

A Monadic Second-Order Definition of the Structure of Convex Hypergraphs

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We consider finite *hypergraphs* with hyperedges defined as sets of vertices of unbounded cardinality. Each such hypergraph has a *unique modular decomposition*, which is a tree, the nodes of which correspond to certain subhypergraphs (induced by certain sets of vertices called *strong modules*) of the considered hypergraph. One can define this decomposition by monadic second-order (MS) logical formulas. Such a hypergraph is *convex* if the vertices are linearly ordered in such a way that the hyperedges form intervals. Our main result says that the unique linear order witnessing the convexity of a *prime hypergraph* (i.e., of one, the modular decomposition of which is trivial) can be defined in MS logic. As a consequence, we obtain that if a set of bipartite graphs that correspond (in the usual way) to convex hypergraphs has a *decidable monadic second-order theory* (which means that one can decide whether a given MS formula is satisfied in some graph of the set) then it has *bounded clique-width*. This yields a new case of validity of a conjecture which is still open. © 2002 Elsevier Science (USA)

Key Words: hypergraph; bipartite graph; monadic second-order logic; clique-width; modular decomposition.

INTRODUCTION

We are interested in the expression in monadic second-order (MS) logic of properties of graphs and hypergraphs, and also in the definition by MS formulas of their relevant structures. We take “structure” as a generic term that can cover several notions like that of a tree-decomposition of width at most k of a graph [15], the representation of a planar drawing of a planar graph (by *rotation schemes*, see [8]), or the modular decomposition of a graph [5, 12, 16], a linear order of a given graph satisfying certain properties like being a topological sorting [5], a depth-first spanning tree [4, 9] just to take a few examples.

Certain graphs have a unique such structure: for instance, every graph has a unique modular decomposition, every connected graph has a unique decomposition in 3-connected components (a result by Tutte [19] used in [7]), and every planar 3-connected graph has a unique planar representation, by a theorem of Whitney (used in [8]). It seems that such structures are easier to define by MS formulas when they are unique. The constructions are otherwise more difficult (as in [15] for tree-decompositions) or impossible (for instance, one cannot define by MS formulas a linear order on a set).

We study here the notion of *modular decomposition of a hypergraph*. To be precise, we consider finite hypergraphs with hyperedges that are unordered and of unbounded *rank*, i.e., that are defined as (unordered) sets of vertices of unbounded cardinality. Each such hypergraph has a *unique modular decomposition*, which is a tree, the nodes of which correspond to certain subhypergraphs (induced by certain sets of vertices called *strong modules*) of the considered hypergraph. One can also consider the tree of the modular decomposition as the syntax tree of an algebraic expression denoting the considered hypergraph. Such an expression uses operations that are defined in terms of substitutions of hypergraphs to vertices in prime hypergraphs. A hypergraph is *prime* if it cannot be obtained from smaller hypergraphs by substitutions.

These notions of module, substitution, prime hypergraph, and modular decomposition are formally similar to the corresponding notions for graphs. A hypergraph can be considered as a bipartite graph in a standard way; however, its modular decomposition is *not* the modular decomposition of the corresponding bipartite graph.

We prove that the modular decomposition of a hypergraph is definable by MS logical formulas. We take advantage of the fact that the logical structure representing a hypergraph has an element in its domain for each hyperedge. This should be contrasted with the case of graphs considered in [5]. This article uses logical structures, the domains of which consist only of vertices. The modular decomposition

of a graph is proved to be definable by MS formulas *with the help of an auxiliary linear order*. In the present case, we need no such order.

We focus then our attention on *convex hypergraphs*, i.e., on those the vertices of which can be linearly ordered in such a way that the hyperedges form intervals. We prove that a prime hypergraph has at most one (up to reversal) linear order witnessing its convexity. This result is similar to the one saying that a prime graph has at most one *transitive orientation* (i.e., an orientation such that the binary edge relation is transitive). See [14] for this result, actually due to Gallai. These results are similar but we do not know whether any of them is derivable from the other.

Our main result says that in a prime convex hypergraph, this unique linear order can be defined by MS formulas.

We derive from this result that, if a set of *convex bipartite graphs* (i.e., of bipartite graphs corresponding to convex hypergraphs) has a decidable monadic second-order theory, which means that one can decide whether a given monadic second-order formula is satisfied in some graph of the set, then it has *bounded clique-width*. This yields a new case of validity of a still open conjecture made by Seese [17] and reformulated in terms of clique-width by means of results by Courcelle and Engelfriet [10]. Previous cases were considered in [4, 9, 17]. The notion of clique-width and a discussion of this conjecture can be found at the beginning of Section 4.

1. PRELIMINARIES

For monadic second-order logic and relational structures, we refer the reader to the Appendix and to previous papers, in particular to the survey [6] and to [5], where modular decompositions of graphs are also considered.

We will consider finite hypergraphs where each hyperedge is a nonempty set of vertices. These sets are not ordered; hence, hypergraphs are undirected.

To a hypergraph H with set of vertices V and set of hyperedges E corresponds a directed bipartite graph $\mathbf{Bip}(H)$ with set of vertices $V \cup E$ and an edge from x to e if and only if x is in V and is a vertex of e ($e \in E$). The requirement that hyperedges are nonempty implies that the corresponding vertices of $\mathbf{Bip}(H)$ are not isolated. A vertex v of V in $\mathbf{Bip}(H)$ can be isolated. The mapping \mathbf{Bip} is thus a bijection of the class of hypergraphs onto a class of directed bipartite graphs.

Trees will be rooted and directed in such a way that every node is reachable from the root (denoted by \mathbf{root}_T for a tree T) by a unique directed path. A tree T will be represented by a structure $\langle N_T, \mathbf{succ}_T \rangle$ where N_T is the set of nodes and $\mathbf{succ}_T \subseteq N_T \times N_T$ is the successor relation in T . Leaves have no successor. Although trees are graphs, it will be convenient to call their vertices *nodes*, especially in the case where we discuss simultaneously a hypergraph and a tree representing its structure.

If A is a set and \sim is an equivalence relation on A a *cross-section* of \sim is a set $B \subseteq A$ which contains one and only one element of each equivalence class of \sim . Thus B is isomorphic to A/\sim in a canonical way. If A is a part of a relational structure and \sim is definable by an MS formula, then the cross-sections of \sim can be characterized by an MS formula with free variable B . It follows that the transformation of A into A/\sim is an MS transduction, which uses a parameter, namely B . See the Appendix or [6] for a review of definitions.

For sets α, β we write $\alpha \perp \beta$ if and only if they *overlap*; that is, if and only if $\alpha - \beta \neq \emptyset$, $\beta - \alpha \neq \emptyset$ and $\alpha \cap \beta \neq \emptyset$. We will also say that α *overlaps* β .

If G is a loop-free undirected graph, a *module* of G is a set X of vertices such that every vertex not in X is adjacent either to no vertex of X or to all of them. A module M is *strong* if no other module overlaps it. The strong modules form a tree for inclusion, which is the *modular decomposition* of the graph. The modular decomposition of a graph can be constructed in linear time [12]; its definability in MS logic is considered in [5].

2. MODULAR DECOMPOSITION OF HYPERGRAPHS

A hypergraph H will be formally handled as a triple $\langle V_H, E_H, \mathbf{inc}_H \rangle$, where V_H is the finite set of vertices, E_H is the finite set of hyperedges ($V_H \cap E_H = \emptyset$), and $\mathbf{inc}_H \subseteq V_H \times E_H$ is the incidence relation; we assume that each $e \in E_H$ belongs to some pair in \mathbf{inc}_H .

We let $H(a) = \{v \in V_H / (v, a) \in \mathbf{inc}_H\}$ for $a \in E_H$. This set is thus nonempty. We may have $H(a) = H(b)$ with $a \neq b$. The rank of $a \in E_H$ is $\mathbf{Card}(H(a))$, the cardinality of $H(a)$.

Let H and K be hypergraphs. We write $H \subseteq K$ if $V_H \subseteq V_K, E_H \subseteq E_K, \mathbf{inc}_H = \mathbf{inc}_K \cap (V_H \times E_H)$. We say then that H is a subhypergraph of K . If $X \subseteq V_H$, we let $H[X] = \langle X, E', \mathbf{inc}_H \cap (X \times E') \rangle$ where $E' = \{a \in E / H(a) \subseteq X\}$. We say that $H[X]$ is the subhypergraph of H induced by X .

We denote the hypergraph $\langle \{x\}, \emptyset, \emptyset \rangle$ by $\mathbf{1}_x$ (or by $\mathbf{1}$ if it is not important to specify x).

Let H, K be nonempty hypergraphs. We write $G = H[K/x]$ if $x \in V_H, V_H \cap V_K = \emptyset, E_H \cap E_K = \emptyset, V_G = V_H \cup V_K - \{x\}, E_G = E_H \cup E_K$, and $\mathbf{inc}_G = \mathbf{inc}_K \cup (V_K \times \{a \in E_H / (x, a) \in \mathbf{inc}_H\}) \cup (\mathbf{inc}_H \cap ((V_H - \{x\}) \times E_H))$.

We say that G is the result of the substitution of K in H for x . It is easy to see that

$$H[K/x][K'/y] = H[K'/y][K/x]$$

if H, K, K' are pairwise disjoint and $x, y \in V_H, x \neq y$. We have also

$$H[K/x][K'/y] = H[K[K'/y]/x]$$

if $x \in V_H$ and $y \in V_K$ and both sides of this equality are well defined. We will use the notation $H[K_1/x_1, \dots, K_n/x_n]$ for $H[K_1/x_1][K_2/x_2] \dots [K_n/x_n]$ if $x_1, \dots, x_n \in V_H$ are pairwise distinct.

We will use two special notations: $H \oplus K$ for $B[H/x, K/y]$ where $B = \langle \{x, y\}, \emptyset, \emptyset \rangle$ and $a * H$ for $L[H/x]$ where $L = \langle \{x\}, \{a\}, \{(x, a)\} \rangle$. Hence $H \oplus K$ is the union of two disjoint hypergraphs H and K and $a * H$ is obtained from H by the addition of a new hyperedge a containing all V_H . We will write

$$A * H = H \quad \text{if } A = \emptyset,$$

and

$$A * H = a_1 * (a_2 * \dots (a_k * H)) \quad \text{if } A = \{a_1, \dots, a_k\}.$$

We say that a hypergraph H is *connected* if the corresponding bipartite graph $\mathbf{Bip}(H)$ is connected.

Although hypergraphs correspond to bipartite graphs (by \mathbf{Bip}), the above definitions for hypergraphs are not just the corresponding ones for bipartite graphs. In particular, a subhypergraph of H corresponds to an induced subgraph of $\mathbf{Bip}(H)$. An induced subhypergraph of H consists of a set X of vertices and all hyperedges that have all their vertices in X . In the hypergraph $H[K/x]$ obtained by substitution, where the vertices of K replace x , the rank of a hyperedge a of H containing x is increased by $\mathbf{Card}(V_K) - 1$, because in $H[K/x]$, its set of vertices is $H(a) \cup V_K - \{x\}$. The substitution of hypergraphs is *not* the substitution of the corresponding bipartite graphs.

LEMMA 2.1. *If $G = H[K_1/x_1, \dots, K_k/x_k]$ then each K_i is a subhypergraph of G and H is isomorphic to a subhypergraph of G .*

Proof. The first assertion is straightforward from the definitions. For the second, observe that one can recover H from G by eliminating all the hyperedges from the K_i 's, and for each i , all the vertices from K_i except one (we can do this because the hypergraphs K_i are nonempty). Hence H is isomorphic to a subgraph of G . ■

LEMMA 2.2. (1) *If H is not connected, it can be written in a unique way as $H_1 \oplus \dots \oplus H_k$ where each H_i is connected and nonempty. Unicity holds up to the ordering of the list H_1, \dots, H_k .*

(2) *If H is connected, it can be written in a unique way as $A * K$ where K is **-atomic*, i.e., is not of the form $a * K'$ for any a and K' . Furthermore $A = \{a \in E_H / H(a) = V_H\}$.*

We omit the easy proof. We only observe that the H_i 's correspond to the connected components of $\mathbf{Bip}(H)$.

We will draw hypergraphs as bipartite graphs, with vertices represented by black dots and hyperedges by white circles. See Fig. 1.

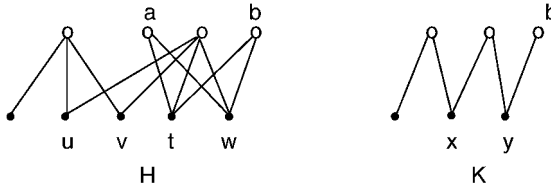


FIGURE 1

A hypergraph is an *atom* if it is nonempty, connected, and $*$ -atomic. In the example of Fig. 1, we have $H = K[L/x, M/y]$, $L = \mathbf{1}_u \oplus \mathbf{1}_v$, $M = a * (\mathbf{1}_t \oplus \mathbf{1}_w)$. Both hypergraphs H and K are atoms.

We are interested in canonical expressions of atoms of the form $P[H_1/x_1, \dots, H_k/x_k]$ where P is as small as possible.

We introduce the notion of a *module*. Let H be a hypergraph. A *module* in H is a subset X of V_H that does not overlap $H(a)$ for any a in E_H , i.e., such that for all such a :

$$H(a) \subseteq X \quad \text{or} \quad X \subseteq H(a) \quad \text{or} \quad X \cap H(a) = \emptyset.$$

In the hypergraph H of Fig. 1, $\{u, v\}$ and $\{t, w\}$ are modules.

If $G = H[K/x]$ for G, H, K hypergraphs, then V_K is a module of G . If X is a module of G , one can express G as $H[G[X]/x]$ for some hypergraph H with $V_H = (V_G - X) \cup \{x\}$ where $x \notin V_G$. Hence, modules are useful for the study of substitutions.

A module of a hypergraph H is not in general a module of the graph $\mathbf{Bip}(H)$: In the hypergraph H' of Fig. 2a, the set $\{5, 6, 7\}$ is a module. It is not a module of the graph $\mathbf{Bip}(H')$ (actually shown in Fig. 2a since we represent hypergraphs as bipartite graphs), because the vertex m of $\mathbf{Bip}(H')$ is linked to 5 and not to 7.

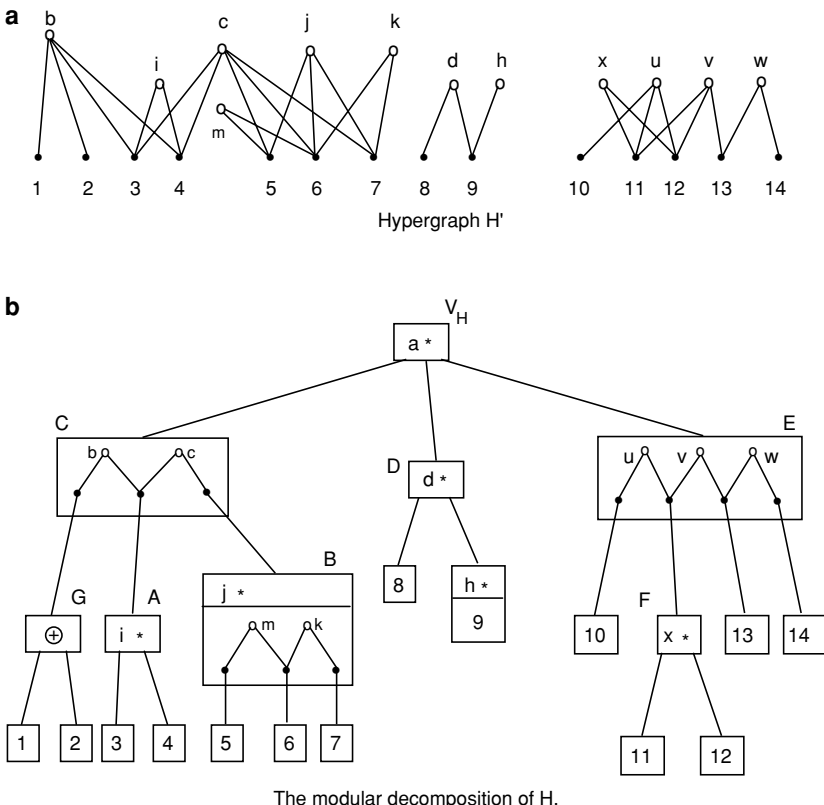


FIGURE 2

A hypergraph H is *prime* if it is an atom, is not of the form $\mathbf{1}_x$, has no hyperedge of rank 1 and no modules other than V_H, \emptyset , and the singletons.

The smallest prime hypergraph is $\bullet \text{---} \circ \text{---} \bullet \text{---} \circ \text{---} \bullet$ because no hypergraph with one or two vertices is prime. If we delete the hyperedge b from K of Fig. 1, we obtain this prime hypergraph.

A prime hypergraph is connected (because it is an atom), and for any two vertices, there is a hyperedge containing one of them and not the other, because otherwise, these two vertices would form a module.

The forthcoming definitions and lemmas will be fully similar to the corresponding ones for graphs. The objective is to have for hypergraphs a notion of modular decomposition.

LEMMA 2.3. *If X and Y are modules in H and $X \cap Y \neq \emptyset$, then $X \cup Y$ is a module in H .*

Proof. Let $a \in E_H$ with $H(a) \cap (X \cup Y) \neq \emptyset$. If $H(a) \cap X \neq \emptyset$ then:

(1) either $H(a) \subseteq X$ and then $H(a) \subseteq X \cup Y$,

(2) or $X \subseteq H(a)$, and then $Y \cap H(a) \neq \emptyset$ since $X \cap Y \neq \emptyset$; if $H(a) \subseteq Y$ then $H(a) \subseteq X \cup Y$; otherwise, $Y \subseteq H(a)$, and then $X \cup Y \subseteq H(a)$. ■

LEMMA 2.4. (1) *Let X be a module of hypergraph H ; let $G = H[K_1/x_1, \dots, K_k/x_k]$. If $x_1, \dots, x_k \notin X$, then X is a module of G . Otherwise, let $X' = X - \{x_1, \dots, x_k\}$. Then $X' \cup \bigcup \{V_{K_i}/x_i \in X\}$ is a module of G .*

(2) *Let X be a module of $G = H[K/x]$. If $V_K \subseteq X$, then $X \cup \{x\} - V_K$ is a module of H .*

The proof is a verification from the definitions.

A module X in a hypergraph H is *strong* if no other module overlaps X . In other words, for every module Y , either $X \subseteq Y, Y \subseteq X$, or $X \cap Y = \emptyset$. A module X in H is *proper* if $X \neq V_H, X \neq \emptyset$. If X, Y are maximal proper strong modules, then either $X \cap Y = \emptyset$ or $X = Y$.

LEMMA 2.5. *Let H be an atom and $X \subseteq V_H$ be a proper module. There exists a maximal proper strong module Y such that $X \subseteq Y$.*

Proof. There exists a maximal proper module Y such that $X \subseteq Y$. We prove that it is strong. Let Z be a module with $Y \cap Z \neq \emptyset, Y$ not a subset of Z and Z not a subset of Y . Then $Y \cup Z$ is a module by Lemma 2.3.

If $Y \cup Z \neq V_H$ then Y is not maximal, a contradiction. Hence $Y \cup Z = V_H$.

Since H is connected, there is a hyperedge $a \in E_H$ such that

$$H(a) \cap (Y - Z) \neq \emptyset \quad \text{and} \quad H(a) \cap Z \neq \emptyset.$$

If $H(a) \subseteq Z$ we cannot have $H(a) \cap (Y - Z) \neq \emptyset$. Hence, since Z is a module, $Z \subseteq H(a)$. We have $H(a) \cap Y \neq \emptyset$. Since $Z \subseteq H(a)$, we cannot have $H(a) \subseteq Y$. Hence $Y \subseteq H(a)$ and $Y \cup Z = V_H \subseteq H(a)$. Hence $H = a * H'$ and is not an atom, a contradiction.

Hence, for every module Z , either $Z \cap Y = \emptyset, Y \subseteq Z$, or $Z \subseteq Y$. This shows that Y is strong. ■

PROPOSITION 2.6. *Let H be an atom, $H \neq \mathbf{1}$. Let C_1, \dots, C_k be its maximal proper strong modules and $K_i = H[C_i]$ for $i = 1, \dots, k$.*

(1) *There exists a prime hypergraph P such that $H = P[K_1/x_1, \dots, K_k/x_k]$.*

(2) *Conversely, if $H = P[L_1/x_1, \dots, L_n/x_n]$ and P is prime, then $\{V_{L_1}, \dots, V_{L_n}\} = \{C_1, \dots, C_k\}, n = k$, and $\{L_1, \dots, L_n\} = \{K_1, \dots, K_k\}$.*

This proposition says in particular that every atom which is not a singleton can be expressed in a unique way as $P[K_1/x_1, \dots, K_k/x_k]$ where P is prime. As in Lemma 2.2 we have a unique decomposition.

Proof. Let H, C_1, \dots, C_k , and $\{K_1, \dots, K_k\}$ be as stated.

(1) By Lemma 2.5, every vertex which is by itself a module is contained in some C_i . Since the C_i 's are disjoint, $\{C_1, \dots, C_k\}$ is a partition of V_H . Hence, there exists P such that $H = P[K_1/x_1, \dots, K_k/x_k]$. Clearly $k \geq 2$. Hence $P \neq \mathbf{1}$, P is an atom (otherwise H would not be an atom), and it has no hyperedge of rank 1 (since K_i contains all hyperedges a of H such that $H(a) \subseteq C_i$).

It remains to prove that P has no module M such that $1 < \mathbf{Card}(M) < \mathbf{Card}(P) = k$. Assume it has one, say M . Let M' be the union of the sets C_i such that x_i belongs to M . It is a proper module of H that contains at least two of the modules C_1, \dots, C_k . This holds by Lemma 2.4.1. By Lemma 2.5, it is contained in one of C_1, \dots, C_k ; hence one of these sets is properly contained in another. This contradicts their definition. Hence, we get that P is prime.

(2) Let us conversely assume that $H = P[L_1/x_1, \dots, L_n/x_n]$ where P is prime. Then V_{L_1}, \dots, V_{L_n} are proper modules. We prove they are maximal. If one of them is properly included in a proper module X of H , then Lemma 2.4.2 yields a proper module of P , and P is not prime.

Hence $\{V_{L_1}, \dots, V_{L_n}\} \subseteq \{C_1, \dots, C_k\}$ by Lemma 2.5. The other inclusion follows from Lemma 2.4.2. This completes the proof. ■

LEMMA 2.7. *If $H = H_1 \oplus \dots \oplus H_k$, with H_1, \dots, H_k connected, $k \geq 2$, or if $H = P[H_1/x_1, \dots, H_k/x_k]$ with P is prime, the strong modules of H are V_H and the strong modules of H_1, \dots, H_k .*

Proof. The set V_H is trivially a strong module in both cases.

Let us consider the case $H = H_1 \oplus \dots \oplus H_k$. Let X be a strong module of H_i . It is a module of H . Let us check that it is strong. Let Y be a module of H that overlaps X . The set $Y \cap V_{H_i}$ is a module of H_i and hence does not overlap X . Hence Y is not a subset of V_{H_i} . Since H_i is connected and Y overlaps X , there is a hyperedge a of H_i such that $H(a)$ overlaps Y . Since Y is a module of H we must have $Y \subseteq H(a)$, contradicting the fact that Y is not a subset of V_{H_i} . Hence X is a strong module of H .

Conversely, let X be a strong module of H , $X \neq V_H$. Let $X_i = X \cap V_{H_i}$. It is a strong module of H_i . If two of the sets X_i are not empty, say X_1 and X_2 , and X_j is such that $Z = V_{H_j} \cap (V_H - X) \neq \emptyset$ (w.l.o.g. $j \neq 1$), then Y defined as $V_{H_2} \cap V_{H_j}$ is a module such that $X - Y$ includes $X_1 \neq \emptyset$, $Y - X$ includes $Z \neq \emptyset$, and $X \cap Y$ includes $X_2 \neq \emptyset$; hence X is not strong.

Hence $X \subseteq V_{H_i}$ for some i , and X is a strong module of H_i .

We now consider the case $H = P[H_1/x_1, \dots, H_k/x_k]$ with P prime. Every strong module X in some H_i is a module in H . Let us prove it is strong in H . If it is not, some module Y in H overlaps X . If Y is in H_i then X is not strong in H_i . Hence Y overlaps H_i , and this contradicts Proposition 2.6.2, saying that V_{H_i} is a strong module of H .

Conversely, by Proposition 2.6.2 and since P is prime, every strong module in H not equal to V_H is in some H_i . It is easy to see that it is strong in H_i . ■

A *component* of a hypergraph H is a subhypergraph of H of the form $H[X]$ where X is a nonempty strong module of H . We let $\mathbf{Comp}(H)$ denote the set of components of H .

THEOREM 2.8. *Every hypergraph H has a unique hierarchical decomposition defined as follows:*

- (1) *if H is not connected, then $H = H_1 \oplus \dots \oplus H_k, k \geq 2, H_1, \dots, H_k$ are connected;*
- (2) *if $V_H = \{x\}$, then $H = E_H * \mathbf{1}$;*
- (3) *if H is connected and V_H not singleton, then $H = A * P[H_1/x_1, \dots, H_k/x_k]$ where*
 - (3.1) *either $P = \mathbf{1}_{x_1}, k = 1, A \neq \emptyset, H_1$ is not connected;*
 - (3.2) *or P is prime, $H_i = H[X_i]$ where $\{X_1, \dots, X_k\}$ is the set of maximal proper strong modules of H .*

Furthermore, we have in these cases, respectively,

- (1) $\mathbf{Comp}(H) = \{H\} \cup \mathbf{Comp}(H_1) \cup \dots \cup \mathbf{Comp}(H_k),$
- (2) $\mathbf{Comp}(H) = \{H\},$
- (3.1) $\mathbf{Comp}(H) = \{H\} \cup \mathbf{Comp}(H_1) - \{H_1\},$
- (3.2) $\mathbf{Comp}(H) = \{H\} \cup \mathbf{Comp}(H_1) \cup \dots \cup \mathbf{Comp}(H_k).$

Proof. Case (1) follows from Lemma 2.2.1; the expression of $\mathbf{Comp}(H)$ follows from Lemma 2.7. Case (2) is immediate from the definitions.

We now consider Case (3). By Lemma 2.2.2, H can be written in a unique way as $A * H'$ where $A = \{a \in E_H / H(a) = V_H\}$ and H' is a $*$ -atom. If H' is not connected we are in Case (3.1), and the

expression of $\mathbf{Comp}(H)$ follows from Lemma 2.7. The set V'_H cannot be a singleton because Case (2) handles this case.

Hence the remaining possibility is when H' is connected and $*$ -atomic. Then Proposition 2.6 applies and yields Case (3.2) together with the unicity of P . The characterization of $\mathbf{Comp}(H)$ follows from the definitions and Lemma 2.7. ■

DEFINITION 2.9 (Modular decomposition). From Theorem 2.8 we get a tree $T = T(H)$ representing this decomposition, called the *modular decomposition* of H and defined as follows:

- (1) $N_T = \mathbf{Comp}(H)$,
- (2) $\mathbf{root}_T = H$,
- (3) the leaves of T are the components with a single vertex,
- (4) for every component $C \in \mathbf{Comp}(H)$, where V_C is not a singleton, we have the following cases where, in each of them, C_1, \dots, C_k are the successors of C :

(4.1) $C = D * P[C_1, \dots, C_k]$, (cf. Case (3.2) of Theorem 2.8) where P is prime, $k \geq 3$ (because prime hypergraphs have at least three vertices), and D may be empty,

(4.2) $C = D * (C_1 \oplus \dots \oplus C_k)$, $k \geq 2$ (cf. Case (3.1) of Theorem 2.8), C_1, \dots, C_k are connected, $D \neq \emptyset$,

(4.3) $C = C_1 \oplus \dots \oplus C_k$, $k \geq 2$, C_1, \dots, C_k are connected (cf. Case (1) of Theorem 2.8).

In Cases (4.1) and (4.2), we will denote D by $Full(C)$ and we call it the set of *full hyperedges* of C .

By Lemma 2.1, a hypergraph P as in Case (4.1) is isomorphic to a prime subhypergraph of C whence of H . We will call these hypergraphs the *principal prime subhypergraphs* of H . We denote their set by $PPrime(H)$.

EXAMPLE. The proper strong modules of H' (shown in Fig. 2a) are $A = \{3, 4\}$, $B = \{5, 6, 7\}$, $C = \{1, 2, 3, 4, 5, 6, 7\}$, $D = \{8, 9\}$, $E = \{10, 11, 12, 13, 14\}$, $F = \{11, 12\}$, $G = \{1, 2\}$, and the singletons.

Let us define $H = a * H'$. The modular decomposition of H is shown in Fig. 2b. The nodes of type (4.1) are B, C, E , and $Full(C) = Full(E) = \emptyset$, $Full(B) = \{j\}$. The nodes of type (4.2) are A, D, F , and the root which corresponds to the module V_H of H . We have $Full(A) = \{i\}$, $Full(D) = \{d\}$, $Full(F) = \{x\}$, $Full(V_H) = \{a\}$. There is only one node of type (4.3), namely G .

The principal prime hypergraphs are shown in boxes B, C , and E . The edges between these nodes and their successors start inside the boxes from the vertices to which substitutions are made (see Case (4.1) of the definition).

The box B contains not only the prime hypergraph but also the hyperedge j which forms $Full(B)$.

We now consider the formalization of this definition in MS logic.

A hypergraph H is represented by the relational structure $|H|_2 = \langle V_H \cup E_H, \mathbf{inc}_H \rangle$. The subscript 2 is a reminder of the *two* possibilities of quantifications arising from this representation: on vertices *and* on hyperedges (see the Appendix). Our objective is to build from $|H|_2$ a structure

$$\mathbf{Dec}(H) = \langle V_H \cup E_H \cup N_T, \mathbf{inc}_H, \mathbf{succ}_T, \mathbf{comp}_T \rangle$$

such that $N_T \cap (V_H \cup E_H) = \emptyset$, $\langle N_T, \mathbf{succ}_T \rangle = T(H)$, and $\mathbf{comp}_T = \{(x, w) / x \in N_T, w \in V_C \cup E_C$ where C is the component of H corresponding to node x of $T(H)\}$.

Hence this structure contains $|H|_2$ and, in addition, a structure representing $T(H)$ together with the relation \mathbf{comp}_T between the nodes of $T(H)$ and the corresponding components of H . Our aim is to construct the nodes of $T(H)$ from $V_H \cup E_H$ and to define its relations by MS formulas. Formally this means that $\mathbf{Dec}(H)$ is obtained from $|H|_2$ by an MS transduction. See the Appendix (or the survey papers [3, 6]) for formal definitions.

In [5], we proved a similar result for graphs G , represented by the less informative structures $|G|_1 = \langle V_G, \mathbf{edg}_G \rangle$, that do not permit quantifications on sets of edges, but given *with a linear order on the vertices*. In contrast, in the following proof, we need no linear order but we use quantifications on sets of hyperedges.

THEOREM 2.10. *There exists an MS-transduction associating $\mathbf{Dec}(H)$ with $|H|_2$ for every hypergraph H .*

Proof. The leaves of T are in bijection with the vertices of H : a leaf of T is a component of H of the form $H[\{x\}]$ for $x \in V_H$ and hence consists of x and the hyperedges a of H such that $H(a) = \{x\}$.

We let $E_1 \subseteq E_H$ be the set of hyperedges a such that $H(a)$ (the set of vertices of a) is a strong module of cardinality at least 2, and for $a, b \in E_1$ we let $a \sim_1 b$ if and only if $H(a) = H(b)$. There is a bijection between E_1/\sim_1 and the set of components C of H that are either of the form (4.2) of Definition 2.9 or of the form (4.1) (also of Definition 2.9) with $Full(C) \neq \emptyset$. Hence, one can select a cross-section X of \sim_1 , $X \subseteq E_1$ and we get in this way a representation of the nodes of $T(H)$ of these two forms.

We let $E_2 = E_H - E_1 - \{\text{the hyperedges of rank 1}\}$. For each $a \in E_2$ we let $C(a)$ be the smallest component containing a . It is necessarily of the form (4.1), and $a \in Full(C(a)) \cup E_P$ (because if $C(a) = Full(C(a)) * P[C_1, \dots, C_k]$ and $a \in C_i$, then C_i and not $C(a)$ would be the smallest component containing a).

Let $E'_2 = \{a \in E_2 / Full(C(a)) = \emptyset\}$. We let for $a, b \in E'_2$: $a \sim_2 b$ if and only if $C(a) = C(b)$. We choose a cross-section $Y \subseteq E'_2$ of \sim_2 . The elements of Y can be used to define the nodes C of type (4.1) such that $Full(C) = \emptyset$. Note that $X \cap Y = \emptyset$.

It remains to define the nodes of type (4.3), i.e., those corresponding to the strong modules that are not connected.

The connected strong modules are represented by the elements of the set $Z = X \cup Y \cup V_H$. We denote by $M(x)$ the module represented by $x \in Z$. We let $Z_3 \subseteq Z$ be the set of $x \in Z$ such that $M(x)$ is some C_i in the unique strong module of the form $C_1 \oplus \dots \oplus C_k$ with $k \geq 2$. We will denote by $N(x)$ this strong module. We let $x \sim_3 y$ if and only if $N(x) = N(y)$.

A cross-section U of \sim_3 , $U \subseteq Z_3$ can be used to represent the strong modules that are not connected (i.e., those of the form (4.3) of Definition 2.9). If $x \in U$, then the module it represents is $N(x)$.

Finally the structure $\mathbf{Dec}(H)$ can be constructed with domain

$$D = (V_H \cup E_H) \times \{1\} \cup (X \cup Y \cup V_H) \times \{2\} \cup U \times \{3\},$$

where $(u, 1)$ in D represents u in $V_H \cup E_H$, where $(u, 2)$ represents the node of $T(H)$ corresponding to $M(u)$, and $(u, 3)$ represents the node $N(u)$.

It is routine to see that $\mathbf{comp}_{T(H)}$ and $\mathbf{succ}_{T(H)}$ are definable by MS formulas. ■

For the hypergraph of the example of Definition 2.9 one obtains, for instance, the tree shown in Fig. 3.

Its internal nodes a, b, d, u, i, j, x are copies of the hyperedges with the same names. Leaves $1, 2, \dots, 14$ are copies of the vertices of H . We have $Z = \{1\}$. We use 1^+ to denote the copy of 1 used to represent $N(1) = G$ (see Fig. 2b); we have $M(1) = H[\{1\}]$.

We conjecture that the complexity of constructing the modular decomposition of a hypergraph is linear, by a suitable adaptation of the algorithm for graphs [12].

Möhring and Radermacher [16] review modular decompositions for several types of structures: graphs, set systems, and Boolean functions. Their set systems are hypergraphs such that no two

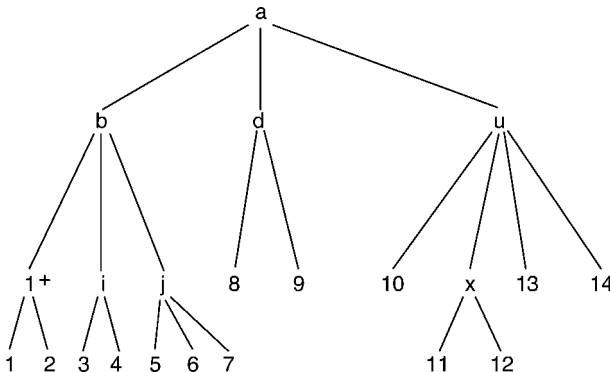


FIGURE 3

hyperedges have the same set of vertices, and the notion of modular decomposition they consider coincides with ours on these hypergraphs.

Bonizzoni and Della Vedova consider in [1] a different notion of modular decomposition for hypergraphs. (Their hypergraphs are actually set systems as in [16].) It is based on a notion of substitution such that, if K is a hypergraph substituted for a vertex x in a hypergraph H , then x is replaced by K in a natural way, and a hyperedge α of the form $\alpha' \cup \{x\}$ of H with α' nonempty is replaced by all the hyperedges of the form $\alpha' \cup \beta$, where β is a nonempty subset of V_K . The corresponding notion of modular decomposition generalizes that for graphs (if we consider graphs as hypergraphs with hyperedges of cardinality 2).

3. CONVEX HYPERGRAPHS

An *ordered hypergraph* (H, \leq) , i.e., a hypergraph H given with a linear order \leq on V_H , is *convex* if $H(a)$ is an interval for each $a \in E_H$. We say that a hypergraph H is *convex* if (H, \leq) is convex for some linear order \leq . We say that such an order *witnesses* the convexity of H . The hypergraph of Fig. 2a is convex.

The undirected graphs of the directed graphs $\mathbf{Bip}(H)$ for some convex hypergraph H are characterized by forbidden induced subgraphs by Tucker [18]. We will not use such characterization. The complexity of recognizing them is linear (Booth and Lueker [2]).

The notion of convexity is compatible with substitutions, modules, and modular decompositions. For example, if H and K are convex hypergraphs, then so is $H[K/x]$. If (H, \leq) is convex, and X is a strong module of H , then X is an interval for \leq . If H is convex, all its components corresponding to the nodes of its modular decomposition are convex. The verifications are easy, and we will not need these facts in the remainder of the paper.

If \leq is a linear order on V_H , we denote by \leq^{-1} the opposite linear order. If (H, \leq) is convex, then so is (H, \leq^{-1}) .

PROPOSITION 3.1. *If (H, \leq) is convex and H is prime, there are only two orders witnessing the convexity of H , namely \leq and \leq^{-1} .*

The proof technique for this proposition will be refined later in order to prove that the two linear orderings are definable by monadic second-order formulas.

This result is also interesting because of its similarity with the one saying that a prime undirected graph has at most one *transitive orientation* (i.e., an orientation such that if $x \rightarrow y$ and $y \rightarrow z$ are directed edges, then there is also a directed edge $x \rightarrow z$). See Kelly [14] for a detailed presentation of this result, actually due to Gallai.

For proving the proposition, we need some notation. We recall that for sets α, β we write $\alpha \perp \beta$ if and only if they *overlap*, i.e., if and only if $\alpha - \beta \neq \emptyset$, $\beta - \alpha \neq \emptyset$, and $\alpha \cap \beta \neq \emptyset$.

Let V be linearly ordered by \leq . If $\alpha, \beta \subseteq V$, we write $\alpha < \beta$ if and only if $\alpha \cap \beta = \emptyset$ and $x < y$ for every $x \in \alpha, y \in \beta$. We write $x|y|z$ if and only if either $x < y < z$ or $z < y < x$. This ternary relation called the *betweenness relation* of $<$, should be read as follows: y is *between* z and x . With its extension to pairwise disjoint sets we obtain the following obvious lemma.

LEMMA 3.2. *If α, β are intervals in (V, \leq) where \leq is a linear order, and $\alpha \perp \beta$, then the intervals $\alpha - \beta, \beta - \alpha$, and $\alpha \cap \beta$ satisfy $(\alpha - \beta) | \alpha \cap \beta | (\beta - \alpha)$.*

We will prove in particular that if H is prime, \leq witnesses its convexity, and m is the least element of V_H , then this order can be defined in a unique way from m and the memberships of the vertices in the hyperedges. The above lemma is a first step in the proof. It proves in particular that if m is in $\alpha - \beta$, then $m \leq x < y < z$ for every x in $\alpha - \beta$, every y in $\alpha \cap \beta$, every z in $\beta - \alpha$, where α and β are overlapping hyperedges. This gives a first approximation of \leq that can be refined into a complete characterization.

A *hyperpath* in a hypergraph H is a nonempty sequence of hyperedges (a_1, a_2, \dots, a_n) such that, if $n \geq 2$:

$$\begin{aligned} H(a_i) \perp H(a_{i+1}) & \quad \text{for } i = 1, \dots, n - 1, \\ H(a_i) \cap H(a_j) = \emptyset & \quad \text{for } 1 \leq i < i + 2 \leq j \leq n. \end{aligned}$$

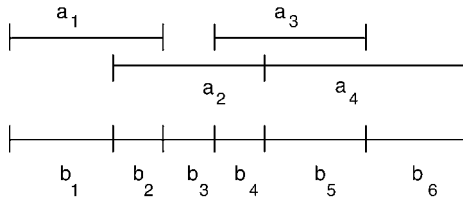


FIGURE 4

LEMMA 3.3. *If H is a connected hypergraph (not necessarily convex) and if x, y are vertices not in a same hyperedge, there exists a hyperpath (a_1, \dots, a_n) in H with $x \in H(a_1), \dots, y \in H(a_n)$.*

Proof. This follows from Lemma 1.2 of [9], but a direct proof is actually easy. ■

Proof of Proposition 3.1. Let (H, \leq) be convex with min and max as least and greatest elements. They are not in a same hyperedge a , because otherwise, $H(a)$ would contain all vertices and H would not be prime.

By Lemma 3.3, there exists a hyperpath (a_1, \dots, a_n) such that $\min \in H(a_1), \max \in H(a_n)$. We have $n \geq 2$. Assume (H, \leq') is convex where \leq' is another linear order. Consider the following sets:

$$\begin{aligned}
 &H(a_1) - H(a_2), \quad H(a_n) - H(a_{n-1}), \\
 &H(a_i) \cap H(a_{i+1}) \quad \text{for } i = 1, \dots, n - 1 \\
 &H(a_i) - (H(a_{i-1}) \cup H(a_{i+1})) \quad \text{for } i = 2, \dots, n - 1.
 \end{aligned}$$

Some of these sets may be empty; see the example of Fig. 4.

They form a partition of V_H in intervals relative to \leq . Let B be the set of these intervals that are nonempty. It is linearly ordered under $<$. It follows from Lemma 3.2 that B is also linearly ordered under $<'$ with the same betweenness relation, since this relation is definable in terms of memberships of vertices in the sets $H(a)$.

Figure 4 shows an example, with four hyperedges a_1, \dots, a_4 represented as intervals. For readability, we denote by a_i the set $H(a_i)$. (We will do the same in Figs. 5 and 6.) We have $H(a_3) - (H(a_2) \cup H(a_4)) = \emptyset$ in the case shown in Fig. 4.

We continue the proof. Without loss of generality we can assume that $H(a_1) - H(a_2) <' H(a_n) - H(a_{n-1})$. If this is not the case, we replace $<'$ by the opposite order. Hence $<$ and $<'$ coincide on B since they are both characterized by the same membership relations and $H(a_1) - H(a_2)$ is smaller than $H(a_n) - H(a_{n-1})$ for the both of them.

It remains to prove that $<$ and $<'$ coincide on each set in B . This is trivial if each set of B is a singleton. Let $b \in B$ have cardinality ≥ 2 . Since H is prime, b is not a module; hence $H(a) \perp b$ for some $a \in E_H$. Let us choose one such a and consider the set B' of sets $c \cap H(a), c - H(a)$ for all $c \in B$. Since $H(a)$ is not a subset of any $c \in B$ (because the sets in B are pairwise disjoint and $H(a) \perp b$), the set B' is a set of \leq -intervals which forms a partition of V_H . The \leq -interval b of B is replaced in B' by two \leq -intervals b' and b'' (with $b = b' \cup b'', b' \cap b'' = \emptyset$); at most one other set of B is divided into two sets.

The set B' is again linearly ordered by $<$, and also, in the same way, by $<'$ since the relative orders of b' and b'' under $<$ and $<'$ are defined by membership relations (by Lemma 3.2) in the same ways. Letting $b' = b \cap H(a), b'' = b - H(a)$, let c be an interval of $B - \{b\}$ such that $H(a) \cap c \neq \emptyset$. If $b < c$ we have $b'' < b' < H(a) \cap c$. If $c < b$ we have $H(a) \cap c < b' < b''$. The same holds with $<'$ instead of $<$. Hence the ordering of B' is the same with respect to $<$ and $<'$.

By repeating this step one reaches a partition B^* of V_H in \leq -intervals which are all singletons. The orders $<$ and $<'$ coincide on the set B^* (by induction) and hence on V_H . This completes the proof of Proposition 3.1. ■

We will now refine this argument and make it into an MS definition of the two orderings of Proposition 3.1.

The MS formulas used in the remainder of this section are intended for the structures $|H|_2$ representing hypergraphs H and will use quantifications on hyperedges and sets of hyperedges.

PROPOSITION 3.4. *There exist MS formulas $\mu(m)$ and $\omega(m, x, y)$ such that, for every prime hypergraph H :*

- (1) $|H|_2 \models \mu(m)$ if and only if (H, \leq) is convex for some linear order \leq with least element m ;
- (2) if \leq is a linear order such that (H, \leq) is convex, if its least element is m , for every two vertices x, y , then $|H|_2 \models \omega(m, x, y)$ if and only if $x \leq y$.

Again, we need some definitions. A *chain* is a sequence (a_1, a_2, \dots, a_n) of hyperedges such that $H(a_i) \perp H(a_{i+1})$ for each $i = 1, \dots, n - 1$. (A hyperpath is thus a chain.)

Let m, x, y be pairwise distinct elements of V_H . An (m, x, y) -*separating chain* is a chain (a_1, a_2, \dots, a_n) such that:

- (i) $m \in H(a)$ where a is the first element,
- (ii) $\{x, y\} \perp H(b)$ where b is the last element,
- (iii) no subsequence $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ with $i_1 < i_2 < \dots < i_k$ is a chain having properties (i) and (ii).

It is clear that, from a chain satisfying properties (i) and (ii), one can extract an (m, x, y) -separating chain.

LEMMA 3.5. *Let H be a hypergraph. Let (a_1, \dots, a_n) be an (m, x, y) -separating chain with $n \geq 2$.*

- (1) We do not have $H(a_i) \perp H(a_j)$ for any $i < i + 2 \leq j$.
- (2) We do not have $H(a_i) \subseteq H(a_j)$ for any $i < j$.

Proof. (1) If we have $H(a_i) \perp H(a_j)$ for some $i < i + 2 \leq j$ then we can delete a_{i+1}, \dots, a_{j-1} which contradicts the *minimality condition* of the definition of separating chains (condition (iii)).

(2) Assume on the contrary that we have $H(a_i) \subseteq H(a_j)$ for some $i < j$. Let (i, j) be the lexicographically first pair with $H(a_i) \subseteq H(a_j), i < j$.

If $i = 1$ then $m \in H(a_j)$ and we can delete a_1, \dots, a_{j-1} from the chain, contradicting the *minimality condition*. Hence $i > 1$. Since $H(a_{i-1}) \perp H(a_i)$, and $H(a_{i-1}) - H(a_j) \neq \emptyset$ by the minimality of i we obtain $H(a_j) \perp H(a_{i-1})$ contradicting (1). Hence no such pair (i, j) can exist. ■

LEMMA 3.6. *Let H be a hypergraph, and let (a_1, \dots, a_n) be an (m, x, y) -separating chain. If $H(a_j) \subseteq H(a_i)$ for $i < i + 2 \leq j$ then $H(a_k) \subseteq H(a_i)$ for all $k, i + 2 \leq k \leq n$.*

Proof. Let us assume $H(a_j) \subseteq H(a_i)$ and $i < i + 2 \leq j$, where j is minimal with these properties. Since $H(a_j) \perp H(a_{j-1})$, and $H(a_{j-1}) - H(a_i) \neq \emptyset$ by the minimality of j , we obtain $H(a_i) \perp H(a_{j-1})$. Hence $j = i + 2$ by Lemma 3.5.1 and so $H(a_{i+2}) \subseteq H(a_i)$. Assuming the conclusion is false, let k be the smallest integer such that $k > i + 2$ and $H(a_k)$ is not included in $H(a_i)$. Since $H(a_{k-1}) \subseteq H(a_i)$ and $H(a_k) \perp H(a_{k-1})$ we have $H(a_i) \perp H(a_k)$ but this contradicts Lemma 3.5.1. ■

An index $i + 1$ such that $H(a_{i+2}) \subseteq H(a_i)$ is called a *turn*. (In the example of Fig. 5, the indices 3, 7, and 10 are turns.)

LEMMA 3.7. *Let (H, \leq) be convex and $m = \min_H$ be the least element of V_H . Let (a_1, \dots, a_n) be an (m, x, y) -separating chain. Let i_1, \dots, i_k be the turns with $i_1 < i_2 < \dots < i_k$ (we may have $k = 0$ and if $k \geq 1$ we have $i_1 > 1$ and $i_k < n$).*

- (1) Each subsequence $(a_1, \dots, a_{i_1}), (a_{i_1}, \dots, a_{i_2}), \dots, (a_{i_{k-1}}, \dots, a_{i_k}), (a_{i_k}, \dots, a_n)$ is a hyperpath.
- (2) For each j we have

$$H(a_{i_j + 1}) \cup \dots \cup H(a_{n-1}) \cup H(a_n) \subseteq H(a_{i_j - 1}).$$

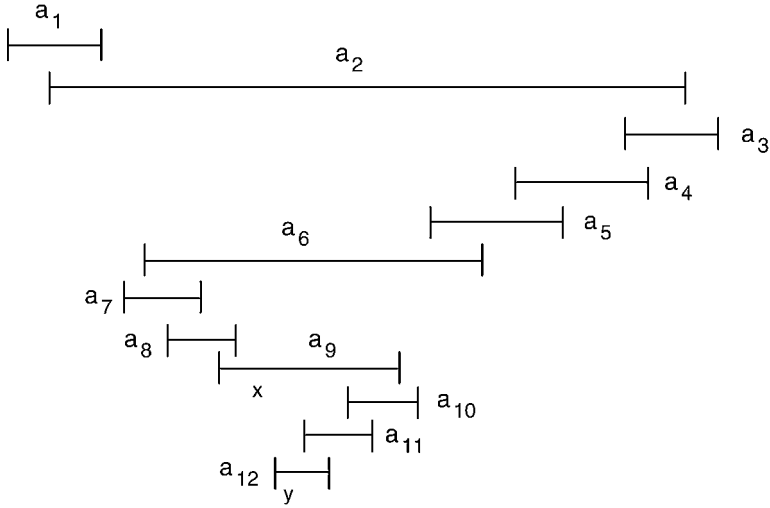


FIGURE 5

(3) Assume that $y \in H(a_n)$, $x \notin H(a_n)$. If $x \notin H(a_1) \cup \dots \cup H(a_n)$, then $k = 0$ and $y < x$; otherwise:

(3.1) either $x \in H(a_{n-1})$ and $y < x$ if k is odd, and $x < y$ if k is even,

(3.2) or $x \in H(a_{i_k-1})$, and $x < y$ if k is odd, and $y < x$ if k is even.

Proof. (1) By Lemmas 3.5 and 3.6, if $H(a_i) \cap H(a_j) \neq \emptyset$ for $i + 1 < j$ then $H(a_j) \subseteq H(a_i)$ and there is a turn between i and j . Hence if a subsequence $(a_i, a_{i+1}, \dots, a_j)$ has no turn we have $H(a_p) \cap H(a_q) = \emptyset$ for $i \leq p < p + 1 < q \leq j$ and this sequence is a hyperpath.

(2) Immediate consequence of Lemma 3.6.

(3) Observe that $H(a_1) \cup \dots \cup H(a_n) = H(a_1) \cup \dots \cup H(a_{i_1})$ by Lemma 3.6.

First case: $x \notin H(a_1) \cup \dots \cup H(a_n)$. Then $k = 0$ because otherwise, (a_1, \dots, a_n) could be shortened into (a_1, \dots, a_j) for some $j \leq i_1$.

Since $H(a_1) \cup \dots \cup H(a_n)$ is an interval that contains $m = \min_H$ and y , and that does not contain x , we have $y < x$. See Fig. 6a.

Second case: $x \in H(a_1) \cup \dots \cup H(a_{n-1})$ and $y \in H(a_n)$. Let j be the largest index such that $x \in H(a_j)$, $j < n$. We must have $y \in H(a_j)$; otherwise (a_1, a_2, \dots, a_j) is a shorter (m, x, y) -separating chain.

Note that $x \notin H(a_t)$ for any $t > j$ and that $y \notin H(a_t)$ for any t with $j + 1 < t < n$ (otherwise the (m, x, y) -chain could be shortened into (a_1, \dots, a_t)).

Subcase 1: $y \in H(a_{j+1})$. Hence $y \in H(a_j) \cap H(a_{j+1})$ and $n = j + 1$. We have $x \in H(a_{n-1})$, and this corresponds to Subcase (3.1) of the statement.

We may have $k = 0$ or $i_k \leq j$ with k even and, in both cases,

$$H(a_j) - H(a_{j+1}) < H(a_j) \cap H(a_{j+1});$$

hence $x < y$. See Fig. 6b.

Otherwise $i_k \leq j$ with k odd, then $H(a_j) \cap H(a_{j+1}) < H(a_j) - H(a_{j+1})$, and hence $y < x$. See Fig. 6c.

Subcase 2: $y \notin H(a_{j+1})$. This means that $n \geq j + 2$ and $H(a_n) \cap H(a_j) \neq \emptyset$. Hence by Lemma 3.6, we have $H(a_{j+2}) \subseteq H(a_j)$; i.e., $j + 1$ is a turn say i_p .

It follows that i_p is the last turn, i.e., $p = k$ and $j = i_k - 1$. Otherwise, the chain could be shortened into (a_1, \dots, a_{i_k-1}) because $H(a_{i_k-1})$ overlaps $\{x, y\}$: $i_k - 1 > j$ implies that $x \notin H(a_{i_k-1})$ and the fact that i_k is a turn implies that $H(a_n) \subseteq H(a_{i_k-1})$ and so $y \in H(a_{i_k-1})$. We are in Subcase (3.2) of the statement. Now there are two cases.

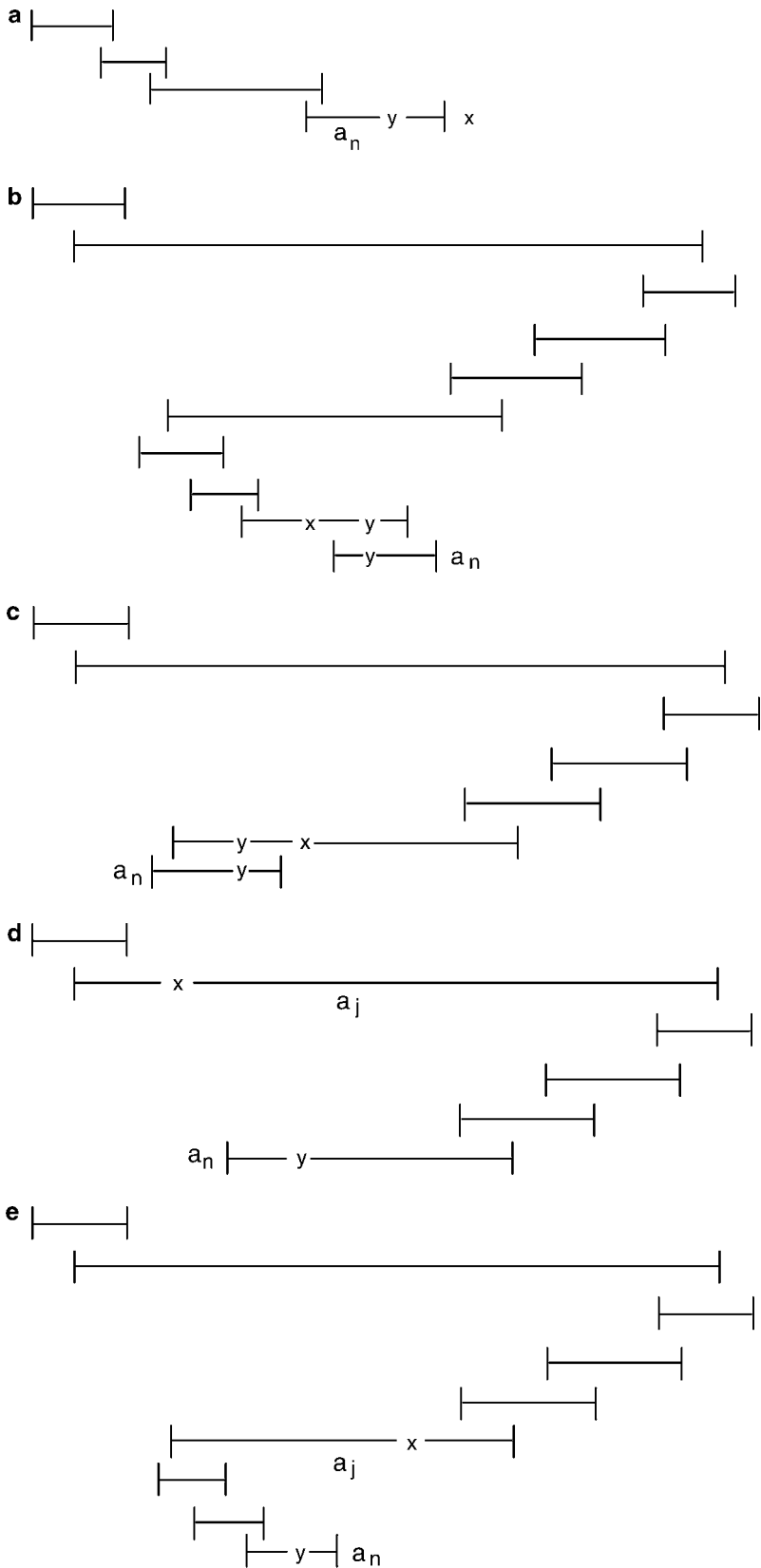


FIGURE 6

If k is odd then $x < z$ for all $z \in H(a_t)$, $t > j$; hence $x < y$. (See Figs. 6d and 5). If k is even then $z < x$ for all such z and $y < x$. See Fig. 6e. ■

The last lemma shows that, if we know an (m, x, y) -separating chain, we can compare x and y . The next lemma proves the existence of such separating chains in prime convex hypergraphs for any x and y .

Let $a \in E_H$, where H is any hypergraph and $H(a)$ has at least two vertices. Let $E(a)$ be the set of hyperedges b such that there is a chain $(a_1, a_2, a_3, \dots, a_n)$ with $a_1 = a, a_n = b$. Let $W(a) = \bigcup \{H(b)/b \in E(a)\}$. With this notation we have:

LEMMA 3.8. *The set $W(a)$ is a module of H .*

Proof. Let $W = W(a)$. Let $c \in E_H$ such that $H(c) \perp W$. There exist $x \in W - H(c)$, $y \in H(c) - W$, and $z \in W \cap H(c)$. There exists a chain (b_1, \dots, b_m) of elements of $E(a)$ such that $x \in H(b_1)$ and $z \in H(b_m)$ (because $x, z \in W$).

There exists a smallest i such that $H(b_i) \cap H(c) \neq \emptyset$. We have $y \in H(c) - H(b_i)$. If $i = 1$ then $x \in H(b_1) - H(c)$. If $i > 1$ then $H(b_i) - H(c) \neq \emptyset$ because if $H(b_i) \subseteq H(c)$ we have $H(b_{i-1}) \cap H(c) \neq \emptyset$ and i is not minimal. Hence $H(c) \perp H(b)$ for some $b \in E(a)$. Hence $c \in E(a)$ and $H(c) \subseteq W$, a contradiction. Hence $W(a)$ is a module. ■

Proof of Proposition 3.4. Let H be prime, let \leq witness the convexity of H , and let m be the least element of V_H .

CLAIM. *For every x, y with $x \neq y$, $x \neq m$, and $y \neq m$, there exists an (m, x, y) -separating chain.*

Proof of claim. Since H is prime, there exists $a \in E_H$ such that $H(a) \perp \{x, y\}$. We let $W(a)$ be as in Lemma 3.8. Hence $W(a)$ is a module with at least two vertices (because H is prime and hence its hyperedges have rank at least 2). Hence $W(a) = V_H$ and $m \in W(a)$. Hence there exists a chain a_1, \dots, a_n such that $a_n = a$ and $m \in H(a_1)$. By removing some elements if necessary, we make it into an (m, x, y) -separating chain. This concludes the proof of the claim. ■

One can write an MS_2 -formula with free variables m, x, y expressing the following:

(1) m, x, y are pairwise distinct vertices,

(2) there exists a subset A of E_H such that

(*) there is $a \in A$ with $m \in H(a)$, there is $b \in A$ with $\{x, y\} \perp H(b)$, and (a, b) belongs to the restriction to A of the reflexive and transitive closure of the relation \perp ,

(**) no proper subset of A satisfies (*).

It follows then that the elements of A as specified by (2) form an (m, x, y) -separating chain.

An MS formula $\rho(A, m, x, y, u, v)$ can express that u is before v on this chain. Another formula $\rho'(A, B, C, m, x, y)$ can express that $B \cup C$ is the set of hyperedges the indices of which are turns of this chain, that B is the set of those with odd indices, and C is the set of those with even indices.

An MS formula $\varphi(m, x, y)$ can be written that uses ρ and ρ' as subformulas and expresses, under the assumption that H is prime and m is the least element of V_H for a linear order \leq such that (H, \leq) is convex, that there exists a separating chain for (m, x, y) with turns of such ranks that we can ensure that $x < y$ by means of the conditions of Lemma 3.7.3.

We let $\mu(m)$ be the formula that holds if and only if

“the relation R defined by $x R y$

if and only if

$x = m$ or $x = y$ or $x \neq y \neq m, x \neq m$, and $\varphi(m, x, y)$ ”

is a linear order on V_H for which (H, R) is convex.

We take for $\omega(m, x, y)$ the formula $\mu(m) \wedge (x = m \vee x = y \vee \varphi(m, x, y))$.

We now verify that, assuming H prime:

(1) $|H|_2 \models \mu(m)$ if and only if (H, \leq) is convex for some linear order \leq with least element m ;

(2) *if \leq is a linear order such that (H, \leq) is convex, if its least element is m , for every two vertices x, y , then $|H|_2 \models \omega(m, x, y)$ if and only if $x \leq y$.*

We first consider (1). The “only if” direction is clear from the definition of μ . Conversely, if (H, \leq) is convex with least element m , then for any two elements x and y (which are distinct and distinct from m) either $\varphi(m, x, y)$ or $\varphi(m, y, x)$ holds, and we have respectively $x < y$ or $y < x$. Hence the relation R specified as in the definition of formula μ is the linear order \leq and $\mu(m)$ holds.

We now check (2). If (H, \leq) is convex with least element m , then from the observations made for the proof of (1), and the definitions, we get that $|H|_2 \models \omega(m, x, y)$ if and only if $x \leq y$.

If conversely $\omega(m, x, y)$ holds, then because of the validity of $\mu(m)$, the condition $\varphi(m, x, y)$ defines a linear order R (see the definition of μ) for which H is convex with least element m . ■

We thus have the following result:

MAIN THEOREM 3.9. *There exists MS_2 -formulas $\theta(m)$ and $\omega(m, x, y)$ satisfying the following conditions, for every hypergraph H :*

(1) *For every $m \in V_H$, $|H|_2 \models \theta(m)$ if and only if H is prime and (H, \leq) is convex for some linear order \leq with least element m*

(2) *H is prime and convex if and only if $|H|_2 \models \exists m. \theta(m)$*

(3) *For every $m \in V_H$, such that $|H|_2 \models \theta(m)$, for every $x, y \in D_H$, we have*

$$|H|_2 \models \omega(m, x, y) \quad \text{if and only if} \quad x \leq y,$$

where \leq is the unique linear order with least element m such that (H, \leq) is convex.

Proof. We let $\theta(m)$ express that the considered hypergraph is prime (which needs no reference to any order on the vertices) and that $\mu(m)$ holds.

Assertions (1)–(3) hold by Proposition 3.4. ■

This proof actually gives another proof of Proposition 3.1.

4. DECIDABLE MONADIC THEORIES OF SETS OF BIPARTITE GRAPHS

We recall that MS_2 refers to MS logic over structures representing graphs in such a way that quantified variables denote edges, vertices, and sets thereof, whereas MS_1 refers to MS logic where quantified variables can only denote vertices and sets of vertices.

We say that a set of finite graphs has a *decidable MS_1 -theory* (resp. *MS_2 -theory*) if there exists an algorithm that decides whether a given MS_1 (resp. MS_2) closed formula is true for some graph in this set.

Seese has proved [17, Theorem 8] that a set of graphs L having a decidable MS_2 -theory has bounded tree-width, and he has conjectured that a set having a decidable MS_1 theory is of the form $\tau(B)$ where B is the set of finite binary trees and τ is a $(1, 1)$ -definable MS transduction (see the Appendix for definitions). Intuitively, this means that each graph in L can be defined “inside” some tree from B , by MS formulas evaluated in this tree. By Theorem 5.6.8 of [6], a set of graphs is of the form $\tau(B)$ as above if and only if it has bounded *clique-width*.

This conjecture has been proved for specific classes of graphs: for planar graphs [17, Theorem 7], for uniformly k -sparse graphs, i.e., graphs such that every subgraph has a number of edges at most k times the number of vertices (this is the main theorem of [9] and concerns the class of planar graphs) for chordal graphs such that each vertex belongs to at most k maximal cliques [4, Theorem 4.8]. None of the last two cases subsumes the other.

We review the notion of clique-width studied in detail in [11].

A k -graph is a simple graph given with a coloring of its vertices by colors among $1, \dots, k$. (We do not require that neighbor vertices have different colors, we only require that every vertex has one and only one color.) We use the following operations on k -graphs:

- (1) the disjoint union of two k -graphs, denoted by \oplus (if the two argument graphs are not disjoint, we replace one of them by a disjoint copy),
- (2) the unary operation $\rho_{i \rightarrow j}$ that replaces every color label i by j , and
- (3) the unary operation $\alpha_{i,j}$ that adds to a graph new directed edges from any vertex with color i to any vertex with color j (since we consider simple graphs, a new edge from x to y is added only if there is not already one).

If we want to construct undirected graphs, we use instead of $\alpha_{i,j}$:

- (3') the unary operation $\eta_{i,j}$ that adds to a graph new undirected edges between any vertex with color i and any vertex with color j (a new edge is added only if there is not already one).

The *clique-width* of a k -graph is the minimum number of colors used by an algebraic expression defining this graph and built with these operations from the elementary graph consisting of one vertex colored by 1. A graph is considered identical to the 1-graph obtained by labeling all its vertices by 1.

We denote by $\mathbf{cwd}(G)$ the clique-width of a graph G . Trees have \mathbf{cwd} at most 3, and the cographs (see for instance [5]) are the graphs of \mathbf{cwd} with at most 2. The set of all graphs having \mathbf{cwd} at most any fixed k has a decidable MS_1 -theory. The conjecture by Seese is thus a kind of converse.

Here, we will consider this conjecture for the class \mathbb{B} of directed bipartite graphs of the form $\mathbf{Bip}(H)$ for some hypergraph. These graphs are the finite directed simple graphs G such that V_G is partitioned into two sets V and W (corresponding respectively to the vertices and to the hyperedges of the considered hypergraph) with V nonempty, every edge is directed from a vertex in V to one in W , and every vertex in W is the end of at least one edge. The sets V and W are uniquely defined for each G in \mathbb{B} .

From the logical point of view we will consider that $|H|_2$ coincides with $|\mathbf{Bip}(H)|_1$: the relations \mathbf{inc}_H of $|H|_2$ and \mathbf{edg}_G of $|G|_1$ where $G = \mathbf{Bip}(H)$ are the same except for their names.

The substitution $H = G[K/x]$ where H , G , and K are bipartite graphs is defined by:

$$H = \mathbf{Bip}(\mathbf{Bip}^{-1}(G)[\mathbf{Bip}^{-1}(K)/x]).$$

Hence it is defined only if x is a vertex of G that corresponds to a vertex of $\mathbf{Bip}^{-1}(G)$ (and not to a hyperedge).

The notions of a prime bipartite graph, of a convex bipartite graph, or of a module or a component in a bipartite graph follow immediately via \mathbf{Bip} from the corresponding notions for hypergraphs. In particular, we let $P\text{Prime}(G) = \mathbf{Bip}(P\text{Prime}(\mathbf{Bip}^{-1}(G)))$ where G is a graph in \mathbb{B} . We will consider in some detail the convex bipartite graphs.

A useful tool will be the representation of a convex hypergraph (H, \leq) by a labeled directed graph $D = \mathbf{DG}(H, \leq)$ defined as follows. We let:

- (1) $V_D = V_H \cup E_H$,
- (2) $p_D \subseteq V_D$ be the unary relation such that, for $x \in V_D$, $p_D(x)$ holds if and only if $x \in V_H$,
- (3) $\mathbf{edg}_D(x, y)$ hold if and only if either $x, y \in V_H$ and y is the successor of x with respect to \leq , or $y \in E_H$ and x is the \leq -smallest element of $H(y)$, or $y \in E_H$ and x is the \leq -largest element of $H(y)$.

It follows that D is a directed graph of indegree at most 2. We consider p_D as a labeling relation. Hence, D is isomorphic to a 2-graph (where 1 (resp. 2) labels the vertices in p_D (resp. not in p_D)). We let $|D|_1 = \langle V_D, p_D, \mathbf{edg}_D \rangle$. (The labeling relation p_D makes it possible to distinguish the edges representing the successor relation of \leq from the others. In many cases this relation can be reconstructed from the degrees of vertices but not always. Consider for instance the graph $x \leftarrow y \rightarrow z$: each of x or z can be considered as a hyperedge of rank 1.)

The structure $|H|_2 = \langle V_H \cup E_H, \mathbf{inc}_H \rangle$ can be reconstructed from $|\mathbf{DG}(H, \leq)|_1$ as follows:

- (4) $V_H \cup E_H = V_D$ (with $D = \mathbf{DG}(H, \leq)$),
- (5) $\mathbf{inc}_H(x, y) \Leftrightarrow p_D(x) \wedge \neg p_D(y)$ holds, and either $\mathbf{edg}_D(x, y)$ holds or there exist u, v such that

$$p_D(u) \wedge p_D(v) \wedge \mathbf{edg}_D(u, y) \wedge \mathbf{edg}_D(v, y)$$

holds and x is on a path in D from u to v all vertices of which satisfy p_D .

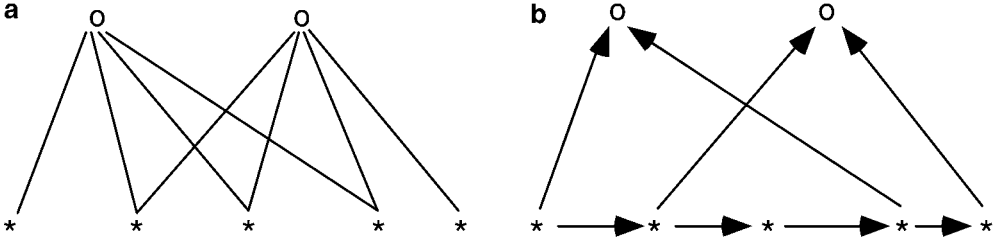


FIG. 7. (a) A convex hypergraph H . (b) The graph $\mathbf{DG}(H, \leq)$.

An example can be seen in Fig. 7.

The transformations of structures of $(|H|_2, \leq)$ into $|\mathbf{DG}(H, \leq)|_1$ and of $|\mathbf{DG}(H, \leq)|_1$ into $|H|_2$ (for every convex hypergraph (H, \leq)) are MS transductions, because the definitions of the relations of one structure in terms of the other (see clauses (1)–(3) and (4)–(5)) are expressible in MS logic.

PROPOSITION 4.1. *If a set of convex prime bipartite graphs has a decidable MS_1 theory, then it has bounded clique-width.*

Proof. Let L be a set of convex prime bipartite graphs; L is a subset of \mathbb{B} . Let L' be the corresponding set of convex prime hypergraphs. Consider the following transformation of structures

$$|H|_2 \mapsto (|H|_2, \leq) \mapsto |\mathbf{DG}(H, \leq)|_1,$$

where H is in L' , and for each H, \leq is one of the two linear orders witnessing that H is convex (by Proposition 3.1).

Since \leq is MS definable in $|H|_2$ by Theorem 3.9, the first transformation is an MS transduction. It uses as parameter a set M intended to be $\{m\}$ for m the least element of the order \leq to describe; the formula μ of Proposition 3.4 verifies the correctness of the choice of a parameter.

The second transformation is also an MS transduction, hence so is their composition β by Theorem A.2.1 of the Appendix.

It follows from Theorem A.2.2 that the set of structures $|\mathbf{DG}(H, \leq)|_1$ for H in L' has a decidable MS theory since it is the image under an MS transduction (namely β) of a set having a decidable MS theory (the set of graphs $\mathbf{DG}(H, \leq)$ for H in L' and \leq witnessing its convexity).

The 2-graphs $\mathbf{DG}(H, \leq)$ represented by these structures have indegree at most 2; hence it follows from Lemma 3.3 and Theorem 4.1 of [9] (but a direct proof is easy) that they have a decidable MS_2 theory (because every MS_2 formula can be transformed into an equivalent MS_1 formula).

From Theorem 8 of [17], or Theorem 4.1.3 of [4], it follows that they have bounded tree-width. They have thus bounded clique-width by Theorem 5.5 of [11]. Now the transformation of $|\mathbf{DG}(H, \leq)|_1$ into $|H|_2 = |\mathbf{Bip}(H)|_1$ is an MS transduction. It follows from Corollary 5.6.9 of [6] (saying that the image of a set of graphs of bounded clique-width under a (1,1)-definable MS transduction has bounded clique-width) that the set $\mathbf{Bip}(H)$ for H in L' , i.e., the set L , has bounded clique-width. ■

The following fact shows that this proposition is not trivial.

FACT 4.2. *The set of all prime convex bipartite graphs does not have a decidable MS_1 theory.*

Proof. Let L be the set of all prime convex bipartite graphs and let M be its image under β . If L had a decidable MS_1 theory, then M would have bounded tree-width, by the proof of Proposition 4.1.

However, consider a complete graph K with a linearly ordered set of vertices $\{v_1, v_2, \dots, v_n\}$. Let E be the set of edges of K . One has in M the incidence graph of K , namely, a graph H with set of vertices $V \cup E$ and an edge from x to y if and only if y is in E and x is one of the vertices of y . The graph K is a minor of this graph; hence H has tree-width at least the tree-width of K which is $n - 1$. Hence, the set M has unbounded tree-width, a contradiction. ■

LEMMA 4.3. *There exists a (2, 2)-definable MS transduction associating with every hypergraph the set of its prime subhypergraphs. There exists a (1, 1)-definable MS transduction doing the same for bipartite graphs.*

Proof. For the first assertion, a transduction τ_2 on hypergraphs can be constructed so as to work as follows. Let H be a hypergraph. The transduction τ_2 takes as its parameter two sets X and Y subject to the following monadic second-order conditions

- (1) $X \subseteq V_H$
- (2) $Y \subseteq E_{H[X]}$
- (3) the subhypergraph of H with X as a set of vertices and Y as a set of hyperedges is prime.

Condition (3) is actually first order, by straightforward translation from the definitions. As output, τ_2 is defined to produce the structure $\langle X \cup Y, R \rangle$, where R is the restriction of \mathbf{inc}_H to $X \cup Y$. We can obtain in this way all prime subhypergraphs of H . A corresponding transduction for bipartite graphs (defined via the bijection **Bip**) is denoted by τ_1 . ■

PROPOSITION 4.4. *Let L be a set of convex bipartite graphs having a decidable MS_1 theory. Then*

- (1) *the set of subgraphs of the graphs in L that are of the form **Bip**(H) for some prime hypergraph H has a decidable MS_1 theory and has bounded clique-width.*
- (2) *L has bounded clique-width.*

Proof. Let L be as in the statement.

(1) It follows that $\tau_1(L)$ (where τ_1 is as in Lemma 4.3) is the set of its prime subgraphs and also has a decidable MS_1 theory (by Theorem A.2.2). Since its elements are convex (because convexity is preserved by taking subhypergraphs), they have bounded clique-width by Proposition 4.1. Let m be this bound. Hence, the set $P\text{Prime}(L)$ defined as the union of the sets $P\text{Prime}(G)$ for G in L has bounded clique-width.

(2) It follows from Theorem 2.8 and Definition 2.9 that every bipartite graph G can be constructed in terms of disjoint unions, of additions of vertices corresponding to “full hyperedges,” and of substitutions of bipartite graphs for vertices in the bipartite graphs belonging to $P\text{Prime}(G)$.

We now observe that these three types of operations on graphs do not increase clique-width. For defining bipartite graphs, we will consider expressions that produce labeled graphs such that the vertices corresponding to hyperedges in the transformation **Bip** have a special label, say \S , not used to label the others. We let $\mathbf{cwd}'(G)$ be the least number of labels apart from \S used in the construction of such a special expression defining G . A routine proof yields:

$$\mathbf{cwd}(G) \leq \mathbf{cwd}'(G) + 1 \leq 2 \mathbf{cwd}(G). \quad (1)$$

We have easily:

$$\mathbf{cwd}'(G \oplus G') = \mathbf{Max}\{\mathbf{cwd}'(G), \mathbf{cwd}'(G')\} \quad (2)$$

$$\mathbf{cwd}'(A * G) = \mathbf{cwd}'(G). \quad (3)$$

Now if $G = H[K/x]$ (where this substitution corresponds to hypergraph substitution via **Bip**, see the beginning of this section) we have:

$$\mathbf{cwd}'(G[H/x]) = \mathbf{Max}\{\mathbf{cwd}'(G), \mathbf{cwd}'(H)\}. \quad (4)$$

For proving the last assertion, it is enough to substitute for the constant denoting x in an expression denoting G the expression $\rho^*(f)$ where f is an expression that defines H and ρ^* is a sequence of operations of the form $\rho_{p \rightarrow 1}$ that rename into 1 all the labels of H which are not the special label \S . (We recall that every vertex is defined in an expression as a singleton subgraph with label 1. This label is changed in the expression by the operations of the form $\rho_{1 \rightarrow i}$ or $\rho_{1 \rightarrow \S}$.)

We now complete the main proof. The prime subgraphs of the graphs in L have \mathbf{cwd} bounded by m and hence \mathbf{cwd}' bounded by $2m - 1$ (by (1)). The components of the graphs in L have \mathbf{cwd}' bounded

by $2m - 1$. This follows from their inductive characterizations obtained in Theorem 2.8 and the above properties (2–4). Hence the graphs in L have **cwd** bounded by $2m$, by (1) again. ■

The class of convex bipartite graphs for which Seese’s conjecture holds is incomparable with the class of chordal graphs, and with the class of uniformly k -sparse graphs, for any k . Hence, we have obtained a new case of validity of this conjecture, incomparable with the previously known ones.

Remark. The proof of Proposition 4.4 applies to slightly more than the convex bipartite graphs. Let \mathcal{L}_k be the family of bipartite graphs ($\mathcal{L}_k \subseteq \mathbb{B}$), the prime subgraphs of which are either convex or uniformly k -sparse. Then a subset L of \mathcal{L}_k having MS_1 theory has bounded clique-width. This is so because by the main theorem of [9], if a set of uniformly k -sparse graphs has a decidable MS_1 theory, then it also has a decidable MS_2 theory and hence bounded tree-width and also bounded clique-width.

Uniformly k -sparse bipartite graphs form a somewhat natural family. A bipartite graph is uniformly k -sparse if and only if the corresponding hypergraph is m -rank-degree-bounded, for some m . We say that a hypergraph H is m -rank-degree-bounded if $E_H = A \cup B$, the hypergraph $\langle V_H, A \rangle$ has rank at most m (each hyperedge has at most m vertices), and the hypergraph $\langle V_H, B \rangle$ has degree at most m (each vertex belongs to at most m hyperedges).

FACT 4.5. (1) If H is k -rank-degree-bounded then $\mathbf{Bip}(H)$ is uniformly $2k$ -sparse.

(2) If $\mathbf{Bip}(H)$ is uniformly k -sparse then H is k -rank-degree-bounded.

The proof is easy. (For (2), we use the fact that a graph is uniformly k -sparse if and only if it has an orientation of indegree at most k . A proof of this well-known lemma is given in [9, Lemma 3.1]). This fact gives a concrete understanding of uniform sparseness for bipartite graphs.

OPEN PROBLEM 4.6. It remains open to extend these results to larger classes of bipartite graphs.

A natural candidate is the class of bipartite graphs we obtain by replacing intervals relative to a linear order on vertices by paths of a tree on the set of vertices. The corresponding hypergraphs have been considered in [13].

APPENDIX: A REVIEW OF DEFINITIONS CONCERNING MONADIC SECOND-ORDER LOGIC

Let R be a finite ranked set of symbols where each element r in R has a rank $\rho(r)$ in \mathbb{N}_+ . A symbol r in R is a $\rho(r)$ -ary relation symbol. An R -(*relational*) *structure* is a tuple $S = \langle D_S, (r_S)_{r \in R} \rangle$ where D_S is a finite set, called the *domain* of S , and r_S is a subset of $D_S^{\rho(r)}$ for each r in R . We will denote by $S(R)$ the class of finite R -structures.

The *monadic second-order formulas* (*MS formulas* for short), intended to describe properties of R -structures S (for fixed R), are written with variables of two types, namely lower case letters x, x', y, \dots denoting elements of D_S and upper case letters X, Y, Y', \dots denoting subsets of D_S . The atomic formulas are of the forms $x = y, x \in X, r(x_1, \dots, x_n)$ (where r is in R and $n = \rho(r)$), and formulas are formed with propositional connectives and quantifications over the two kinds of variables. For every finite set W of object and set variables, we denote by $\mathcal{L}(R, W)$ the set of all formulas that are written with relational symbols from R and have their free variables in W . If S is an R -structure, if $\varphi \in \mathcal{L}(R, W)$, and γ is a W -assignment in S (i.e., $\gamma(X)$ is a subset of D_S for a set variable X , and $\gamma(x) \in D_S$ for an object variable x ; we write this $\gamma : W \rightarrow S$ to be short), we write $(S, \gamma) \models \varphi$ if and only if φ holds in S with the values of the free variables of φ being defined by γ . We write $S \models \varphi$ in the case where φ has no free variable.

Graphs and hypergraphs can be represented in several ways by relational structures.

For a directed graph G , we let $|G|_1 = \langle V_G, \mathbf{edg}_G \rangle$ and $|G|_2 = \langle D_G, \mathbf{inc}_G \rangle$ where $D_G := V_G \cup E_G$, \mathbf{edg}_G is the set of pairs (x, y) such that some edge links x to y , and \mathbf{inc}_G is the set of triples (e, x, y) such that the edge e links x to y . If G is undirected, the definitions are similar with “ x and y ” instead of “ x to y .” Thus \mathbf{edg}_G is symmetric.

For a hypergraph H , we use the structure $|H|_2$ with the *binary* relation \mathbf{inc} , presented in Section 2 before Theorem 2.10.

An MS_1 formula (MS_2 formula) is an MS formula written with the relation symbol **edg** (the relation symbol **inc**). It is intended to express a property of a structure of the form $|G|_1$ ($|G|_2$), where G is a graph (a graph or a hypergraph).

We will use transformations of relational structures, called MS (*definable*) *transductions of relational structures*. (See [3, 6]).

Let R and Q be two finite ranked sets of relation symbols. Let W be a finite set of set variables, called here the set of *parameters*. A (Q, R) -*definition scheme* is a tuple of formulas of the form

$$\Delta = (\varphi, \psi_1, \dots, \psi_k, (\theta_w)_{w \in Q * k}),$$

where

$$k > 0, Q * k := \{(q, j)/q \in Q, j \in \{1, \dots, k\}^{\rho(q)}\},$$

$$\varphi \in \mathcal{L}(R, W),$$

$$\psi_i \in \mathcal{L}(R, W \cup \{x_1\}) \quad \text{for } i = 1, \dots, k,$$

$$\theta_w \in \mathcal{L}(R, W \cup \{x_1, \dots, x_{\rho(q)}\}), \quad \text{for } w = (q, j) \in Q * k.$$

Let $S \in \mathcal{S}(R)$, let γ be a W -assignment in S . A Q -structure T with domain $D_T \subseteq D_S \times \{1, \dots, k\}$ is *defined by Δ in (S, γ)* if:

- (i) $(S, \gamma) \models \varphi$,
- (ii) $D_T = \{(d, i)/d \in D_S, i \in \{1, \dots, k\}, (S, \gamma, d) \models \psi_i\}$
- (iii) for each q in Q :

$$q_T = \{((d_1, i_1), \dots, (d_t, i_t)) \in D_T^t / (S, \gamma, d_1, \dots, d_t) \models \theta_{(q, j)}\},$$

where $j = (i_1, \dots, i_t)$ and $t = \rho(q)$.

(By $(S, \gamma, d_1, \dots, d_t) \models \theta_{(q, j)}$, we mean $(S, \gamma') \models \theta_{(q, j)}$, where γ' is the assignment extending γ , such that $\gamma'(x_i) = d_i$ for all $i = 1, \dots, t$ and similarly for $(S, \gamma, d) \models \psi_i$.) Since T is associated in a unique way with S, γ , and Δ whenever it is defined, i.e., whenever $(S, \gamma) \models \varphi$, we can use the functional notation $\mathbf{def}_\Delta(S, \gamma)$ for T .

The *transduction defined by Δ* is the relation $\mathbf{def}_D := \subseteq \mathcal{S}(R) \times \mathcal{S}(Q)$.

A relation $f \subseteq \mathcal{S}(R) \times \mathcal{S}(Q)$ is an *MS transduction* if and only if it is equal to

$$\{(S, T)/T = \mathbf{def}_\Delta(S, \gamma) \text{ for some } W\text{-assignment } \gamma \text{ in } S\}$$

for some (Q, R) -definition scheme Δ .

These definitions apply to graphs and hypergraphs via their representation by relational structures as explained above. We say that a binary relation R on graphs or hypergraphs is an (i, j) -*definable MS transduction* where i and j belong to $\{1, 2\}$ if and only if $\{(|H|_i, |H'|_j) / (H, H') \in R\}$ is an MS transduction.

Theorem A.1 says that if $T = \mathbf{def}_\Delta(S, \mu)$ then the monadic second-order properties of T can be expressed as monadic second-order properties of (S, μ) .

Let $\Delta = (\varphi, \psi_1, \dots, \psi_k, (\theta_w)_{w \in Q * k})$ be a (Q, R) -definition scheme, written with a set of parameters W . Let V be a set of set variables disjoint from W . For every variable X in V , for every $i = 1, \dots, k$, we let X_i be a new variable. We let $V' := \{X_i / X \in V, i = 1, \dots, k\}$. For every mapping $\eta : V' \rightarrow \mathcal{P}(D_S)$, we let $\eta_k : V \rightarrow \mathcal{P}(D_S \times \{1, \dots, k\})$ be defined by $\eta_k(X) = \eta(X_1) \times \{1\} \cup \dots \cup \eta(X_k) \times \{k\}$. (Note that every mapping from V to $\mathcal{P}(D_S \times \{1, \dots, k\})$ is of this form.) With these notations we can state the

BACKWARDS TRANSLATION THEOREM A.1. *For every formula β in $\mathcal{L}(Q, V)$, one can construct a formula β' in $\mathcal{L}(R, V' \cup W)$ such that, for every S in $\mathcal{S}(R)$, for every assignment $\mu : W \rightarrow S$, for every*

assignment $\eta : V' \rightarrow S$, we have:

def $_{\Delta}(S, \mu)$ is defined (if it is, we denote it by T),

η_k is a V -assignment in T and $(T, \eta_k) \models \beta$ if and only if $(S, \eta \cup \mu) \models \beta'$.

The following is a consequence.

THEOREM A.2. (1) *The composition of two MS transductions is an MS transduction.*

(2) *If a class L of relational structures has a decidable MS theory and if τ is an MS transduction, then $\tau(L)$ has also a decidable MS theory.*

See [3, 6] for the proof.

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