by Theorem 3.4 contains strictly \mathcal{F} . Hence, since Lovasz established in [14] that the family of perfect graphs is closed under substitution-composition, each \mathcal{F}^* will be a class of perfect graphs.

By Theorem 3.6 and the fact that a graph G is perfect if and only if \overline{G} is perfect (proved by Lovász in [14]), we can also characterize by forbidden configurations each perfect class $\overline{\mathcal{F}^*}$, the family of the complementary graphs of \mathcal{F}^* .

We shall focus on the forbidden configurations of \mathcal{F}_1^* . The enumeration of the remaining configurations does not raise any particular problem and is left to the reader.

Definition. A graph G is called *Berge graph*, if none of its induced subgraph is an odd hole or the complement of an odd hole.

5.2. Subclasses of P_5 -free graphs

We note by \mathcal{F}_i a family of Berge graphs that are (P_5, Z_i) -free, where Z_i , $1 \leq i \leq 7$, is depicted in Fig. 3 at the end of this paper.

The perfection of these classes was established in the following references: \mathcal{F}_1 in [18], \mathcal{F}_2 in [13], \mathcal{F}_3 in [11] and $\mathcal{F}_4, \dots, \mathcal{F}_7$ in [1].

We can easily verify that \mathcal{F}_1^* is the family of Berge graphs that are P_5 , Z_i^* -free, where Z_i^* , $1 \leq i \leq 4$, is depicted in Fig. 3.

5.3. Welsh–Powell perfect graphs

Given a graph G , a graph-coloring heuristics consists in defining first a linear order $<$ on the set of vertices of G and next by assigning to each vertex x the smallest positive integer assigned to no neighbour y of x ($y < x$). Chvátal [6] proposed to call $<$ a *perfect order*, if for each induced subgraph H of G , the number of colors used by the above heuristic on H , equals the chromatic number of H .

Welsh and Powell [21] define $<$ in such a way that

$$d(x) \geq d(y) \text{ whenever } x < y, \quad \text{where } d(x) \text{ is the degree of } x \text{ in } G. \quad (*)$$

A graph will be called Welsh–Powell perfect if the order $<$ satisfying $(*)$ is perfect.

In [7] it is proved that the Welsh–Powell perfect graphs can be defined by a set of 17 forbidden configurations denoted by F_1, F_2, \dots, F_{17} . We can easily verify that F_8, F_9 , and F_{11}, \dots, F_{16} are decomposable graphs such that each nontrivial module of any of these graphs has exactly two vertices, while the remaining forbidden configurations are prime graphs.

The family \mathcal{F}_8 :

In [11] the Berge graphs that are (Z_1, Z_8) -free are shown to be perfect. We can easily verify that Z_8 is a prime graph.

The family \mathcal{F}_9 :

In [22] the Berge graphs that are C_4 and Z_9 -free are shown to be perfect. We can easily verify that every nontrivial module of Z_9 and of a C_4 has exactly two vertices.

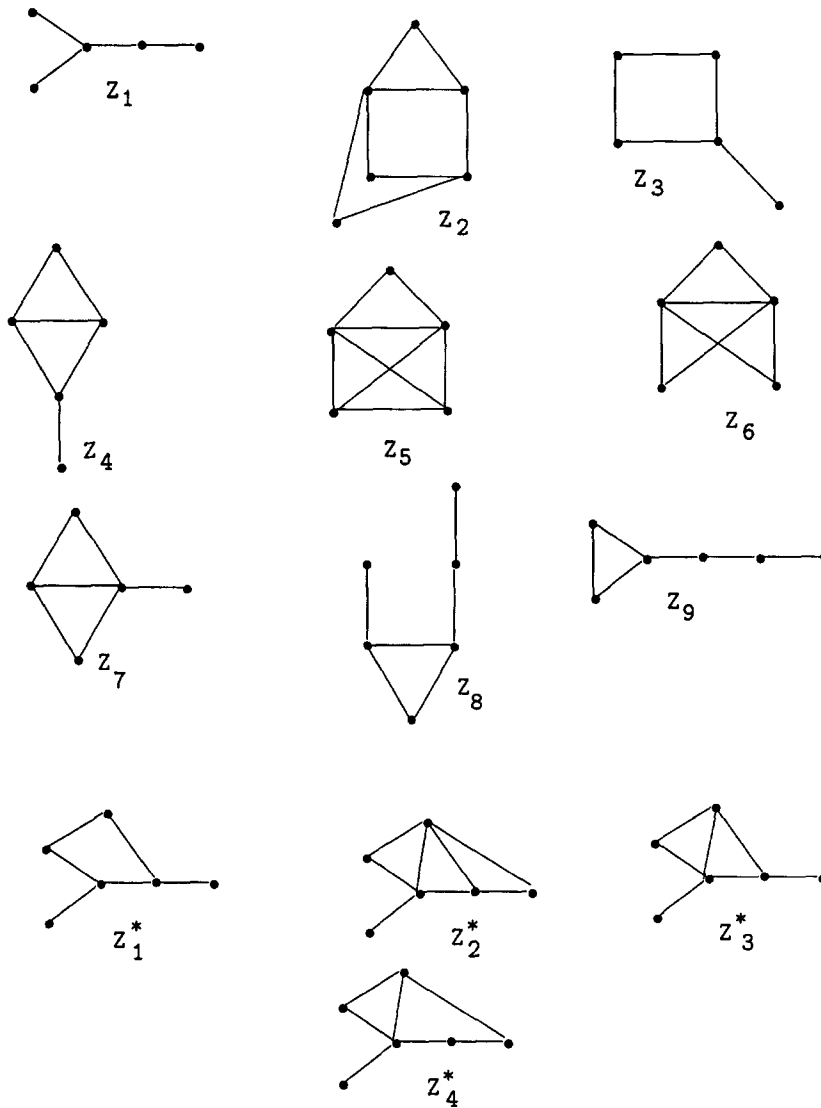


Fig. 3.

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