
Finite set interpretations

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INTERPRETATIONS

Interpretations: logically defined mappings from structures to structures.

Property: interpretations are closed under composition.

Property: $\mathcal{S} \models \tilde{I}(\phi)$ iff $I(\mathcal{S}) \models \phi$.

Consequence: Transfer of decidability results (model checking, sat).

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Example:

Theorem (Rabin): The full binary tree has a decidable MSO-theory.

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Expressive power? Completeness?

Question: are all structures of decidable MSO-theory prefix recognizable?

Seese conjecture: are all structures of decidable MSO-theory MSO-interpretations of a tree? **Partially solved (Courcelle & Oum).**

OVERVIEW

- Finite set interpretations & Elementary results
- Finite set interpretations of trees
- Focus on car parking
- Conclusion

Finite set interpretations

STRUCTURES AND LOGICS

Def: A relational structure $\mathcal{S} = (\mathcal{U}, R_1, \dots, R_k)$

- \mathcal{U} is a set called the universe of the structure
- each R_i is a relation of arity r_i over \mathcal{U}

Isomorphism: $\mathcal{S} \simeq \mathcal{S}'$ if $R(u_1, \dots, u_n) \leftrightarrow R'(f(u_1), \dots, f(u_n))$ (for f bijective)

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First-order logic (FO)

FO variables: x, y, x_1, \dots ranging over \mathcal{U}

FO logic: $x = y, R(x_1, \dots, x_k), \wedge, \vee, \neg, \exists x, \forall x$

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Model: $\mathcal{S} \models \phi, \mathcal{S} \models \phi(a, b, c), \mathcal{S} \models \phi(A, B, C)$

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Def: First-order interpretation $(\delta(x), \phi_1(x_1, \dots, x_{r_1}), \dots, \phi_k(x_1, \dots, x_{r_k}))$

- $\delta, \phi_1, \dots, \phi_k$ are first-order formulas
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Def: Finite set interpretation $(\Delta(X), \Phi_1(X_1, \dots, X_{r_1}), \dots, \Phi_k(X_1, \dots, X_{r_k}))$

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SEMANTIC OF INTERPRETATIONS

Given an interpretation I and a structure \mathcal{S} .

FO interpretation: $I_{\text{FO}}(\mathcal{S})$ is defined by:

Universe: $\{a \in \mathcal{U}_{\mathcal{S}} : \mathcal{S} \models \delta(a)\}$

Relations: $R_i(a_1, \dots, a_{r_i})$ iff $\mathcal{S} \models \phi(a_1, \dots, a_{r_i})$

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Motto: Elements of $I_{\text{FS}}(\mathcal{S})$ are finite sets of elements of \mathcal{S} .

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Isomorphism: $f(E) = \sum_{n \in E} 2^n$

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Prop: $(\mathbb{N}, \text{Succ}) \models \text{Plus}(A, B, C)$ iff $f(A) + f(B) = f(C)$

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Conclusion: $I_{\text{FS}}(\mathbb{N}, \text{Succ}) \simeq (\mathbb{N}, +)$

ELEMENTARY PROPERTIES

Prop: $I_{\text{FS}}(\mathcal{S}) \models \phi$ iff $\mathcal{S} \models \widetilde{I_{\text{FS}}}(\phi)$.

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Proof:

Transform in ϕ every FO-variable into a WMSO-variable.

Relativise the quantification to Δ .

Substitute the corresponding formula in I_{FS} for every relational symbol.

E.g. $\exists x. \forall y. \exists z. x + y = z$ becomes $\exists X. \forall Y. \exists Z. \text{Plus}(X, Y, Z)$

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Prop: If I_{FO} is first-order, I_{FS} is finite set and I_{WMSO} is weak monadic then

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Corollary: The set $\text{FS}(\mathcal{S})$ of structures obtainable by finite set interpretations from \mathcal{S} is closed under FO-interpretations.

UNIVERSAL INTERPRETATION/GENERATOR

Def: The **universal FS-interpretation** P_F is defined by:

$P_F = (\Delta(X), \Phi_{\subseteq}(X, Y), \Phi_1(X_1, \dots, X_{r_1}), \dots, \Phi_k(X_k, \dots, X_{r_k}))$ with:

$$\Delta(X) = \text{True}$$

$$\Phi_{\subseteq}(X, Y) = \forall x. x \in X \rightarrow x \in Y$$

$$\Phi_i(X_1, \dots, X_{r_i}) = \exists x_1 \dots \exists x_{r_i}. \bigwedge_j X_j = \{x_j\} \wedge R_i(x_1, \dots, x_{r_i})$$

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Description: $P_F(\mathcal{S})$ is the structure such that:

- The universe is the set of finite sets of elements of \mathcal{S} .
- The relation R_{\subseteq} coincide with the inclusion.
- The structure \mathcal{S} is replicated over singletons.

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$$I_{FS} = I_{FO} \circ P_F .$$

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Proof: Eliminate FO-variables using \subseteq .

Replace syntactically WMSO-variables by FO-variables.

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Corollary: $FS(\mathcal{S})$ is the set of structures FO-interpretable in $P_F(\mathcal{S})$ (called the **generator** of $FS(\mathcal{S})$)

AUTOMATIC/TREE AUTOMATIC STRUCTURES

Def(Hodgson, Dauchet&Tison, Khoussainov&Nerode, Blumensath&Grädel):
A structure is **automatic** if its universe is a regular language of words, and its relations are definable by left synchronized finite state automata.
A structure **admits an automatic presentation** if it is isomorphic to an automatic structure. (Similarly for **tree-automatic**).

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Proof: Consequence of the closure of tree-automata under projection, union, intersection and complementation, and the decidability of the emptiness problem.

Prop: A structure admits an automatic presentation iff it is isomorphic to $I_{\text{FS}}(\mathbb{N}, +)$ for some finite set interpretation I_{FS} .

A structure admits a tree-automatic presentation iff it is isomorphic to $I_{\text{FS}}(\Delta_2)$ for some finite set interpretation I_{FS} .

Finite set interpretations of trees

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Def: By **tree** we intend (possibly infinite) binary rooted labeled trees. In the signature it is possible to distinguish left-child and right child.

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Proof: $I_{\text{FS}}(\mathcal{S}) = I_{\text{FS}}(I_{\text{WMSO}}(t)) = I'_{\text{FS}}(t)$

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Conclusion: Structures obtainable by finite set interpretations of trees are an important class to study.

Goal: Study $\text{FS}(t)$ for t a **deterministic tree**.

QUOTIENT OF STRUCTURES

Def: An equivalence relation \sim is a **congruence** over a structure \mathcal{S} if for every n and $u_1 \sim v_1, \dots, u_{r_n} \sim v_{r_n}$:

$$R_n(u_1, \dots, u_{r_n}) \quad \text{iff} \quad R_n(v_1, \dots, v_{r_n})$$

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Prop: If $\mathcal{S} = (\mathcal{U}, R_1, \dots, R_k, \sim)$ has a decidable FO-theory, \mathcal{S}/\sim also has a decidable FO-theory.

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Corollary: The tree-automatic structures are closed under quotient.

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Corollary of Main Theorem: Fix I_{FS} . There exists I_{WMSO} such that for every \mathcal{S} and t ,

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Corollary of corollary: \mathcal{S} is a structure, t a tree then

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Focus on car parking

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$f : \text{Atoms} \subseteq \mathcal{P}(\text{dom}(t)) \rightarrow \text{dom}(t)$ is an injection.
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OVERVIEW OF THE PROOF

Corollary of Main Theorem: Fix I_{FS} . There exists I_{WMSO} such that for every \mathcal{S} and t ,

$$P_{\text{F}}(\mathcal{S}) \simeq I_{\text{FS}}(t) \quad \text{iff} \quad \mathcal{S} \simeq I_{\text{WMSO}}(t) .$$

Remark: R_{\subseteq} is an order on $I_{\text{FS}}(t)$ which has all the properties of \subseteq over finite sets. In particular it has a minimal element \perp .

Def: Call **Atoms** the set of R_{\subseteq} -minimal non- \perp elements (FO-definable).

Fact: $I_{\text{FS}}(t)|_{\text{Atoms}} \simeq \mathcal{S}$.

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$f : \text{Atoms} \subseteq \mathcal{P}(\text{dom}(t)) \rightarrow \text{dom}(t)$ is an injection.
 $A \mapsto u$ with $t \models \Psi(A, u)$

Conclusion: By sending $I_{\text{FS}}(t)|_{\text{Atoms}} \simeq \mathcal{S}$ via f , one obtains $I_{\text{WMSO}}(t) \simeq \mathcal{S}$.

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Idea: Each $f(X)$ is a car. Compute the flow of cars, and an itinerary for each car.

At the end, one must have at most one car per node.

Conclusion

CONCLUSION

New object: finite set interpretations

Elementary properties:

- Composition: $I_{FO} \circ I_{FS}$ and $I_{FS} \circ I_{WMSO}$ are FS-interpretation.
- Transfer: $I_{FS}(\mathcal{S}) \models \phi$ iff $\mathcal{S} \models \widetilde{I_{FS}}(\phi)$.
- Existence of a universal FS-interpretation P_F .

Advanced results:

- Closure of $FS(t)$ under quotient.
- Main result: $P_F(\mathcal{S}) \simeq I_{FS}(t)$ iff $\mathcal{S} \simeq I_{WMSO}(t)$.

Applications:

- The free monoid (two elements) non FS-interpretable in a tree.
- Random graph non FS-interpretable in a tree.
- Strictness of the tree-automatic hierarchy (variant by Blumensath).
- Intrinsic regularity in $P_F(\mathbb{N}, \text{Succ})$ (extension of (Barany)).