The monadic second-order logic of graphs VIII: Orientations

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Abstract

In every undirected graph or, more generally, in every undirected hypergraph of bounded rank, one can specify an orientation of the edges or hyperedges by monadic second-order formulas using quantifications on sets of edges or hyperedges. The proof uses an extension to hypergraphs of the classical notion of a depth-first spanning tree. Applications are given to the characterization of the classes of graphs and hypergraphs having decidable monadic theories.

1. Introduction

Graphs and hypergraphs of various kinds can be represented by logical structures. These representations make it possible to express some of their properties by logical formulas and to classify these properties according to the syntactical complexity of the formulas expressing them. Monadic second-order logic (i.e., first-order logic of "power-set" structures) is a language of particular interest in this respect because many graph properties (like k-colorability for fixed k, planarity, connectivity, Hamiltonicity) can be expressed, whereas several general decidability and complexity results hold for all graph properties that are expressible in this language.

We shall review very shortly the main results concerning decidability (and we refer the reader to [2,5,6,15] for complexity results). We first recall that a graph or hypergraph $G$ can be represented in several ways by a relational structure: as domain of the structure one can take either the set $V_G$ of vertices of $G$, or the set $V_G \cup E_G$ consisting of vertices and hyperedges (edges are particular hyperedges), and in either cases, relations express adjacency and incidence. In the first case quantified variables

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can denote vertices or sets of vertices whereas in the second, they can also denote hyperedges or sets of hyperedges. Hence, we get two ways of expressing graph properties by monadic second-order formulas, depending on the chosen representation of graphs or hypergraphs. We shall denote by \( MS_2 \) the monadic second-order language using vertex and hyperedge quantifications and by \( MS_1 \) the less expressive one using only vertex quantifications. Some graph properties like the existence of a Hamiltonian cycle are expressible in \( MS_2 \) but not in \( MS_1 \). The investigation of the border between the class of \( MS_1 \)-expressible graph properties and the class of those that are \( MS_2 \)-expressible but not \( MS_1 \)-expressible has been already addressed in [7] and the present work continues this investigation. Three general questions motivate the present work.

**Question 1.** What is the rôle of orientation in the expressibility of graph properties in \( MS_1 \) or in \( MS_2 \)?

Certain results are somewhat surprising in this respect. For instance Ajtai and Fagin [1] have proved that the existence of a directed path in a finite graph linking two given vertices is not expressible by a monadic \( \Sigma^1_1 \) formula, i.e., by a formula of the language \( MS_1 \), beginning with a sequence of existential set quantifications, the remaining part of the formula being first order. By contrast, the undirected version of this problem is expressible by a monadic \( \Sigma^1_1 \) formula.

Here is another case where the presence of orientation makes a difference. The existence of a Hamiltonian cycle in a finite undirected graph \( G \) is not \( MS_1 \)-expressible (see the proof of Proposition 2.5). However this property is expressible as follows: \( G \) has a Hamiltonian cycle if and only if it has an orientation \( G' \) that satisfies the following \( MS_1 \)-expressible property: it is acyclic, linearly ordered by the transitive closure of the (directed) adjacency relation, and its first and last vertices (with respect to this linear order) are adjacent (see the proof of Proposition 2.4 for details). (We shall say that the existence of a Hamiltonian cycle in a finite undirected graph is expressible in \( MS_1 \) with the help of an auxiliary orientation.) Or in other words,

- if \( L \) is an \( MS_1 \)-definable class of finite directed graphs,
- then, the class \( \text{und}(L) := \{ \text{und}(G) | G \in L \} \) is not necessarily \( MS_1 \)-definable,

where we let \( \text{und}(G) \) denote the undirected graph underlying a directed graph \( G \) (i.e., every directed edge \( (x, y) \) of \( G \) is replaced by the undirected edge \( \{x, y\} \)) and we say that a class of finite graphs is \( MS_1 \)-definable if the class of structures representing its elements is the class of all finite models of some closed formula of the language \( MS_1 \). By contrast, we shall obtain (as a consequence of Theorem 3.2, with a similar notion of \( MS_2 \)-definability) that:

- if \( L \) is an \( MS_2 \)-definable class of finite directed graphs,
- then \( \text{und}(L) \) is always \( MS_2 \)-definable.
In intuitive words, $\mathbf{MS}_1$-definability is "sensible to orientation" whereas $\mathbf{MS}_2$-definability is not.

In order to state the second (very general) question which motivates this work, we need some terminology. Let $\mathcal{C}$ be a class of logical structures and let $\mathcal{L}$ be a language expressing properties of the structures of $\mathcal{C}$. We shall say that the $\mathcal{L}$-theory of a subclass $L$ of $\mathcal{C}$ is decidable if there exists an algorithm that decides whether any given closed formula of $\mathcal{L}$ holds in all structures of $L$. We shall use the same terminology for a class $\mathcal{C}$ of graphs or hypergraphs, assuming that the elements of $\mathcal{C}$ are represented in a fixed way by relational structures using a fixed finite set of relations and that the language $\mathcal{L}$ uses only these relations. (The monadic second-order languages $\mathbf{MS}_1$ and $\mathbf{MS}_2$ refer to two distinct representations of graphs and hypergraphs by relational structures.)

**Question 2.** Which classes of graphs and hypergraphs have a decidable $\mathbf{MS}_2$-theory? Which classes of graphs and hypergraphs have a decidable $\mathbf{MS}_1$-theory without having a decidable $\mathbf{MS}_2$-theory?

We shall review some known results relevant to this question. Since we shall restrict our attention to finite graphs and hypergraphs in this article (the extension of our results to infinite graphs and hypergraphs being not completely immediate), we state them for finite graphs, even if they hold in more general cases.

**Result 1** (Courcelle [6, 9]). For each integer $k$, the $\mathbf{MS}_2$-theory of the class of finite directed graphs of tree-width at most $k$ is decidable.

The *tree-width* of a graph or hypergraph is an integer measuring how far it is from being a tree; trees have tree-width 1; tree-width is defined in terms of certain hierarchical decompositions of graphs and hypergraphs called *tree-decompositions* reviewed at the end of Section 1. The notion of tree-width has been introduced by Robertson and Seymour [23] in their study of graph minors. The following result is some sort of converse.

**Result 2** (Seese [25, Theorem 8]). If the $\mathbf{MS}_2$-theory of a class of finite graphs $L$ is decidable, then the graphs in $L$ have uniformly bounded tree-width.

These two results also hold for directed hypergraphs of bounded rank (see [6, 9]) and, in the present paper, we extend Result 2 to undirected hypergraphs of bounded rank. (The extension of Result 1 from the directed to the undirected case is straightforward.) One of our aims is also to extend Result 2 to the language $\mathbf{MS}_1$, for an appropriate complexity measure on graphs and hypergraphs instead of tree-width. (The set of finite cliques has a decidable $\mathbf{MS}_1$-theory but finite cliques have unbounded tree-width; hence, we must use another complexity measure than tree-width to get the analogous of Result 2 for the language $\mathbf{MS}_1$.)
We shall say that results of this type characterize the classes of finite graphs and hypergraphs of bounded rank having a decidable \( \text{MS}_2 \)-theory, where "characterize" has a precise meaning that we now define. By a complexity measure on a class of finite graphs or hypergraphs \( \mathcal{C} \), we mean a computable mapping \( \mu : \mathcal{C} \to \mathbb{N} \). Let us fix a representation of the elements of \( \mathcal{C} \) by relational structures and a language \( \mathcal{L} \) expressing properties of the corresponding structures. We shall say that \( \mu \) characterizes the subclasses of \( \mathcal{C} \) having a decidable \( \mathcal{L} \)-theory if and only if:

1. for every \( k \in \mathbb{N} \) the \( \mathcal{L} \)-theory of the class
   \[ \mathcal{C}(\mu, k) := \{ G \in \mathcal{C} | \mu(G) \leq k \} \]
   is decidable, and

2. for every \( L \subseteq \mathcal{C} \), if the \( \mathcal{L} \)-theory of \( L \) is decidable then
   \[ L \subseteq \mathcal{C}(\mu, k) \] for some \( k \in \mathbb{N} \).

Hence Results 1 and 2 together with their generalizations to directed and undirected hypergraphs of bounded rank establish that tree-width characterizes the classes of finite graphs and hypergraphs of bounded rank having a decidable \( \text{MS}_2 \)-theory. For the language \( \text{MS}_1 \), we have a candidate complexity measure, related to the so-called vertex replacement (VR) graph grammars (see the survey papers [12, 19]), and we conjecture (see (4.7) below) that it characterizes the classes of finite graphs having a decidable \( \text{MS}_1 \)-theory. We have proofs of special cases of this conjecture, among which the following result showing that orientation plays a certain rôlé in this question:

if a class of directed graphs is closed under arbitrary changes of edge directions
(i.e., if it is of the form \( \text{und}^{-1}(L) \) for some class \( L \) of undirected graphs) and has
a decidable \( \text{MS}_1 \)-theory, then its elements have uniformly bounded tree-width.

Here is the last of our general questions.

**Question 3.** Which additional features of the graphs or hypergraphs of some fixed class can one define uniformly, by fixed monadic second-order formulas?

By an "additional feature", we mean for instance an orientation (if the given graphs or hypergraphs are undirected), a linear ordering of the set of vertices, a planar layout, a tree-decomposition of width at most some fixed \( k \) (if the given graphs or hypergraphs have tree-width at most \( k \)).

We shall consider the possibility of defining orientation. We shall establish positive results for the language \( \text{MS}_2 \) and negative ones for the language \( \text{MS}_1 \). Let us mention that a linear ordering of the set of vertices can be defined in particular cases, for instance for graphs having spanning trees of bounded degree, see [7], and that the definability of tree-decompositions is actually a conjecture, established for \( k \leq 2 \) only (see [8]).

This paper is structured as follows. Section 1 presents the basic definitions and notation. It can be read very quickly by the reader knowing any of [7, 8, 10–12, 14]. In Section 2 we establish the results relating orientations and \( \text{MS}_1 \). In Section 3 we establish that \( \text{MS}_2 \) formulas can define all orientations of undirected graphs and
hypergraphs of bounded rank. In Section 4 we review the known answers to Question 2 and provide some new ones by using the results of Section 3.

1. Preliminaries

A hypergraph is a tuple \( H = \langle V_H, E_H, \text{Vert}_H \rangle \) consisting of a set of vertices \( V_H \), a set of hyperedges \( E_H \) (disjoint with \( V_H \)), a mapping \( \text{Vert}_H \) with domain \( E_H \) describing the hyperedges. There are two cases. If \( H \) is undirected then for every \( e \in E_H \), the object \( \text{Vert}_H(e) \) is a finite subset of \( V_H \), called the set of vertices of \( e \). If \( H \) is directed then \( \text{Vert}_H(e) \) is a finite sequence of elements of \( V_H \), where no vertex occurs twice. In both cases we say that \( H \) is simple if the mapping \( \text{Vert}_H \) is one-to-one. If \( \text{Vert}_H(e) = \text{Vert}_H(e') \) and \( e \neq e' \), we say that \( e \) and \( e' \) form a pair of multiple hyperedges. The rank of \( e \in E_H \) is the cardinality of \( \text{Vert}_H(e) \) in the first case and its length in the second. The rank of \( H \) is the maximal rank of its hyperedges.

In the present paper, all hypergraphs will be finite, i.e., the sets \( V_H \) and \( E_H \) will be finite.

In other papers [3, 6, 8, 9, 11, 13, 14, 21] hyperedges have labels and hypergraphs have distinguished vertices called sources or ports; hyperedges may have no vertex and be of rank 0; one could also extend the above definitions by letting \( \text{Vert}_H(e) \) be a sequence where vertices may occur several times in the directed case, and be a multiset in the undirected case; this extension would formalize a notion of loop. However, the results of the present paper extend to all these (slightly) more general cases, without any significant change in the proofs. So we take the simplest definitions.

We shall denote by \( \text{UH} \) the class of undirected hypergraphs, by \( \text{UH}_k \) the subclass of those of rank at most \( k \), and we shall denote by \( \text{H} \) and \( \text{H}_k \) the corresponding classes of directed hypergraphs. The subscript \( s \) in notations like \( \text{UH}_{s:k} \) or \( \text{H}_s \) refers to the corresponding classes of simple hypergraphs.

We let \( \text{und}: \text{H} \to \text{UH} \) be the mapping such that for every \( H \in \text{H} \), \( \text{und}(H) \) is the hypergraph \( H' \) such that \( V_{H'} = V_H, E_{H'} = E_H, \text{Vert}_{H'}(e) \) is the set of vertices occurring in the sequence \( \text{Vert}_H(e) \). Hence, hyperedges have the same rank in \( H \) and in \( \text{und}(H) \) since these sequences have no repetitions.

A hypergraph all edges of which are of rank 2 is thus a graph. Hence, graphs may be directed or undirected, they may have multiple edges but they will have no loop. We shall denote by \( \text{UG} \) and \( \text{G} \) the classes of undirected and directed graphs, respectively. It is clear that \( \text{und} \) maps \( \text{G} \) onto \( \text{UG} \). The corresponding classes of simple graphs will be denoted by \( \text{UG}_s \) and \( \text{G}_s \).

Two isomorphic hypergraphs will not be considered as equal, by contrast with what is done in [6–10]. (Considering two isomorphic hypergraphs as equal is convenient when one deals with the algebraic structure on hypergraphs, like in [3, 6], but not when one performs concrete constructions like deleting an edge or reversing its direction as we shall do in the sequel).
We now review relational structures and monadic second-order logic. Let \( R \) be a finite ranked set of symbols where each element \( r \in R \) has a rank \( \rho(r) \in \mathbb{N}_+ \). A symbol \( r \in R \) is a \( \rho(r) \)-ary relation symbol. An \( R \)-relational structure is a tuple \( S = \langle D_S, (r_S)_{r \in R} \rangle \) where \( D_S \) is a finite (possibly empty) set, called the domain of \( S \), and \( r_S \) is a subset of \( D_S^{\rho(r)} \) for each \( r \in R \). We shall denote by \( \mathcal{L}(R) \) the class of \( R \)-structures.

The monadic second-order formulas (MS formulas), intended to describe properties of \( R \)-structures \( S \) (for fixed \( R \)), are written with variables of two types, namely lower case letters \( x, x', y, \ldots \) called object variables, denoting elements of \( D_S \), and upper case letters \( X, Y, Y', \ldots \) called set variables, denoting subsets of \( D_S \). The atomic formulas are of the forms \( x = y \), \( x \in X \), \( r(x_1, \ldots, x_n) \) (where \( r \) is in \( R \) and \( n = \rho(r) \)), and formulas are formed with propositional connectives and quantifications over the two kinds of variables. For every finite set \( W \) of object and set variables, we denote by \( \mathcal{L}(R, W) \) the set of all formulas that are written with relational symbols from \( R \) and have their free variables in \( W \); we also let \( \mathcal{L}(R) := \mathcal{L}(R, \emptyset) \) denote the set of closed formulas. If \( S \) is an \( R \)-structure, if \( \varphi \in \mathcal{L}(R, W) \), and \( \gamma \) is a \( W \)-assignment in \( S \) (i.e., \( \gamma(X) \) is a subset of \( D_S \) for every set variable \( X \) in \( W \), and \( \gamma(x) \in D_S \) for every object variable \( x \) in \( W \); we write this \( \gamma : W \to S \) to be short), we write \( (S, \gamma) \models \varphi \) if and only if \( \varphi \) holds in \( S \) with the values of the free variables of \( \varphi \) being defined by \( \gamma \). We write \( S \vDash \varphi \) in the case where \( \varphi \) has no free variable. A class of \( R \)-structures \( L \) is definable if there is a formula \( \varphi \) in \( \mathcal{L}(R) \) such that \( L \) is the set of all \( R \)-structures \( S \) such that \( S \vDash \varphi \). Since any two isomorphic structures satisfy the same formulas, definable classes of structures are closed under isomorphism. Note that all structures are by definition finite. (It follows in particular that there is no distinction to make between monadic second-order and weak monadic second-order logic.)

Graphs and hypergraphs can be represented in several ways by relational structures. Our purpose is to use monadic second-order formulas to write some of their properties through their various representations. For an undirected hypergraph \( H \), we let

\[
|H|_1 = \langle V_H, (\text{edg}_H)_{k=1} \rangle
\]

and

\[
|H|_2 = \langle D_H, \text{inc}_H \rangle,
\]

where \( D_H := V_H \cup E_H \) (let us recall that \( E_H \cap V_H = \emptyset \)); the relation symbol \( \text{edg}_k \) has arity \( k \) and we define:

\[
\text{edg}_H(x_1, \ldots, x_k) : \iff \{ x_1, \ldots, x_k \} = \text{Vert}_H(e) \quad \text{for some hyperedge } e \text{ of rank } k.
\]

On the other hand, \( \text{inc} \) is a binary relation symbol expressing incidence as follows:

\[
\text{inc}_H(x, e) : \iff e \in E_H, x \in V_H \text{ and } x \in \text{Vert}_H(e).
\]

For a directed hypergraph \( H \), we shall take

\[
|H|_1 = \langle V_H, (\text{edg}_H)_{k=1} \rangle, \quad |H|_2 = \langle V_H \cup E_H, \text{next}_H \rangle,
\]
where
\[ \text{edg}_H(x_1, \ldots, x_k) : \iff (x_1, \ldots, x_k) = \text{Vert}_H(e) \text{ for some } e \in E_H \]
and \text{next} is a ternary relation such that:
\[ \text{next}_H(x, y, e) : \iff e \in E_H, x, y \text{ are vertices of } e \text{ and } y \text{ is the successor of } x \text{ in the sequence } \text{Vert}_H(e). \]

An MS$_1$-formula is an MS-formula written with the relation symbols edg$_k$, $k \geq 1$. It is intended to express a property of a structure of the form $|H|_1$. An MS$_2$-formula is an MS-formula written with the relation symbol inc or next. It is intended to express a property of a structure $|H|_2$ where $H$ is undirected in the first case and directed in the second.

The structures $|H|_1$ are inadequate for expressing properties of hypergraphs $H$, either directed or undirected, of unbounded rank, because an MS$_1$-formula contains a finite number of relation symbols edg$_k$, hence "cannot say anything" concerning hyperedges of arbitrary large rank. (We want to express properties of hypergraphs by single finite formulas. The situation would be different if we would use infinite formulas.)

It is clear that $|H|_2$ is isomorphic to $|H'|_2$ if and only if the hypergraphs $H$ and $H'$ are isomorphic. Two simple hypergraphs $H$ and $H'$ are isomorphic if and only if $|H|_1$ is isomorphic to $|H'|_1$. Hence, simple hypergraphs $H$ are unambiguously represented (up to isomorphism) by the structures $|H|_1$, whereas arbitrary hypergraphs are not.

We shall say that a property $P$ of the hypergraphs $H$ of a class $\mathcal{C}$ is expressed by a logical formula $\varphi$ via the representation $|H|$ if, for every $H$ in $\mathcal{C}$, the property $P(H)$ holds if and only if $|H| \models \varphi$. In particular, we shall say that a property of hypergraphs is $i$-definable (where $i$ is 1 or 2), if it is expressible by an MS$_1$ formula, via the representation $|\cdot|_i$. To give an example, the following formula expresses that an undirected graph $G$ represented by the structure $|G|_1$ is connected:

\[ \forall x \forall y \forall X \left[ (\forall u \in X \land \text{edg}_1(u, v) \Rightarrow v \in X) \land x \in X \Rightarrow y \in X \right]. \]

The structure $|H|_1$ is less expressive than $|H|_2$ for representing properties of a hypergraph $H$ by MS formulas for the obvious reason that one cannot express "in" $|H|_1$ properties dealing with multiple edges. However, this is also the case if $H$ is assumed to be simple. For instance, the existence of a Hamiltonian cycle in a simple graph is a 2-definable property that is not 1-definable. (Proof will be given below).

Our proofs will make an essential use of (monadic second-order) definable transductions of hypergraphs, already used in [7, 8, 10] and surveyed in [11]. We first introduce definable transductions of relational structures.

Let $R$ and $Q$ be two finite ranked sets of relation symbols. Let $W$ be a finite set of set variables, called here the set of parameters. (It is not a loss of generality to assume that all parameters are set variables.) A $(Q,R)$-definition scheme is a tuple of formulas of the form

\[ \Delta = (\varphi, \psi_1, \ldots, \psi_k, (\theta_w)_{w \in Q^{**}}), \]
where
\[ k > 0, \quad Q^* k := \{ (q, j) | q \in Q, j \in \{1, \ldots, k\}^{\varphi(q)} \}, \]
\[ \varphi \in \mathcal{L}(R, \mathcal{W}), \quad \psi_i \in \mathcal{L}(R, \mathcal{W} \cup \{x_1\}) \quad \text{for } i = 1, \ldots, k, \]
\[ \theta_{w} \in \mathcal{L}(R, \mathcal{W} \cup \{x_1, \ldots, x_{\rho(w)}\}) \quad \text{for } w = (q, j) \in Q^* k. \]

Let \( S \in \mathcal{S}(R) \), let \( \gamma \) be a \( W \)-assignment in \( S \). A \( Q \)-structure \( T \) with domain \( D_T \subseteq D_S \times \{1, \ldots, k\} \) is defined by \( \Delta \) in \( (S, \gamma) \) if:

(i) \( (S, \gamma) \vdash \varphi \),
(ii) \( D_T = \{ (d, i) | d \in D_S, i \in \{1, \ldots, k\}, (S, \gamma, d) \vdash \psi_i \} \)
(iii) for each \( q \) in \( Q \):
\[ q_T = \{ (d_1, i_1, \ldots, d_i, i_i) \in D_T^i | (S, \gamma, d_1, \ldots, d_i) \vdash \theta_{q, j}, \} \]
where \( j = (i_1, \ldots, i_i) \) and \( t = \rho(q) \).

(By \( (S, \gamma, d_1, \ldots, d_i) \vdash \theta_{q, j} \), we mean \( (S, \gamma') \vdash \theta_{q, j} \), where \( \gamma' \) is the assignment extending \( \gamma \) such that \( \gamma'(x_i) = d_i \) for all \( i = 1, \ldots, t \); a similar convention is used for \( (S, \gamma, d) \vdash \psi_i \).)

Since \( T \) is associated in a unique way with \( S, \gamma \) and \( \Delta \) whenever it is defined, i.e., whenever \( (S, \gamma) \vdash \varphi \), we can use the functional notation \( \text{def}_\Delta(S, \gamma) \) for \( T \).

The transduction defined by \( \Delta \) is the relation:
\[ \text{def}_\Delta := \{ (S, T) | T = \text{def}_\Delta(S, \gamma) \text{ for some } W \text{-assignment } \gamma \text{ in } S \} \subseteq \mathcal{S}(R) \times \mathcal{S}(Q). \]

A transduction \( f \subseteq \mathcal{S}(R) \times \mathcal{S}(Q) \) is definable if and only if it is equal to \( \text{def}_\Delta \) for some \( (Q, R) \)-definition scheme \( \Delta \). We shall also consider \( \text{def}_\Delta \) as a mapping from \( \mathcal{S}(R) \) to \( \mathcal{P}(\mathcal{S}(Q)) \) (the power set of \( \mathcal{S}(Q) \)) by letting \( \text{def}_\Delta(S) = \{ T | (S, T) \in \text{def}_\Delta \} \). If \( \text{def}_\Delta(S) \) is the singleton set \( \{ T \} \), we shall write \( \text{def}_\Delta(S) = T \) instead of \( \text{def}_\Delta(S) = \{ T \} \). This is necessarily the case when \( \Delta \) has no parameter. However, we shall sometimes use transductions defined with parameters that are nevertheless functional. This means that the result is the same for every choice of the parameters. In the special case where \( k = 1 \), we shall say that \( \text{def}_\Delta \) is noncopying, we shall write more simply \( \Delta \) as \( (\varphi, \psi, (\theta_{q, \rho(Q)}) \), and we shall have:
\[ D_T = \{ d \in D_S | (S, \gamma, d) \vdash \psi \} \]
and for each \( q \) in \( Q \):
\[ q_T = \{ (d_1, \ldots, d_t) \in D_T^t | (S, \gamma, d_1, \ldots, d_t) \vdash \theta_{q, j} \}, \]
where \( t = \rho(q) \).

Note in particular that \( D_T \subseteq D_S \), whereas \( D_S \) and \( D_T \) are disjoint in the general case.

These definitions apply to hypergraphs via their representation by relational structures as explained above. However, since we have two representations of hypergraphs by logical structures, we must be more precise. We say that a hypergraph transduction, i.e., a binary relation \( f \) on hypergraphs is \( (i, j) \)-definable, where \( i \) and
j belong to \{1, 2\} if and only if the transduction of structures \(\{(|H|_1, |H'|_2) | (H, H') \in f\}\) is definable. We shall also use transductions from trees to hypergraphs. Since a tree \(t\) is a graph, it can be represented by either \(|t|_1\) or \(|t|_2\). However, both structures are equally powerful for expressing monadic second-order properties of trees, and definable transductions from trees to hypergraphs (see [7]), hence, we shall use the former which is simpler.

To take a few examples, the transductions that associate with a simple directed graph \(G\) its underlying undirected graph \(\text{und}(G)\) is \((1, 1)\)-definable and noncopying (its effect is just a redefinition by a first-order formula of the \(\text{edg}_2\) relation). The identity on the class of directed graphs is \((2, 1)\)-definable and noncopying: its effect is to restrict the domain of the representing structure by eliminating edges and to define \(\text{edg}_2\) in terms of \(\text{next}\): all this can be done by the following first-order formulas, without parameters:

\[
\begin{align*}
\varphi(x_1) : & \quad \neg (\exists u, v \ \text{next}(u, v, x_1)), \\
\theta_{\text{edg}_2}(x_1, x_2) : & \quad \exists e \ \text{next}(x_1, x_2, e).
\end{align*}
\]

The transductions that associate with a graph \(G\) its spanning forests, or its connected components, or its minors (see Section 2 for minors) or the graph consisting of the union of two disjoint copies of \(G\), are all \((2, 2)\)-definable (but first-order formulas are insufficient in the first three cases).

The following three propositions are adaptations to the present definitions of classical results by Tarski et al. [27] or Rabin [22] concerning interpretation of structures. Proposition 1.1 says that if \(T = \text{def}_\Delta(S, \mu)\) then the monadic second-order properties of \(T\) can be expressed as monadic second-order properties of \((S, \mu)\). Let \(\Delta = (\varphi, \psi_1, \ldots, \psi_k, (\theta_\mu)_\mu, (\theta_\Delta)_\Delta)\) be a \((Q, R)\)-definition scheme, written with a set of parameters \(W\). Let \(V\) be a set of set variables disjoint from \(W\). For every variable \(X\) in \(V\), for every \(i = 1, \ldots, k\), we let \(X_i\) be a new variable. We let \(V' = \{X_i | X \in V, i = 1, \ldots, k\}\). For every mapping \(\eta : V' \rightarrow \mathcal{P}(\mathcal{D}_S)\), we let \(\eta_k : V \rightarrow \mathcal{P}(\mathcal{D}_S \times \{1, \ldots, k\})\) be defined by: \(\eta_k(X) = \eta(X_1) \times \{1\} \cup \ldots \cup \eta(X_k) \times \{k\}\). (Note that every mapping from \(V\) to \(\mathcal{P}(\mathcal{D}_S \times \{1, \ldots, k\})\) is of this form.) With these notations we can state the following proposition.

1.1. Proposition. For every formula \(\beta\) in \(L'(Q, V)\), one can construct a formula \(\beta'\) in \(L'(R, V' \cup W)\) such that, for every \(S\) in \(\mathcal{P}(R)\), for every assignment \(\mu : W \rightarrow S\), for every assignment \(\eta : V' \rightarrow S\), we have:

\[
\text{def}_\Delta(S, \mu) \text{ is defined (if it is, we denote it by } T\text{),}
\]

\[
\eta_k \text{ is a } V\text{-assignment in } T, \text{ and } (T, \eta_k) \models \beta
\]

if and only if

\[
(S, \eta \cup \mu) \models \beta'.
\]

Proof. See [8, Proposition 2.5, p. 166]. \(\square\)
From this proposition, we get easily (see [11]) the following proposition.

1.2. Proposition. (1) The inverse image of a definable class of structures under a definable transduction is a definable class of structures.
(2) The composition of two definable transductions is a definable transduction.

Note that the inverse of a definable transduction is not necessarily a definable transduction (see [11]).

1.3. Corollary. If a class \( L \) of relational structures has a decidable monadic theory and if \( \tau \) is a definable transduction, then \( \tau(L) \) has also a decidable monadic theory.

Proof. Let \( \tau \) be defined by \( \Delta = (\varphi, \psi_1, \ldots, \psi_k, (\theta_m)_{m \in Qr^+}) \) with parameters \( X_1, \ldots, X_n \).
Let \( \beta \) be an arbitrary formula. We can decide as follows whether \( T \models \beta \) for some \( T \) in \( \tau(L) \). We let \( \beta' \) be the formula "translating" \( \beta \) by Proposition 1.1. For every structure \( S \) in \( L \) and assignment \( \mu \) of \( \{X_1, \ldots, X_n\} \) into \( S \), we have:

\[
\text{def}_\tau(S, \mu) \models \beta \quad \text{if and only if} \quad (S, \mu) \models \varphi \land \beta'.
\]

Hence \( T \models \beta \) for some \( T \) in \( \tau(L) \) if and only if \( S \models \exists X_1, \ldots, X_n. \varphi \land \beta' \) for some \( S \) in \( L \).
In order to decide whether a given formula \( \varphi \) holds in all \( T \) in \( \tau(L) \), we apply this construction to \( \beta = \neg \varphi \). Hence \( T \models \varphi \) for all \( T \) in \( \tau(L) \) if and only if it is not true that \( S \models \exists X_1, \ldots, X_n. \varphi \land \beta' \) for some \( S \) in \( L \), if and only if \( S \models \neg \exists X_1, \ldots, X_n. \varphi \land \beta' \) for all \( S \) in \( L \). This is decidable since \( L \) is assumed to have a decidable monadic theory. \( \square \)

We now recall from [23] the definitions of tree-decomposition and tree-width. These notions are essential for the construction of efficient algorithms from specifications expressed by MS formulas (see [2,6,15]) but also for the characterization of classes of graphs having a decidable MS\(_2\)-theory, as we shall recall. Let \( H \) be a hypergraph. A tree-decomposition of \( H \) is a pair \( (T,f) \) consisting of a tree \( T \) and a mapping \( f: V_T \rightarrow \mathcal{P}(V_H) \) such that:

(1) \( V_H = \bigcup \{f(i) | i \in V_T\} \),
(2) every hyperedge of \( H \) has all its vertices in \( f(i) \) for some \( i \),
(3) for every vertex \( v \) of \( H \), the set \( \{ i \in V_T | v \in f(i) \} \) induces a connected subgraph of \( T \).

Each set \( f(i) \) is called a box of the tree-decomposition \((T,f)\). The width of \((T,f)\) is defined as \( \max \{ \text{card}(f(i)) | i \in V_T \} - 1 \). The tree-width of \( H \) is the minimum width of its tree-decompositions. It is denoted by \( \text{twd}(H) \). It is always defined and is at most \( \text{card}(V_H) - 1 \). The following result [24, 25] relates logical and combinatorial notions: If a set of planar graphs has a decidable MS\(_1\)-theory, then its members have uniformly bounded tree-width. In Section 4, we shall establish similar results.

We conclude by recalling the definition of minor inclusion. Let \( G \) and \( H \) be finite undirected graphs. We say that \( H \) is included as a minor in \( G \) or more simply that \( H \) is a minor of \( G \) (this is denoted by \( H \trianglelefteq G \)) if \( H \) is isomorphic to a graph obtained from
a subgraph of $G$ by a finite sequence of edge contractions. If $H$ has degree at most 3, then $H \sqsubseteq G$ if and only if a subgraph $G'$ of $G$ is isomorphic to a subdivision of $H$ (i.e., to a graph obtained from $H$ by the replacement of edges by paths). For a directed graph $G$, we let $H \sqsubseteq G$ if and only if $H \sqsubseteq \text{und}(G)$. We shall use the following result of Robertson and Seymour [23].

1.4. Proposition. For every undirected planar graph $H$ there is an integer $k$ such that for every graph $G$: If $k \leq \text{twd}(G)$, then $H \sqsubseteq G$.

2. MS$_1$ and orientations

An orientation of an undirected hypergraph $G$ is a hypergraph $H$ such that $\text{und}(H) = G$. If $H$ is a directed hypergraph, a reorientation of $H$ is any orientation of $\text{und}(H)$. A graph is complete if any two vertices are linked by a directed or undirected edge. A clique is a simple undirected complete graph. A tournament is any orientation of a clique.

2.1. Theorem. Let $L$ be a class of directed graphs closed under reorientation. If the MS$_1$-theory of $L$ is decidable, then the graphs in $L$ have bounded tree-width.

This result improves some known results (like the result of Seese quoted at the end of the last section) concerning the combinatorial structure of sets of graphs having decidable MS$_1$- or MS$_2$-theories. (See Theorem 4.1 and the tables at the end of Section 4 for a thorough review.) Since the tree-width of a complete graph is equal to the number of its vertices minus 1, we have the following corollary.

2.2. Corollary. The MS$_1$-theory of the class of tournaments is undecidable.

This corollary should be contrasted with the fact that the MS$_1$-theory of the class of cliques (i.e., of “undirected tournaments”) is decidable. (This fact is a consequence of Corollary 1.3 because cliques can be obtained from strings (or even from finite graphs without edges) by a (1,1)-definable transduction and it is known [4, 20] that the MS$_1$-theory of the class of strings is decidable.)

The proof of Theorem 2.1 will use some technical notions. Let $C$ be a finite set called the set of colors. Let $\bar{C}$ be a directed graph with set of vertices $C$. A $\bar{C}$-coloring of a directed graph $G$ is a mapping $\gamma: V_G \rightarrow C$ such that, for every edge $x \rightarrow y$ of $G$ there is an edge $\gamma(x) \rightarrow \gamma(y)$ in $\bar{C}$. If $\gamma$ is a partial function $V_G \rightarrow C$ (that is not necessarily a $\bar{C}$-coloring), we denote by $G[\gamma, \bar{C}]$ the subgraph $H$ of $G$ such that

1. $V_H = \text{dom}(\gamma)$ (i.e., is the set of vertices $x$ such that $\gamma(x)$ is defined)
2. $E_H$ is the set of edges $x \rightarrow y$ of $G$ such that $x, y \in V_H$ and $\gamma(x) \rightarrow \gamma(y)$ is an edge of $\bar{C}$.

By construction, $\gamma$ is thus a $\bar{C}$-coloring of $G[\gamma, \bar{C}]$. 
We now define some planar graphs of special form. We let $G_{n,q}$ denote the $n \times q$-grid; see Fig. 1 which shows $G_{3,4}$. If $n, p$ are odd integers, $n, p \geq 3$, we let $W_{n,p}$ be the directed acyclic planar graph best described by Fig. 2 (which shows $W_{5,7}$) than formally. We call it the $n \times p$ wall.

The graph $W_{n,p}$ has $p$ "rows" of, alternatively, $n$ and $n - 1$ "bricks". The edges are all directed from left to right and from top to bottom. (Fig. 2 does not show all edge directions for the purpose of clarity.) We have $\text{und}(W_{n,p}) \cong G_{n,p+1}$ by some edge contractions illustrated on Fig. 3 for $n = 5, p = 3$ (see Fig. 1 for the grid $G_{3,4}$). The contracted edges are marked in bold.
A subdivision of $W_{n,p}$ is a (directed acyclic) graph obtained from $W_{n,p}$ by substituting directed paths for edges. Fig. 4 shows a subdivision of $W_{3,5}$ together with a coloring of its vertices as explained below. The directions of edges (always from left to right and from top to bottom) have been omitted.

We now let $C = \{1, \ldots, 9\}$ be a set of colors. We let $\bar{C}$ be the graph (actually the orientation of the clique $K_9$) with set of vertices $C$ and edges listed as follows:

\begin{align*}
1 &\to 2 \to 3 \to 1 \\
4 &\to 5 \to 6 \to 4 \\
7 &\to 8 \to 9 \to 7 \\
i &\to j \quad \text{whenever} \quad i \in \{1, 2, 3\} \text{ and } j \in \{4, 5, 6\} \\
&\quad \text{or} \quad i \in \{4, 5, 6\} \text{ and } j \in \{7, 8, 9\} \\
&\quad \text{or} \quad i \in \{7, 8, 9\} \text{ and } j \in \{1, 2, 3\}.
\end{align*}

For every subdivision $\bar{W}$ of $W_{n,p}$, we construct a $\bar{C}$-coloring $\gamma$ of $\bar{W}$ as follows:

1. the vertices of the first, fourth, \ldots, $(3m + 1)$-th horizontal paths are colored with 1, 2, 3, starting with 1;
2. the vertices of the second, fifth, \ldots, $(3m + 2)$-th horizontal paths are colored with 4, 5, 6 starting with 4;
3. the vertices of the third, \ldots, $3m$th horizontal paths are colored with 7, 8, 9 starting with 7;
4. the vertices $y_1, y_2, \ldots, y_k$ of any vertical path $x \to y_1 \to y_2 \to \cdots \to y_k \to z$ where $x$ and $z$ belong to consecutive horizontal paths are colored with the three colors used for the horizontal path to which $x$ belongs.

From our choice of $\bar{C}$, there is one and only one $\bar{C}$-coloring of $W$ satisfying conditions (1)-(4). We shall call it the standard coloring of $W$. Fig. 3 shows the standard coloring of a subdivision of $W_{3,5}$. 
2.3. Lemma. One can construct a noncopying (1,1)-definable transduction $\tau: G_s \rightarrow G_s$ such that for every class $M$ of simple undirected graphs of unbounded tree-width $\tau(\text{und}^{-1}(M))$ is a class of planar directed graphs of unbounded tree-width.

Proof. Let $X_1, X_2, \ldots, X_9$ be set variables; they will be the parameters of the transduction $\tau$. We first define $\tau$ as a partial function, and we shall check later that it is indeed a noncopying (1,1)-definable transduction. If $G \in G_s$ and $X_1, \ldots, X_9 \subseteq V_G$, we let $\tau(G, X_1, \ldots, X_9)$ be defined if and only if the following two conditions hold:

1. $X_i \cap X_j$ for every $i \neq j$;
2. the graph $G[\gamma, \bar{C}]$ is planar, where $\gamma$ is the partial function: $V_G \rightarrow C$ such that $\gamma(x) = i$ if and only if $x \in X_i$.

If these conditions hold, we let $\tau(G, X_1, \ldots, X_9) := G[\gamma, \bar{C}]$.

The mapping $(G, X_1, \ldots, X_9) \mapsto G[\gamma, \bar{C}]$ defined by (1) (with $\gamma$ defined as in (2)) is a noncopying (1,1)-definable transduction: the set of vertices of the result graph $H = G[\gamma, \bar{C}]$ is the union of the sets $X_1, \ldots, X_9$; the $\text{edg}_H$ relation is defined by:

$\text{edg}_H(x, y)$ holds if and only if:
- $\text{edg}_G(x, y)$ holds and $i \rightarrow j$ is an edge of $\bar{C}$,
where $i$ and $j$ are such that $x \in X_i$ and $y \in X_j$. 

Since planarity is expressible in $\text{MS}_1$-logic [9, Lemma 4.2] and by Proposition 1.1, the planarity of $G^{[\gamma, \mathcal{C}]}$ can be expressed by an $\text{MS}_1$-formula $\varphi(X_1, \ldots, X_9)$. It follows that $\tau$ is a noncopying (1,1)-definable transduction.

Let now $M \subseteq \text{UG}$ be a class of simple undirected graphs of unbounded tree-width. By the result of Robertson and Seymour recalled in Proposition 1.4, every undirected planar graph $W$ is isomorphic to a minor of some graph in $M$. We take $W$ to be $\text{und}(W_{n, n})$ for $n$ odd, $n \geq 3$. Let $G \in M$ such that $G \cong W$. Since $W$ has degree 3, $G$ has a subgraph $H$ which is isomorphic to a subdivision of $W$; let $H'$ be the orientation of $H$ such that $H'$ is isomorphic to a subdivision of the directed graph $W_{n, n}$. Let $\gamma$ be the standard $\mathcal{C}$-coloring of $H'$. We let $G'$ be the orientation of $G$ constructed as follows:

For every edge $e$ in $E_G$:
- if $e \in E_H$, we direct it as in $H'$;
- if $e \notin E_H$, if its two ends $x$ and $y$ are in $V_H$ and if $\gamma(x) \neq \gamma(y)$,
  then we direct it such that:
    - $x \to y$ if $\gamma(y) \to \gamma(x)$ in $\mathcal{C}$
    - $y \to x$ if $\gamma(x) \to \gamma(y)$ in $\mathcal{C}$
  (note the directions of edges)
- otherwise we direct $e$ arbitrarily.

The graph $G'$ has been oriented in such a way that $G'[\gamma, \mathcal{C}] = H'$. ($\gamma$ is undefined on the vertices of $G'$ that are not in $H'$, and the edges of $G'$ that are not in $H'$ are directed in such a way that they are not in $G'[\gamma, \mathcal{C}]$.) Since $H'$ is planar, we have $H' \in \tau(G')$, hence $H' \in \tau(\text{und}^{-1}(G)) \subseteq \tau(\text{und}^{-1}(M))$. We have $\text{und}(H') = H \cong W = \text{und}(W_{n, n}) \cong G_{n, n+1} \cong G_{n, n}$. Hence $\text{twd}(H') = \text{twd}(H) \geq \text{twd}(G_{n, n})$ which is equal to $n$. Since $n$ was arbitrary the graphs in $\tau(\text{und}^{-1}(M))$ have unbounded tree-width. 

Proof of Theorem 2.1. Let $L$ be a class of directed graphs closed under reorientation. Then $L = \text{und}^{-1}(M)$ where $M = \text{und}(L)$. Let us assume that $L$ has a decidable $\text{MS}_1$-theory. Then, the class of directed graphs $\tau(L)$ where $\tau$ is the (1,1)-transduction of Lemma 2.3 has also a decidable $\text{MS}_1$-theory by Corollary 1.3. Since it consists of planar graphs, the second assertion of Theorem 4.1 below yields that its graphs have bounded tree-width. By Lemma 2.3, $M$, whence $L$, has not unbounded tree-width.

The use of auxiliary orientation can be seen as a way to extend the powers of the languages $\text{MS}_i$ ($i = 1$ or 2) for expressing properties of undirected graphs. We denote by $\text{MS}_1 + \text{AO}$ the variant of the logic $\text{MS}_1$ consisting of $\text{MS}_1$ formulas written with the relation symbols appropriate for directed hypergraphs but with a special semantics, expressing as follows properties of undirected hypergraphs:

if $\varphi$ is such an $\text{MS}_1$-formula and $H$ is an undirected hypergraph, we say that $\varphi$ holds in $H$, written $H \models \varphi$ if and only if

$|H'| \models \varphi$ for some orientation $H'$ of $H$. 

Note in particular that the negation of a property expressible in MS\(_i\) \(+\) AO is not obviously expressible in MS\(_i\) \(+\) AO. However, it is, as we shall see, in the case \(i = 2\).

We say that a class \(L\) of undirected hypergraphs is \(i'-\text{definable}\), where \(i = 1\) or 2, if it is the class of undirected hypergraphs in which some closed formula of MS\(_i\) \(+\) AO holds, equivalently if and only if \(L = \text{und}(L')\) for some \(i\)-definable class \(L'\) of directed hypergraphs. For \(j\) in \(\{1, 2, 1', 2'\}\), we let \(j\)-Def denote the family of \(j\)-definable classes of hypergraphs. We have the following inclusion diagram:

\[
\text{2-Def} = 2'-\text{Def} \cup 1'-\text{Def} \cup 1\text{-Def}
\]

The equality \(2\text{-Def} = 2'-\text{Def}\) will be proved in the next section. We do not know whether the inclusion of \(1'-\text{Def}\) in \(2'-\text{Def}\) is proper. The class \(\mathcal{H}\) of Hamiltonian undirected graphs is 2-definable (see [6, Proposition (3.8)]) but not 1-definable (see Proposition 2.5). Actually, \(\mathcal{H}\) distinguishes the classes 1'-Def and 1-Def by the following result.

2.4. Proposition. The class \(\mathcal{H}\) of Hamiltonian undirected graphs is 1'-definable.

Proof. Let \(G\) have a Hamiltonian cycle \((x_1, x_2, \ldots, x_n, x_1)\). Let \(G'\) be the orientation of \(G\) such that \(x_i \rightarrow x_j\) if and only if \(i < j\). Then \(G'\) satisfies the following properties:

1. it is acyclic,
2. any two distinct vertices \(x, y\) are linked by a directed path either from \(x\) to \(y\) or from \(y\) to \(x\),
3. there is an edge from the unique vertex of indegree 0 to the unique vertex of outdegree 0.

We claim that an undirected graph \(G\) is Hamiltonian if and only if it has an orientation \(G'\) satisfying conditions (1)–(3) above. Since these conditions are MS\(_i\)-expressible, the verification of the claim completes the proof. Let \(G'\) satisfy (1)–(3). From (1) and (2), it is easy to prove by induction on \(\text{card}(V_G)\) that there exists in \(G'\) a directed Hamiltonian path \(x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots \rightarrow x_n\). From condition (3), we also have \(x_1 \rightarrow x_n\) so that \(\text{und}(G')\) has a Hamiltonian cycle.

We now give several negative results saying, intuitively, that one cannot in general define uniformly by a fixed finite number of MS\(_i\)-formulas

(1) all orientations of an undirected graph \(G\),
(2) some orientation of an undirected graph \(G\)

and finally, that in the graphs of a certain infinite family, one can define by MS\(_i\)-formulas some orientation but not all of them.
Let \( f \) be a partial function from \( \mathcal{S} \) to \( \mathcal{T} \), where \( \mathcal{S} \) and \( \mathcal{T} \) are two classes of hypergraphs. Its inverse \( f^{-1} \) is a transduction from \( \mathcal{T} \) to \( \mathcal{S} \). A transduction \( g \) from \( \mathcal{T} \) to \( \mathcal{S} \) is a partial inverse of \( f \) if \( \text{dom}(g) = \text{dom}(f) \) and \( f \circ g \) is the identity on \( \text{dom}(g) \). (We denote by \( \text{dom}(g) \) the class of elements \( T \) of \( \mathcal{T} \) such that \( g(T) \) is defined.)

2.5. Proposition. (1) The transduction \( \text{und}^{-1}: \text{UG}_s \to \text{G}_s \) is not \((1,1)\)-definable.

(2) The transduction \( \text{und}: \text{G}_s \to \text{UG}_s \) has no noncopying \((1,1)\)-definable partial inverse, i.e., there is no noncopying \((1,1)\)-definable transduction \( \omega: \text{UG}_s \to \text{G}_s \) defining orientations of all graphs in \( \text{UG}_s \).

(3) There is a \( 1\)-definable class of undirected graphs \( K \) such that the restriction of \( \text{und}^{-1} \) to \( K \) is not \((1,1)\)-definable, whereas \( \text{und} \) has a partial inverse, the restriction to \( K \) of which is \((1,1)\)-definable and noncopying.

Proof. (1) We first sketch the proof that the class \( \mathcal{H} \) of Hamiltonian undirected graphs is not \( 1\)-definable. We let \( \phi \) be the definable transduction that maps a structure of the form \((D, R)\) where \( R \) is a proper nonempty subset of \( D \) (\( R \) is a unary relation) to the structure \([H]_1\) where \( H \) is the complete bipartite graph with set of vertices \( D \) and with an edge between any vertex in \( R \) and any vertex not in \( R \). Then \( H \) is Hamiltonian if and only if \( \text{card}(R) = \text{card}(D - R) \). If \( \mathcal{H} \) would be \( 1\)-definable, then the class of structures \((D, R)\) such that \( \text{card}(R) = \text{card}(D - R) \) would be definable (by Proposition 1.2), which is not the case because no MS-formula can express that two arbitrary sets have equal cardinality (see [6]).

If the transduction \( \text{und}^{-1}: \text{UG}_s \to \text{G}_s \) would be \((1,1)\)-definable, then every \( 1'\)-definable class \( L \subseteq \text{UG}_s \), i.e., every class \( L \) of the form \( \text{und}(L') \) where \( L' \) is defined in \( \text{G}_s \) by an MS\(_1\)-formula \( \phi \) would be \( 1\)-definable in \( \text{UG}_s \), by Proposition 1.2. This would imply that the \( 1'\)-definable class \( \mathcal{H} \) of Hamiltonian undirected graphs is \( 1\)-definable, contradiction.

(2) A noncopying \((1,1)\)-definable partial inverse of \( \text{und} \), say \( \omega: \text{UG}_s \to \text{G}_s \), would be defined by MS\(_1\)-formulas \( \phi(X_1, \ldots, X_n) \) and \( \theta(X_1, \ldots, X_m, x, y) \) such that

(a) for every \( H \in \text{UG}_s \), there exists \( X_1, \ldots, X_n \subseteq V_H \) such that

\[
[H]_1 \models \phi(X_1, \ldots, X_n),
\]

(b) for every such \( H \) and \( X_1, \ldots, X_n \) we have, for every \( x, y \in V_H \)

(i) \( [H]_1 \models \theta(X_1, \ldots, X_m, x, y) \lor \theta(X_1, \ldots, X_m, y, x) \)

if and only if \( x \) and \( y \) are adjacent in \( H \).

(ii) \( [H]_1 \models \theta(X_1, \ldots, X_m, x, y) \Rightarrow x \neq y \lor \neg \theta(X_1, \ldots, X_m, y, x) \).

Assuming that we have such formulas, we let \( H \) be a clique with more than \( 2^n \) vertices, and we let \( X_1, \ldots, X_n \) satisfy \( \phi \) in \( H \). There exist two vertices \( x \) and \( y \) such that for every \( i \in \{1, \ldots, n\}, x \in X_i \) if and only if \( y \in X_i \). Then the bijection \( h: V_H \to V_H \) that exchanges \( x \) and \( y \) is an automorphism of \( H \) that leaves \( X_1, \ldots, X_n \) invariant. Hence, if

\[
[H]_1 \models \theta(X_1, \ldots, X_m, x, y)
\]
we get also

\[ |H|_1 \models \theta(h(X_1), \ldots, h(X_n), h(x), h(y)), \]

i.e.

\[ |H|_1 \models \theta(X_1, \ldots, X_n, y, x). \]

But this contradicts condition (ii) on \( \theta \).

(3) Finally, we consider the set \( K \) of all undirected graphs consisting of a rooted binary tree with at least four leaves, augmented with edges between any two leaves. By \( \text{MS}_1 \)-formulas one can define the root (this is the only vertex of degree 2; the internal vertices of the tree have degree 3 and the leaves have degree at least 4); one can define the directions of the edges of the tree in such a way that every vertex is reachable from the root by a directed path in the tree; one can define an ordering of the tree by choosing a partition \((X, Y)\) of its set of nodes such that, for every node \( z \) of the tree, one of its successors is in \( X \) and the other is in \( Y \), and by deciding that the successor of a node \( z \) belonging to \( X \) is the left successor of \( z \) and that that in \( Y \) is the right one; from this choice we get a linear order \( \leq \) on the set of leaves of the tree, namely the usual left-right order, which is definable by an \( \text{MS}_1 \)-formula. Hence, we can direct the remaining edges which connect the leaves of the tree; an edge \( x \rightarrow y \) is directed from \( x \) to \( y \) if and only if \( x < y \). This proves that one can specify by \( \text{MS}_1 \)-formulas orientations of every graph in \( K \).

Assume now that all orientations of such a graph \( G \) could be characterized by formulas in terms of \( n \) parameters \( X_1, \ldots, X_n \). Every graph \( G \) in \( K \) has \( 2m - 1 \) vertices for some \( m \), and \( m \) of these vertices induce a complete subgraph of \( G \). Hence \( G \) has \( 2(m - 1) + m(m - 1)/2 \) edges and, hence, \( 2^{(m^2)/2} + (3m/2)^{-2} \) orientations. There are only \( (2^{(m^2)/2})^m = 2^{(2m^2 - 1)/2} \) different \( n \)-tuples of sets \( X_1, \ldots, X_n \) with \( X_1 \subseteq V_G \). Since \( 2^{(m^2)/2} + (3m/2)^{-2} \) is larger than \( 2^{(2m^2 - 1)/2} \) for large enough \( m \), one cannot specify all orientations with \( n \)-tuples of sets of vertices. This proves that no \((1,1)\)-definable transduction can produce \( \text{und}^{-1}(G) \) for all \( G \in K \). \( \square \)

Let us remark that for certain classes of graphs, it is very easy to specify orientations by means of \( \text{MS}_1 \)-formulas. Take for instance the class of \( k \)-colorable graphs, for fixed \( k \). Using a coloring of the graph with colors 1, \ldots, \( k \), (such a coloring can be given by \( k \) sets of vertices as in the proof of Lemma 2.3), one directs the edges in such a way that \( x \rightarrow y \) if and only if \( x \) has color \( i \) and \( y \) has color \( j \) with \( i < j \).

We conclude this section with an open problem concerning the use of arbitrary auxiliary orientations in the hope of increasing the expressive power of \( \text{MS}_1 \)-formulas while keeping closure of the language by negation. We denote by \( \text{MS}_1 + \text{AO} \) the sublanguage of \( \text{MS}_1 + \text{AO} \) consisting of formulas \( \phi \) (written with the relation symbols appropriate for directed hypergraphs of rank at most \( k \), for fixed \( k \)), that are invariant under reorientation, i.e., that are such that for every \( H \) in \( H_{ik} \) and \( H' \) in \( \text{und}^{-1}(\text{und}(H)) \):

\[ |H|_1 \models \phi \quad \text{if and only if} \quad |H'|_1 \models \phi. \]
This means that any two auxiliary orientations of the considered undirected hypergraph yield the same answer. It follows in particular that the negation of a property expressible in $\text{MS}_1 + A'O$ is also expressible in $\text{MS}_1 + A'O$.

**Question 2.6.** Is $\text{MS}_1 + A'O$ strictly more expressive than $\text{MS}_1$?

Or equivalently the following question.

**Question 2.6'.** Does there exist a subset $L$ of $UH_{id}$ that is not 1-definable but is such that $\text{und}^{-1}(L)$ is 1-definable?

Note that in Proposition 2.4, we established the related (but different) property that there does exist a 1-definable subset $L$ of $G$, such that $\text{und}(L)$ is not 1-definable.

If instead of "arbitrary auxiliary orientation", we take "arbitrary auxiliary linear ordering", the answer to the analogue of Question 2.6 is "Yes": that a set has even cardinality is $\text{MS}_1$-expressible in terms of an arbitrary linear ordering but is not $\text{MS}_1$-expressible for "pure" (unordered) sets (see [6]).

3. **$\text{MS}_2$ definition of orientations**

This section contains the main technical result of this paper, stating that in undirected hypergraphs of bounded rank one can define some orientations by $\text{MS}_2$-formulas.

A directed hypergraph $H$ is **partially ordered** if there exists a partial order on its set of vertices such that the sequence of vertices of each hyperedge is increasing with respect to that order. The notion of a **partially ordered orientation** of an undirected hypergraph follows immediately.

3.1. **Theorem.** For every $k \geq 2$, the mapping $\text{und}: H_k \to \text{UH}_k$ has a $(2,2)$-definable partial inverse. More precisely, one can construct a $(2,2)$-definable transduction $\omega_k$ such that, for every $H$ in $\text{UH}_k$, $\omega_k(H)$ is a nonempty set of partially ordered orientations of $H$.

The proof of this theorem will be easier to understand if we first sketch the simpler proof of its special case concerning graphs.

3.2. **Theorem.** One can construct a $(2,2)$-definable transduction $\omega: \text{UG} \to G$ that associates with every undirected graph $G$ a nonempty set $\omega(G)$ of partially ordered, acyclic orientations of $G$.

**Proof (sketch).** Let $G \in \text{UG}$ be connected. A spanning tree $T$ of $G$ is **depth-first** if it has a distinguished vertex (called its **root**) such that, if $x$ and $y$ are any two vertices adjacent in $G$, then either $x$ is on the unique path in the tree linking the root to $y$ or $y$ is
on the unique path in the tree linking the root to $x$. Since $G$ is finite and connected, it has a depth-first spanning tree $T$ (see [26]). From $T$ we get a unique orientation $G'$ of $G$ such that, for every edge $x \to y$ in $G'$, the vertex $x$ belongs to the unique path in $T$ from the root to $y$.

One can write MS$_2$-formulas $\varphi(X), \varphi'(X, u)$ and $\theta(X, u, x, y)$ such that for every undirected graph $G$, for every $u$ and $X$:

1. $|G|_2 \models \varphi(X)$ if and only if $X$ is a set of edges of $G$ forming a spanning tree,
2. $|G|_2 \models \varphi'(X, u)$ if and only if $u$ is a vertex, $|G|_2 \models \varphi(X)$, and the spanning tree of $G$ with set of edges $X$ and root $u$ is depth-first,
3. if $|G|_2 \models \varphi'(X, u)$ then, for every two vertices $x$ and $y$:

$$|G|_2 \models \theta(X, u, x, y)$$

if and only if $x \neq y$, $x$ belongs to the path in $G$ that links $u$ to $y$ and has all its edges in $X$.

For each edge $(x, y)$ of $G$, we have either $\theta(X, u, x, y)$ or $\theta(X, u, y, x)$ (and not both); hence, $\theta$ defines an orientation of each edge; however, this orientation depends on two parameters $X$ and $u$ that must satisfy $\varphi'$. Finally, if $G$ is connected, there exist $X$ and $u$ satisfying $\varphi'$.

If $G$ is not connected, one takes a depth-first spanning forest (i.e., a set of trees consisting of one depth-first spanning tree of each connected component of $G$). The corresponding formulas are written with two parameters $X$ and $U$ where $U$ is intended to be the set of roots of the trees of the spanning forest of which $X$ is the set of edges. We omit the details. □

Our proof of Theorem 3.1 will somehow extend this proof. Our first task is to find some notion of depth-first spanning tree for hypergraphs.

It will be convenient to call the vertices of trees *nodes*. A *directed tree* is a directed graph $T$ such that $\text{und}(T)$ is a tree, and that is oriented in such a way that there exists a node called the root (we shall denote it by $\text{rt}_T$) such that every node of $T$ is reachable from the root by a directed path. The root is uniquely defined from the orientation, and, conversely, for every node taken as root, there is a unique orientation of the tree making it into a directed tree. The *successors* of a node $x$ are the nodes $y$ such that there is in $T$ a directed edge from $x$ to $y$. Every node except the root has a unique *predecessor*, of which it is a successor. The iterated successors are called the *descendants* (a node is not a descendant of itself), and the *ancestors* of a node $x$ are the nodes of which $x$ is a descendant.

A directed tree can be specified by a relational structure $T = \langle N_T, \text{Suc}_T \rangle$ where $N_T$ is the set of nodes and $\text{Suc}_T$ is the successor relation. We write also $x \in \text{Suc}_T(y)$ to state that $x$ is a successor of $y$. It is easy to write an MS formula expressing that a finite structure $T = \langle N, S \rangle$ represents a directed tree.

Let $H$ be an undirected hypergraph. A *hypertree in $H$* is a directed tree $T = \langle N_T, \text{Suc}_T \rangle$ such that:

(i) $N_T \subseteq E_H$
(ii) for every $e \in N_T$, if $e' \in \text{Succ}_T(e)$ then $\text{Vert}_H(e) \cap \text{Vert}_H(e') \neq \emptyset$

(iii) for every $e, e' \in N_T$, if $e \neq e'$ and $\text{Vert}_H(e) \cap \text{Vert}_H(e') \neq \emptyset$ then either $e$ and $e'$ are adjacent as nodes of $T$ or they are successors of some $e'' \in N_T$ and $\text{Vert}_H(e) \cap \text{Vert}_H(e') \subseteq \text{Vert}_H(e'')$.

If $N$ is a set of hyperedges of $H$, an $N$-hyperpath in $H$ is a hypertree such that every node belongs to $N$ and has at most one successor. A hyperpath will be denoted by $(e_1, e_2, \ldots, e_n)$ where $e_1$ is the root and $e_{i+1}$ is the successor of $e_i$, for every $i = 1, \ldots, n-1$. We shall say that this hyperpath is from $e_1$ to $e_n$.

We come back to hypertrees. From Condition (iii), it follows in particular that $\text{Vert}_H(e) \cap \text{Vert}_H(e') = \emptyset$ if $e'$ is a successor of a descendent of $e$. Fig. 5 illustrates the notion of a hypertree. The hyperedges of $H$, all of rank 3 or 4, are numbered from 1 to 11, and hyperedge 9 is the root of $T$. Note that hyperedge 2 is not a node of $T$.

3.3. Lemma. Let $T$ be a hypertree in an undirected hypergraph $H$. For every $e, e' \in N_T$ with $e' \neq e, e' \in \text{Succ}_T(e)$ if and only if $\text{Vert}_H(e) \cap \text{Vert}_H(e') \neq \emptyset$ and every $N_T$-hyperpath in $H$ from $\text{rt}_T$ to $e'$ contains $e$.

Proof. "Only if". Let $e' \in \text{Succ}_T(e)$. Then $\text{Vert}_H(e) \cap \text{Vert}_H(e') \neq \emptyset$ by condition (ii) of the definition of a hypertree; if $(e_1, e_2, \ldots, e_n)$ is an $N_T$-hyperpath in $H$ from $\text{rt}_T = e$, to $e_n = e'$, then it is equal to the path in $T$ from $\text{rt}_T$ to $e'$, and $e = e_{n-1}$.

"If". Let $e, e' \in N_T$ be such that $e \neq e'$, $\text{Vert}_H(e) \cap \text{Vert}_H(e') \neq \emptyset$ and every $N_T$-hyperpath in $H$ from $\text{rt}_T$ to $e'$ contains $e$. By Condition (iii) there are three cases:

- $e' \in \text{Succ}_T(e)$,
- $e \in \text{Succ}_T(e')$,
- $e, e' \in \text{Succ}_T(e'')$ for some $e'' \in N_T$.

Fig. 5.
In the last two cases, the path in $T$ from $rt_T$ to $e'$ is an $N_T$-hyperpath from $rt_T$ to $e'$ that does not contain $e$; hence, these two cases cannot hold and $e' \in \mathrm{Suc}_T(e)$ as was to be proved. \hfill \Box

3.4. Lemma. One can write $\mathbf{MS}_2$-formulas $\varphi_1(N, r)$ and $\varphi_2(N, r, e, e')$ such that, for every hypergraph $H$:

$$|H|_2 \vDash \varphi_1(N, r)$$

if and only if $N \subseteq E_H$, $r \in N$, and $N$ is the set of nodes of a hypertree $T$ in $H$ with root $r$, and

$$|H|_2 \vDash \varphi_2(N, r, e, e')$$

if and only if $\varphi_1(N, r)$ holds in $|H|_2$, $e, e' \in N$ and $e' \in \mathrm{Suc}_T(e)$.

Proof. One can construct an $\mathbf{MS}_2$-formula $\pi(N, e, e')$ expressing that $N \subseteq E_H$, $e, e' \in N$, and $N$ is the set of nodes of a hyperpath in $H$ from $e$ to $e'$. By using this formula, one can construct an $\mathbf{MS}_2$-formula $\varphi_2(N, r, e, e')$ expressing that $N \subseteq E_H$, $r \in N$, $e, e' \in N$, $\mathrm{Vert}_H(e) \cap \mathrm{Vert}_H(e') \neq \emptyset$, and every $N$-hyperpath in $H$ from $r$ to $e'$ contains $e$.

This formula defines (for fixed $N$ and $r$) a binary relation $S$ on $N$. One can then construct $\varphi_1(N, r)$ expressing the following conditions:

- $N \subseteq E_H$, $r \in N$,
- $T = \langle N, S \rangle$ is a directed tree with successor relation $S$ and root $r$ and
- $T$ satisfies Conditions (ii) and (iii) of the definition of a hypertree.

Hence $\varphi_1(N, r)$ expresses that $T$, specified as above, is a hypertree in $H$ with root $r$.

(Note that by Lemma 3.3, the successor relation of any hypertree is completely defined by its set of nodes and its root). Then we take for $\varphi_2$ the formula $\varphi_1 \land \varphi_2$. It follows from Lemma 3.3 that the formulas $\varphi_1$ and $\varphi_2$ are as desired. \hfill \Box

Let $T$ be a hypertree in $H$ with root $r$. For every $e \in N_T$, we define:

$$V(T, e) = \mathrm{Vert}_H(e) \quad \text{if } e = r,$$

$$V(T, e) = \mathrm{Vert}_H(e) - \mathrm{Vert}_H(e') \quad \text{if } e \neq r, \text{ and } e \in \mathrm{Suc}_T(e').$$

From Condition (iii) of the definition of a hypertree, it follows that $\mathrm{Vert}_H(e) - V(T, e) \subseteq V(T, e')$ whenever $e \in \mathrm{Suc}_T(e')$, and that $V(T, e)$ and $V(T, e')$ are disjoint whenever $e \neq e'$.

We also let $\leq_T$ be the quasiorder on $V_H$ defined as follows:

$$x \leq_T y$$

if and only if

$$x = y, \quad \text{or} \quad x, y \in V(T, e) \quad \text{for some } e \in N_T,$$

or $x \in V(T, e)$, $y \in V(T, e')$ for some $e, e' \in N_T$ such that $e'$ is an ancestor of $e$. 

We let $x <_T y$ mean that $x \leq_T y$ holds and $y \leq_T x$ does not. In the hypergraph of Fig. 5, we have $u \leq_T t \leq_T u <_T z <_T y <_T x$.

It is clear that, with the help of Lemma 3.4, one can construct an MS$_2$-formula $\phi_3(N, r, x, y)$ such that, for every hypergraph $H$:

$$|H|_2 \vDash \phi_3(N, r, x, y)$$

if and only if

$$|H|_2 \vDash \phi_1(N, r), \quad x, y \in V_H \text{ and } x \leq_T y$$

where $T$ is the hypertree defined by $N$ and $r$.

We shall say that a hypertree $T$ in $H$ is a depth-first spanning hypertree if every hyperedge $e$ of $H$ of rank at least 2:

- either has two vertices $x, y$ with $x <_T y$
- or is included in some hyperedge $e'$ in $N_T$ (or both cases hold).

(We say that $e$ is included in $e'$ if $\text{Vert}_H(e) \subseteq \text{Vert}_H(e')$; the second condition holds trivially for every hyperedge $e$ in $N_T$.)

One can construct an MS$_2$-formula $\phi_4(N, r)$ such that:

$$|H|_2 \vDash \phi_4(N, r)$$

if and only if $|H|_2 \vDash \phi_4(N, r)$ and the hypertree $T$ associated with $N, r$ is a depth-first spanning hypertree in $H$.

The hypertree $T$ of Fig. 5 is a depth-first spanning hypertree in the hypergraph $H$. In particular, the vertex common to hyperedges 2 and 3 is smaller with respect to $<_T$ than the one common to hyperedges 2 and 9, so hyperedge 2 meets the first condition.

A connected hypergraph may have no depth-first spanning hypertree. Take for example the hypergraph $H$ with set of vertices $\{1, \ldots, 9\}$ and hyperedges $\{1, 9\}, \{2, 8\}, \{3, 7\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}$; assume that $T$ is a depth-first spanning hypertree of $H$: it must contain all the hyperedges of $H$, but then, it is not a hypertree. However, every "augmented" hypergraph $H^+$ (as defined below), has a depth-first spanning hypertree.

For every undirected hypergraph $H$, we construct as follows an undirected hypergraph $H^+$ such that $H \subseteq H^+$:

- we add to $H$ a new vertex $u_0$,
- we add undirected edges linking $u_0$ to all vertices of $H$,
- we add a hyperedge $e_0$ of rank 1 with vertex $u_0$.

3.5. Proposition. For every undirected hypergraph $H$, the hypergraph $H^+$ has a depth-first spanning hypertree with root $e_0$.

In order to prove this proposition, we shall construct a depth-first spanning hypertree of $H^+$ by a suitable adaptation of the classical depth-first search algorithm that constructs in a graph a depth-first spanning tree. We need a few more definitions.
A directed tree $T$ is ordered if the set of successors of each node is linearly ordered. Hence an ordered directed tree $T$ can be given by a triple $\langle N_T, fsuc_T, next_T \rangle$ where $N_T$ is the set of nodes, $fsuc_T(x)$ is the first successor of node $x$, and $next_T(x)$ is the successor of the predecessor of $x$ that follows $x$ in the considered linear order. The functions $fsuc_T$ and $next_T$ are partial.

We shall define an algorithm that takes as input an arbitrary undirected hypergraph $H$, a hyperedge $r$ of $H$ and constructs the following sequences:

- a sequence of hyperedges $r = e_0, e_1, \ldots, e_m \in E_H$,
- a sequence of ordered hypertrees $T_0, T_1, \ldots, T_m$ in $H$, all with root $r$ and such that $\{e_0, e_1, \ldots, e_i\}$ is the set of nodes of $T_i$.

The output is $T_m$. We shall prove later that when applied to $(H^+, e_0)$ this algorithm produces a depth-first spanning hypertree in $H^+$.

In what follows, we let $E_H^+$ denote the set of hyperedges of $H$ of rank at least 2, and we let $V_i$ denote $\bigcup \{Vert_H(e_j) | 0 \leq j \leq i\}$ for $i \geq 0$.


**Input:** An undirected hypergraph $H$ and a hyperedge $r$ of $H$.

**Initial step:** Construct $T_0$ as the hypertree reduced to $e_0 = r$ (we may have $e_0$ of rank 1).

**For** $i = 0, 1, 2, \ldots$, **construct** $T_{i+1}$ from $T_i$ as follows:

1. Choose $e \in E_H^+$ not in $T_i$ such that
   \[ \emptyset \neq Vert_H(e) \cap Vert_H(e_i) \quad \text{and} \quad Vert_H(e) \cap V_{i-1} = \emptyset \]

2. Let $e_{i+1} = e$ and build $T_{i+1}$ from $T_i$ by adding $e_{i+1}$ as unique (hence first) successor of $e_i$; \{property: $e_i$ is a leaf of $T_i$\}

3. If step 1 fails, let $e_j, j < i$ be the ancestor of $e_i$ in $T_i$ with $j$ maximal such that there exists $e \in E_H^+$ not in $T_i$ with:
   \[ \emptyset \neq Vert_H(e) \cap Vert_H(e_j) \quad \text{and} \quad Vert_H(e) \cap V_i \subseteq V(T_i, e_j) \]

4. Choose $e$ is as in 3, let $e_{i+1} = e$ and build $T_{i+1}$ by adding $e_{i+1}$ as new successor of $e_j$ so that $e_{i+1} = next_{T_i}(e_j)$ where $e_f$ is the last successor of $e_j$ in $T_i$.

5. If step 3 fails then exit the for-loop

**Return:** $T_i$.

Step 3 corresponds to backtracking in the classical depth-first search; the notation $V(T_i, e_j)$ is defined after Lemma 3.4. This algorithm is nondeterministic: “choose” means “choose any object” satisfying the requirements. The termination is ensured by the finiteness of $H$. With the notation of (3.6), we have the following lemma.

3.7. Lemma. For every $i = 0, 1, \ldots, m$, $T_i$ is an ordered hypertree in $H$, its root is $e_0$, its set of nodes is $\{e_0, e_1, \ldots, e_i\}$ and $e_i$ is a leaf of $T_i$. 
Proof. Easy induction on i. □

3.8. Lemma. Let $T = T_m$ be produced from $(H, r)$ by Algorithm 3.6. For every hyperedge $e$ in $E_H$ such that $\text{Vert}_H(e) \cap V_m \neq \emptyset$, either $\text{Vert}_H(e) \subseteq V(T, e_i)$ for some $i = 0, \ldots, m$ or there exists $x, y \in \text{Vert}_H(e)$ with $x <_T y$.

Proof. Let $e \in E_H$.

Case 1: $\text{Vert}_H(e) \subseteq \text{Vert}_H(e_i)$ for some $i$. If $\text{Vert}_H(e) \subseteq V(T, e_i)$, we are done; if $\text{Vert}_H(e) \subseteq \text{Vert}_H(e_i) - V(T, e_i)$ then we are done also because $\text{Vert}_H(e_i) - V(T, e_i) \subseteq V(T, e')$ where $e'$ is the predecessor of $e_i$; the last case is when we have $x, y \in \text{Vert}_H(e)$ such that $x \in V(T, e_i), y \in \text{Vert}_H(e_i) - V(T, e_i)$: then $y \in V(T, e')$ where $e'$ is the predecessor of $e_i$ and we have $x <_T y$ as desired.

Case 2: $\text{Vert}_H(e)$ is not a subset of any set $\text{Vert}_H(e_i), i = 0, \ldots, m$; in particular $e$ is not any of $e_0, \ldots, e_m$. Note also that $V_m$ is the union of the sets $V(T, e_i)$ for $i = 0, \ldots, m$. Since $\text{Vert}_H(e) \cap V_m \neq \emptyset$ there exists a first integer $i \in \{0, \ldots, m\}$ such that $\text{Vert}_H(e) \cap V(T, e_i) = \emptyset$, and we consider two subcases:

Subcase 2.1: $\text{Vert}_H(e) \cap V_m \subseteq V(T, e_i)$.

If $e_i$ is a leaf of $T$, this means that Step 1 in the construction of $T_{i+1}$ from $T_i$ by Algorithm 3.6 has failed. But $e$ meets the requirements of Step 1 (because $\text{Vert}_H(e) \cap V_{i-1} = \emptyset$ by the hypothesis of Subcase 2.1), hence we get a contradiction because $e_i$ should not have remained a leaf.

If $e_i$ is not a leaf of $T$, then let $j$ be the largest integer such that $e_j$ is a descendent of $e_i$ in $T$; this means that in the construction of $T_{j+1}$ from $T_j$ no $e'$ has been found such that:

$$\emptyset \neq \text{Vert}_H(e') \cap \text{Vert}_H(e_i) \text{ and } \text{Vert}_H(e') \cap V_j \subseteq V(T, e_i).$$

But $\text{Vert}_H(e) \cap V_j \subseteq \text{Vert}_H(e) \cap V_m \subseteq V(T, e_i) = V(T, e_j)$, hence $e$ could have been chosen to yield $\text{next}_{T_{j+1}}(e_{j'})$ where $e_{j'}$ is the last successor of $e_j$ in $T_{j}$. But the definition of $j$ yields that $\text{next}_{T_{j+1}}(e_{j'})$ is undefined, hence this subcase cannot happen.

Subcase 2.2: $\text{Vert}_H(e) \cap V_m$ is not a subset of $V(T, e_i)$.

Hence there is a first integer $j > i$ such that $\text{Vert}_H(e) \cap V(T, e_j) \neq \emptyset$. Again we get two subcases.

Subcase 2.2.1: $e_j$ is a descendent of $e_i$. In this case we can take for $y$ any element of $\text{Vert}_H(e) \cap V(T, e_i)$ and for $x$ any element of $\text{Vert}_H(e) \cap V(T, e_j) \cap V(T, e_i)$ and we get $x <_T y$ as desired.

Subcase 2.2.2: $e_j$ is not a descendent of $e_i$. As in Subcase 2.1, we distinguish two subcases, according to whether $e_i$ is a leaf of $T$ or not. In both subcases we get contradictions by essentially the same arguments.

This completes the proof. □

Proof of Proposition 3.5. We now apply Algorithm 3.6 to a hypergraph of the special form $H^+$ by taking $e_0$ as root of the hypertree to be constructed. We let $T = T_m$ be the hypertree so constructed.
Claim. Every hyperedge \( e \in E_H \) has some vertex in \( V_m \).

Proof of Claim. Assume on the contrary \( \{x_1, \ldots, x_n\} = \text{Vert}_H(e) \) and \( x_i \notin V_m \) for each \( i = 1, \ldots, n \). This means that, in Algorithm 3.6, a failure to extend \( T_m \) has occurred, but the edge linking \( u_0 \) and \( x_i \) could have been chosen to extend \( T_m \) by Step 3 of the algorithm. This gives a contradiction. \( \square \)

We can now apply Lemma 3.8 which gives that for every hyperedge \( e \in E_H \) either \( x <_T y \) for some \( x, y \in \text{Vert}_H(e) \) or \( \text{Vert}_H(e) \subseteq V(T, e_i) \) for some \( i = 0, 1, \ldots, m \). This establishes that \( T \) is a depth-first spanning hypertree in \( H \), as desired. \( \square \)

Let us recall that we denote by \( D_H \) the domain of the structure \( |H|_2 \) namely the set \( V_H \cup E_H \).

3.9. Proposition. For every integer \( k \geq 2 \), one can construct MS₂-formulas \( \theta(X_1, \ldots, X_n) \) and \( \mu(X_1, \ldots, X_n, x, y) \) such that, for every undirected hypergraph \( H \) of rank at most \( k \):

(1) there exist \( X_1, \ldots, X_n \subseteq D_H \) such that \( |H|_2 \not\models \theta(X_1, \ldots, X_n) \),
(2) for every such \( n \)-tuple \( X_1, \ldots, X_n \), the binary relation \( R \) on \( D_H \) defined by:

\[
R(x, y) \text{ if and only if } |H|_2 \models \mu(X_1, \ldots, X_n, x, y)
\]

is a partial order on \( V_H \), that is linear on each set \( \text{Vert}_H(e) \) for \( e \in E_H \).

Proof. The proof is by induction on \( k \). The case \( k = 1 \) is obvious because the trivial partial order \( \leq \) such that \( x \leq y \) if and only if \( x = y \) satisfies the desired conditions.

We establish the case \( k \) by assuming the case \( k - 1 \) to hold. We first assume that \( H \) has a depth-first spanning hypertree with set of nodes \( N \subseteq E_H \) and root \( r \in N \). We shall use the notation of the proof of Proposition 3.5 and we let in addition \( V(T) \) denote the union of the sets \( V(T, e) \) for \( e \in N \). We have noted that the sets \( V(T, e) \) form a partition of \( V(T) \). They are all of cardinality at most \( k \). Hence, there exists a \( k \)-tuple \( Y_1, \ldots, Y_k \) of subsets of \( V(T) \) forming a partition of \( V(T) \) such that:

(1) each set \( V(T, e) \cap Y_i \) has at most one element.

From \( N, r \) (which define \( T \)) and \( Y_1, \ldots, Y_k \), one can define a partial order on \( V(T) \) as follows:

\[
x \leq y: \text{ if and only if either } x \ <_T y \\
\text{ or } x \text{ and } y \text{ belong to some set } V(T, e) \text{ and } x \in Y_i, y \in Y_j, \text{ with } i \leq j.
\]

Since \( \leq_T \) is a quasiorder, since \( x \leq_T y \) if \( x \) and \( y \) belong to a same set \( V(T, e) \), and by the hypothesis on \( Y_1, \ldots, Y_k \), \( \leq \) is indeed a partial order. Furthermore, it can be
written in $\text{MS}_2$ that $N, r$ define a depth-first spanning hypertree in the considered hypergraph $H$, and that $Y_1, \ldots, Y_k$ form a partition of $V(T)$ satisfying Condition (1); since $\leq_T$ is definable from $N$ and $r$ by an $\text{MS}$ formula (see Lemma 3.4 and the subsequent remarks), the partial order $\leq$ is definable from $N, r, Y_1, \ldots, Y_k$ by an $\text{MS}$ formula.

The proof is not complete because $V(T)$ is sometimes strictly included in $V_H$. We shall use induction on $k$ to handle the general case.

We let $W(T) = \bigcup \{ \text{Vert}_H(e) / e \in N' \} - V(T)$ where $N'$ is the set of hyperedges $e$ in $E_H$ such that $\text{Vert}_H(e)$ is not included in $V(T)$. We let $K$ be the hypergraph such that:

$V_K = W(T)$,
$E_K = N'$,
$\text{Vert}_K(e) = \text{Vert}_H(e) - V(T)$.

Since $T$ is a depth-first spanning hypertree in $H$, each hyperedge in $N'$ has at least two vertices in $V(T)$ hence at most $k - 2$ vertices not in $V(T)$. Its rank as a hyperedge of $K$ is thus at most $k - 2$. By the induction hypothesis, there exists a pair of $\text{MS}$ formulas that defines on $V_K$ a partial order $\leq'$ which is linear on each set $\text{Vert}_K(e)$. These formulas use parameters, say $X_1, \ldots, X_n$. Since, $K$ is definable "in" $H$ by $\text{MS}_2$ formulas, or more formally, since $K$ is defined from $H$ by a noncopying $(2,2)$-definable transduction, these formulas can be translated (by Proposition 1.1) into formulas to be satisfied in $H$. These new formulas have parameters $N, r, X_1, \ldots, X_n, Y_1, \ldots, Y_k$, so that $\leq'$ is also definable in $H$. We can combine $\leq$ and $\leq'$ into a partial order $\leq''$ on $V_H$:

$x \leq'' y$ if and only if either $x = y$ or $x \leq y$ or $x \leq' y$

or $x \in V(T)$ and $y \in W(T)$.

This partial order is clearly definable by an $\text{MS}_2$ formula using the parameters $N, r, X_1, \ldots, X_n, Y_1, \ldots, Y_k$. Since $\leq$ is linear on each set $\text{Vert}_H(e)$ included in $V(T)$, since $\leq'$ is linear on each set $\text{Vert}_K(e) = \text{Vert}_H(e) \cap W(T)$, the partial order $\leq''$ is linear on each set $\text{Vert}_H(e)$.

This proof works for every hypergraph $H$ of rank at most $k$ having a depth-first spanning hypertree. However, the existence of a depth-first spanning hypertree has been established for $H^*$ and not for $H$. The transformation of $H$ into $H^*$ is a $(2,2)$-definable transduction (for $H$ nonempty which is the only case of interest). One can build a definition scheme for this transduction such that every vertex $v$ of $V_H$ is represented in the structure $|H^*|_2$ by the pair $(v, 1)$. (Let us recall that every vertex of $H$ is a vertex of $H^*$.) The first part of the proof gives us a pair of formulas defining a partial order in $H^*$. Hence, it follows from Proposition 1.1 that these formulas can be translated so as to define it in $H$. Its restriction to the vertices of $V_H$ is the desired partial order. □
**Proof of Theorem 3.1.** From a partial order \( \preceq \) on the set \( V_H \) that is linear on the set of vertices of every hyperedge, one gets immediately an acyclic orientation of the hypergraph: it suffices to order these sets according to \( \preceq \). We have defined such a partial order by MS formulas, hence the corresponding orientation of hyperedges can be defined by such formulas. The corresponding transduction is noncopying: it does not modify the domain of the structure \( |H|_2 \) representing a hypergraph \( H \); it replaces the incidence relation \( \text{inc}_H \) by the "next incident vertex" relation \( \text{next}_H \). □

The following result says that from any orientation of a directed hypergraph of rank at most some fixed \( k \), all other orientations can be defined by \( \text{MS}_2 \) formulas in terms of appropriate parameters.

**3.10. Proposition.** For every integer \( k \), the transduction mapping a hypergraph \( H \) in \( H_k \) to the set \( \text{und}^{-1}(\text{und}(H)) \subseteq H_k \) is \( (2, 2) \)-definable and noncopying.

**Proof.** Reorienting a hyperedge \( e \) with sequence of vertices \( (x_1, \ldots, x_n) \) consists in permuting this sequence, making it into \( (x_{\pi(1)}, \ldots, x_{\pi(n)}) \) for some bijection \( \pi \) of \( \{1, \ldots, n\} \) onto itself. Hence, a reorientation of a hypergraph \( H \) of \( H_k \) is completely defined by pairwise disjoint sets \( X_e \) of hyperedges, where \( \pi \) is a bijection of \( \{1, \ldots, n\} \) onto itself for some \( n \) in \( \{2, \ldots, k\} \) and the hyperedges in \( X_e \) are all of rank \( n \). The corresponding reorientation \( H' \) of \( H \) is obtained by permuting by \( \pi \) the sequence of vertices of the hyperedges in \( X_e \). This can be formalized as a noncopying \( (2,2) \)-definable transduction with parameters \( X_e \). □

From this proposition and Theorem 3.2, we get immediately the following result which says that all orientations of undirected hypergraphs of rank at most \( k \) can be specified by a noncopying \((2,2)\)-definable transduction.

**3.11. Theorem.** For every \( k \geq 2 \), the transduction \( \text{und}^{-1} : UH_k \to H_k \) is \( (2, 2) \)-definable and noncopying.

**Proof.** This follows from Proposition 1.2 since \( \text{und}^{-1} \) is the composition of the transduction \( \omega_k : UH_k \to H_k \) of Theorem 3.1 and of \( \text{und}^{-1} \circ \text{und} : H_k \to H_k \) which are both \((2,2)\)-definable and noncopying (by Proposition 3.10 for the latter). □

**3.12. Corollary.** For each \( k \), the languages \( \text{MS}_2 + \text{AO} \) and \( \text{MS}_2 \) are equally powerful for expressing properties of undirected hypergraphs of rank at most \( k \).

**Proof.** Let \( \psi \) in \( \text{MS}_2 + \text{AO} \) define a set \( L \subseteq UH_k \). Then \( L = \text{und}(L') \), where \( L' \) is the subset of \( H_k \) defined by \( \psi \). Since \( \text{und}^{-1} \) is \((2,2)\) definable, \( L = (\text{und}^{-1})^{-1}(L') \) is 2-definable by Proposition 1.2. □
We have proved in Section 2 that Theorem 3.1 does not hold for MS$_1$: the proof was done for graphs so that it works a fortiori for hypergraphs. In the statement of Theorem 3.1 we cannot eliminate the bound $k$ on the rank of hypergraphs. Otherwise, we would have a definable transduction $\tau: \text{UH} \to \text{H}$ such that $\tau(H)$ is nonempty, $\text{und}(K) = H$ for every $H$ in $\text{UH}$ and every $K$ in $\tau(H)$. For a hypergraph $H$ consisting of a unique hyperedge and an arbitrary set $V$ as set of vertices, $\tau(H)$ would define a linear ordering on $V$, one could express in MS$_1$ that $V$ has an even number of elements and we know that this is impossible (see [6]).

4. The structure of sets of graphs and hypergraphs having a decidable monadic theory

We shall consider the problem of characterizing in terms of complexity measures the sets of graphs and hypergraphs having a decidable MS$_1$ or MS$_2$-theory. This problem is far from being completely solved. In this section, we give new partial results. Our starting point is the following theorem reviewing the known results.

4.1. Theorem. Let $k$ be any integer; let $H$ be any undirected graph. Tree-width characterizes
(1) the sets of simple graphs of degree at most $k$ that have a decidable MS$_1$-theory,
(2) the sets of graphs not containing $H$ as a minor that have a decidable MS$_1$-theory,
(3) the sets of graphs that have a decidable MS$_2$-theory,
(4) the sets of directed hypergraphs of rank at most $k$ that have a decidable MS$_2$-theory.

Seese has established in [24, 25] Assertion 3 and the special case of Assertion 2 for planar graphs (instead of graphs that do not contain some fixed $H$ as a minor).

Proof (sketch). The MS$_1$-theory of any set of arbitrarily large square grids is undecidable because terminating computations of Turing machines can be encoded by MS formulas on large enough square grids. The transduction from a graph to its minors is $(2,2)$-definable (see [9]), and the class of square grids is $2$-definable. Hence, the transduction from a graph to its square grid minors is $(2,2)$-definable. One concludes that $L$ has bounded tree-width by Corollary 1.3 and Proposition 1.4 if $L$ have a decidable MS$_2$-theory. This gives Assertion 3.

It is proved in [14] that the identity is $(1,2)$-definable on the class of simple graphs of degree at most $k$ (for any fixed integer $k$) and on that of simple graphs that do not contain $H$ as a minor (for any fixed undirected graph $H$). If a set $L$ of simple graphs included in any of these classes has a decidable MS$_1$-theory, this means that the set of structures $|L|_1$ has a decidable monadic theory, hence that $|L|_2$ has a decidable monadic theory (by Corollary 1.3). It follows from Assertion 3 that the graphs in $L$ have bounded tree-width. This gives Assertions 1 and 2.

Assertion 4 is proved in [9, Proposition 2.10]. \hfill $\Box$
Our objective is now to consider the case of undirected hypergraphs of bounded rank.

4.2. Theorem. Tree-width characterizes the sets of undirected hypergraphs of bounded rank having a decidable MS$_2$-theory.

Proof. Let $k \in \mathbb{N}$. Since the transduction und is (2,2)-definable and preserves tree-width, since the set of hypergraphs in $H_k$ of tree-width at most some fixed $m$ has a decidable MS$_2$-theory, the set of hypergraphs in $UH_k$ of tree-width at most $m$, which is the image of the former set by und has also a decidable MS$_2$-theory by Corollary 1.3.

Let $L \subseteq UH_k$. By Theorem 3.1 the transduction und has a (2,2) definable inverse $\omega_k$: $UH_k \rightarrow H_k$. If $L$ has decidable MS$_2$-theory, then, so has $\omega_k(L)$ by Corollary 1.3. Hence the graphs in $\omega_k(L)$ have uniformly bounded tree-width by Assertion 4 of Theorem 4.1, and so has $L$ since tree-width does not depend on orientation. □

We consider next the case of undirected hypergraphs of unbounded rank. They can be considered as directed bipartite graphs. We now formalize this observation. For every undirected hypergraph $H$, we let bpg($H$) denote the simple directed graph $G$ such that:

$$V_G = V_H \cup E_H$$

$$\text{edg}_2(x, y) \iff x \in V_H, y \in E_H, x \in \text{Vert}_H(y).$$

It is clear that $|\text{bpg}(H)|_1 = |H|_2$ if we identify inc of $|H|_2$ with $\text{edg}_2$ of $|\text{bpg}(H)|_1$.

4.3. Proposition. Let $L \subseteq UH$. The following conditions are equivalent:
(i) the MS$_2$-theory of $L$ is decidable,
(ii) the MS$_1$-theory of bpg($L$) is decidable.

Proof. The implications (ii) $\Rightarrow$ (i) $\Rightarrow$ (ii) follow from Corollary 1.3 since we have a (2,1)-definable bijection bpg: $L \rightarrow \text{bpg}(L)$, the inverse of which is (1,2)-definable. □

This proposition reduces the characterization of subsets of UH having a decidable MS$_2$-theory to that of directed bipartite graphs having a decidable MS$_1$-theory. This latter case is still open (see Conjectures 4.6 and 4.7) but seems easier to handle than that of hypergraphs of unbounded rank. Here is an application of Theorem 4.2 to sets of graphs.

Let $G$ be a (directed or undirected) graph. The quasi-degree of $G$ is defined as:

$$qdeg(G) = \min \{k \in \mathbb{N} | \text{no two vertices of degree } > k \text{ are adjacent}\}.$$ 

It is clear that $qdeg(G) \leq \deg(G)$ for every graph.

The degree of a hypergraph is the maximum number of hyperedges incident with a same vertex. If $G = \text{bpg}(H)$ for some hypergraph $H$ of rank or of degree at most $k$, then $qdeg(G) \leq k$. 
4.4. Theorem. Let \( L \) be a set of simple (directed or undirected) graphs of bounded quasi-degree. If the MS\(_1\)-theory of \( L \) is decidable, then so is its MS\(_2\)-theory and the graphs in \( L \) have uniformly bounded tree-width.

Proof. We first consider the case where \( L \) is a subset of the set \( \mathcal{Q} \) of simple undirected graphs of quasidegree at most \( k \). We shall establish that the identity on \( \mathcal{Q} \) is (1,2)-definable; in other words, we shall prove that the transduction \( \{(|G_1|, |G_2|) | G \in \mathcal{Q}\} \) is definable. It will follow that \( L \) has a decidable MS\(_2\)-theory by Corollary 1.3 and that the graphs in \( L \) have bounded tree-width by Theorem 4.1.

We first establish this fact with \( B \) instead of \( \mathcal{Q} \), where \( B = \text{und}(\text{bpg}(\text{UH}_k)) \). In other words, \( G \in B \) if and only if \( G \) is a simple undirected graph and:

(B) there is a partition \((X, Y)\) of \( V_G \) such that all vertices in \( Y \) have degree at most \( k \) and all edges link a vertex in \( X \) and a vertex in \( Y \).

Fig. 6 shows a graph \( G \) in \( B \) (to the left) and a graph \( G' \) in \( \mathcal{Q} \) (to the right) for \( k = 3 \).

For every \( G \in B \):

\[
G = \text{und}(\text{bpg}(\omega_k(\sigma(G, X, Y))))
\]

for some subsets \( X \) and \( Y \) of \( V_G \), where \( \omega_k \) is the transduction of Theorem 3.1 and \( \sigma \) is the transduction \( B \rightarrow \text{UH}_k \) defined as follows, with parameters \( X \) and \( Y \):

- for \( G \in B \), we let \( H = \sigma(G, X, Y) \in \text{UH}_k \) be defined if and only if \( X, Y \subseteq V_G \) and satisfy condition (B) and all isolated vertices of \( G \) belong to \( X \),
- we let then \( V_H = X \), \( E_H = Y \),
- and \( x \in \text{Vert}_H(y) \) if and only if \( x \) is in \( X \), \( y \) is in \( Y \) and \( x \) and \( y \) are adjacent in \( G \).

This transduction is (1,2)-definable.

We now verify that the transduction \( \text{bpg} \) is (2,2)-definable: \( \text{H}_k \rightarrow \text{G}_k \). Let \( H \in \text{H}_k \) and \( G = \text{bpg}(H) \). Then:

- \( V_G = V_H \cup E_H \),

Fig. 6.
\[ E_G = \{(e, i) \mid e \in E_H, \ 1 \leq i \leq n \ \text{where} \ n \ \text{is the rank of} \ e\}, \]
\[ (e, i) \ \text{links} \ x \ \text{to} \ y \ \text{in} \ G \ \text{if and only if} \ (e, i) \in E_G, \ y = e, \ \text{and} \ x \ \text{is the} i\text{-th vertex of the sequence} \ \text{Vert}_G(e). \]

From this characterization it is clear that \( \text{bpg} \) is (2,2)-definable: \( H_k \rightarrow G_s. \) Since the composition of definable transductions is definable by Proposition 1.2, it follows from (1) that the identity on \( B \) if (1,2)-definable, as desired. Hence, the transduction \( \{(|G|_1, |G|_2) \mid G \in B\} \) is definable since it is a composition of definable transductions. (Note that (1) holds for any choice of \( X \) and \( Y \) satisfying the required conditions.)

We now establish the same result for \( Q. \) For \( G \in Q, \) we define
\[ X = \{x \in V_G \mid \text{deg}(x) \geq k + 1\}, \]
\[ Y = \{x \in V_G \mid 0 < \text{deg}(x) \leq k\}, \]
\[ Z = \{x \in V_G \mid \text{deg}(x) = 0\}. \]

First, we let \( G' \) be the induced subgraph of \( G \) with set of vertices \( Y. \) Its degree is at most \( k. \) Since the identity is a (1,2)-definable transduction on each set of simple undirected graphs of bounded degree (see [7]) we obtain that the transduction \( \tau_1 = \{(|G|_1, |G'|_2) \mid G \in Q\} \) is definable.

Second, we let \( G'' \) be the subgraph of \( G \) with set of vertices \( X \cup Y \) and set of edges \( E_G - E_G^* \): these edges link vertices of \( X \) and vertices of \( Y, \) hence \( G'' \in B. \) We have seen above that the identity is (1,2)-definable on \( B, \) hence the transduction \( \tau_2 = \{(|G|_1, |G''|_2) \mid G \in Q\} \) is definable. By combining the definition schemes for \( \tau_1 \) and \( \tau_2, \) we can obtain a definition scheme establishing that the identity is (1,2)-definable on \( Q. \)

It follows that if a subclass \( L \) of \( Q \) has a decidable \( \text{MS}_1 \)-theory, it has also a decidable \( \text{MS}_2 \)-theory (by Corollary 1.3) and that its members have uniformly bounded tree-width (by Theorem 4.1).

We consider now the case where \( L \) is a set of directed graphs of bounded quasi-degree having a decidable \( \text{MS}_1 \)-theory. Then \( \text{und}(L) \) has a decidable \( \text{MS}_1 \)-theory by Corollary 1.3 (or an easier more direct argument), hence a bounded tree-width by the previous case. Since, by the results of [7, 14], the identity is (1,2)-definable on the set of simple directed graphs of tree-width at most any fixed \( k, \) it follows from Corollary 1.3 that the \( \text{MS}_2 \)-theory of \( L \) is also decidable. \( \Box \)

**4.5. Corollary.** Let \( k \in \mathbb{N} \) and \( \mathcal{G} \) be the class of undirected hypergraphs of degree at most \( k. \) The subsets of \( \mathcal{G} \) having a decidable \( \text{MS}_2 \)-theory are characterized by the complexity measure \( \mu \) such that \( \mu(H) = \text{twd}(\text{bpg}(H)). \)

**Proof.** For every \( k \) and \( m, \) let \( \mathcal{G}_k \) be the class of hypergraphs of degree at most \( k \) and \( \mathcal{G}_{k,m}, \) the subclass of those, \( H, \) such that \( \mu(H) \leq m. \)

We first prove that the \( \text{MS}_2 \)-theory of \( \mathcal{G}_{k,m} \) is decidable. The class of graphs \( \text{bpg}(\mathcal{G}_{k,m}) \) is 1-definable because the class of graphs of tree-width at most \( m \) is
1-definable (see [9]); since its members have (by definition) tree-width at most \(m\), its \(\mathbf{MS}_1\)-theory is decidable (by the results of [6]). Hence, the \(\mathbf{MS}_2\)-theory of \(\mathcal{C}_{k,m}\) is decidable since the classes of structures \(\mathcal{C}_{k,m,1}^2\) and \(\mathcal{B}_1\sgn(\mathcal{C}_{k,m})\) are equal up to isomorphism.

Let \(L\) be a subclass of \(\mathcal{C}_k\) having a decidable \(\mathbf{MS}_2\)-theory, then \(\mathcal{B}_1\sgn(L)\) is a class of graphs of quasidegree at most \(k\), having a decidable \(\mathbf{MS}_1\)-theory; hence its members have tree-width at most \(m\) for some \(m\) (by Theorem 4.4). Hence, \(L\) is a subclass of \(\mathcal{C}_{k,m}\) for some \(m\). \(\square\)

We consider next the case of sets of graphs having a decidable \(\mathbf{MS}_1\)-theory. We denote by \(\mathcal{B}\) the set of rooted ordered binary trees.

4.6. Conjecture. Let \(L\) be a set of directed or undirected hypergraphs of rank at most \(k\). If the \(\mathbf{MS}_1\)-theory of \(L\) is decidable, then \(L \subseteq \tau(\mathcal{B})\) for some \((1, 1)\)-definable transduction.

Since the \(\mathbf{MS}_1\)-theory of \(\mathcal{B}\) is decidable (see [28]) every set of hypergraphs of the form \(\tau(\mathcal{B})\) as in (4.6) has a decidable \(\mathbf{MS}_1\)-theory by Corollary 1.3. We shall define a complexity measure on graphs called \emph{clique-width} giving an equivalent form of Conjecture 4.6 for sets of graphs. Since the definition of clique-width is quite technical, we postpone to the appendix its definition, and the proof of the equivalence of the two conjectures for the case of graphs.

4.7. Conjecture. For each \(k\), \emph{clique-width} characterizes the sets of graphs having a decidable \(\mathbf{MS}_1\)-theory.

We shall establish Conjecture 4.6 for certain types of chordal graphs. Let \(G\) be a simple undirected graph. A \emph{clique in} \(G\) is a maximal complete subgraph with at least two vertices. (Maximality is relative to subgraph inclusion.) The \emph{clique-degree} of \(G\) is the maximum number of cliques containing any vertex. We denote it by \(\text{cl-deg}(G)\).

A graph \(G\) in \(\mathcal{U}_G\), is \emph{chordal} if all its induced cycles are triangles, equivalently, if every cycle of length \(\geq 4\) has a chord. By a theorem of Dirac [17] (see also [16]), a graph is chordal if and only if it has a tree-decomposition the boxes of which are the cliques. For technical convenience we consider only connected chordal graphs.

Let us note that from the initial definition, the set of chordal graphs is 1-definable. It suffices to observe that a graph \(G\) is not chordal if and only if it has two sets \(X\) and \(Y\) of at least three vertices, the intersection of which is a set \(\{x, y\}\) of two vertices and such that, there is no edge between \(x\) and \(y\), there is no edge between a vertex of \(X - \{x, y\}\) and a vertex of \(Y - \{x, y\}\), and the subgraphs of \(G\) induced by \(X\) and by \(Y\) are connected. These conditions are expressible by a \(\mathbf{MS}_1\)-formula. It follows that the class of chordal graphs is 1-definable.

4.8. Theorem. Let \(k\) be any integer. If a set of chordal graphs of clique-degree at most \(k\) has a decidable \(\mathbf{MS}_1\)-theory, then it is contained in \(\tau(\mathcal{B})\) for some \((1, 1)\)-definable
transduction \( \tau \). It follows that clique-width characterizes the sets of chordal graphs of clique-degree at most \( k \) having a decidable MS\(_1\)-theory.

For every simple undirected graph \( G \), let its hypergraph of cliques, denoted by \( \text{cl}(G) \), be the hypergraph \( H \) constructed as follows:

\[ \mathbf{V}_H = \mathbf{V}_G, \]

\( \mathbf{E}_H \) is the set of cliques of \( G \),

\( x \in \text{Vert}_H(e) \) if and only if \( x \) is a vertex of the clique \( e \).

If the only cliques of \( G \) are its edges, then \( \text{cl}(G) = G \). It is clear that \( \text{cl}^{-1}: \mathbf{U}_H \to \mathbf{U}_G \) is a \((2, 1)\)-definable transduction, consisting in the replacement in a hypergraph of every hyperedge of rank \( k \) by a complete graph \( K_k \), and the deletion of duplicate edges. Note in particular that a hypergraph \( H \) is of the form \( \text{cl}(G) \) for some graph \( G \) if and only if no hyperedge is properly included in any other hyperedge.

4.9. Lemma. For each \( k \in \mathbb{N} \), the transduction \( \text{cl}: \mathbf{U} \to \mathbf{U} \) is \((1, 2)\)-definable on the set of chordal graphs of clique-degree at most \( k \).

Proof. Let us fix \( k \in \mathbb{N} \) with \( k \geq 2 \). Let \( G \) be a chordal graph of clique degree at most \( k \) and \((T, f)\) be a tree-decomposition of \( G \), the boxes of which are the cliques of \( G \). By choosing a root \( r \), we make \( T \) into a directed tree. For every \( x \in \mathbf{N}_T \), we let \( g(x) := f(x) - f(y) \) where \( y \) is the predecessor of \( x \) in \( T \). (If \( x \) is the root, then \( g(x) = f(x) \)).

Claim 1. Let \( v \in \mathbf{V}_G \) and \( x \in \mathbf{N}_T \). Then \( v \in g(x) \) if and only if \( x \) is the node of \( T \) such that \( v \) is closest as possible to the root.

Proof. The case where \( v \in f(r) \) is clear. Let \( v \in g(x) = f(x) - f(y) \) where \( y \) is the predecessor of \( x \) in \( T \).

If \( v \in f(z) \) where \( z \neq x \) and \( z \) is an ancestor of \( x \), then \( v \in f(y) \) by the definition of a tree-decomposition. This contradicts the hypothesis. Hence \( x \) is the node of \( T \) which is closest to the root such that \( v \in f(x) \).

Let conversely assume \( x \) is closest to the root such that \( v \in f(x) \). Then \( v \notin f(y) \) because \( y \) is closer to \( r \) than \( x \), hence \( v \notin f(x) - f(y) = g(x) \).

It follows from this claim that \( g(x) \cap g(y) = \emptyset \) if \( x \neq y \).

Let \( C \) be the set of colors \( \{0, \ldots, 2k\} \). We let \( \gamma: \mathbf{N}_T \to C \) be the coloring of \( T \) defined as follows. The color of the root \( r \) of \( T \) is 0. If a node has color \( i \), then its successors have color \( \text{mod}_{2k+1}(i+1) \), namely, the remaining of the division of \( i+1 \) by \( 2k+1 \).

We let \( \gamma': \mathbf{V}_G \to C \) be such that \( \gamma'(v) = \gamma(x) \) where \( x \) is the unique node of \( T \) such that \( v \in g(x) \). We let \( X_0, \ldots, X_{2k} \) be the subsets of \( \mathbf{V}_G \) such that \( X_i = \gamma'^{-1}(i) \) for each \( i \in C \). We shall establish that \( \mathbf{E}_H, \text{inc}_H \) for \( H = \text{cl}(G) \) can be reconstructed from \( X_0, \ldots, X_{2k} \).
Our coloring \( \gamma \) of the nodes of \( T \) uses the cyclic graph of colors \( 0 \to 1 \to 2 \to \cdots \to 2k \to 0 \). For every \( i \in C \), we let \( p(i) \) denote the set consisting of \( i \) and the \( k - 1 \) colors preceding \( i \) in this graph. Hence, for instance, if \( k \geq 4 \), \( p(2) = \{ k + 4, \ldots , 2k, 0, 1, 2 \} \).

**Claim 2.** Let \( x \in \mathbb{N}_T \), \( i \in C \) and \( v \in g(x) \cap X_i \). Then \( f(x) = \{ v' \in V_G \mid v' \text{ is adjacent to } v \text{ and } v' \in X_j \text{ for some } j \in p(i) \} \).

**Proof.** We let \( x \in \mathbb{N}_T \), \( i \in C \) and \( v \in g(x) \cap X_i \).

Let \( v' \in f(x) \). Then \( v' \) is adjacent to \( v \) since \( f(x) \) induces a complete subgraph of \( G \). If \( v' \in g(x) \), then its color is \( i \). Otherwise, \( v' \) belongs to \( g(y) \) for some ancestor \( y \) of \( x \). Since \((T, f)\) is a tree-decomposition, \( v' \) also belongs to \( f(z) \) for every \( z \) on the path in \( T \) from \( y \) to \( x \). Since \( G \) has clique-degree at most \( k \), this path has at most \( k \) vertices. Hence, from the way \( T \) has been colored, the color of \( y \) belongs to \( p(i) \). So does the color of \( v' \), hence \( v' \in X_j \) for some \( j \in p(i) \).

Let us assume conversely that \( v' \) is adjacent to \( v \) and \( v' \in X_j \) for some \( j \in p(i) \). We shall prove that \( v' \in f(x) \). By the definition of a tree-decomposition, \( v \) and \( v' \) both belong to \( f(z) \) for some \( z \).

**Case 1:** \( z = x \). We have the desired result.

**Case 2:** \( z \neq x \) and \( z \) is not a descendent of \( x \).

Then, the unique undirected path in \( T \) from \( x \) to \( z \) contains \( y \), the predecessor of \( z \) (that must exist since \( z \) is not a descendent of \( x \)). Since \( v \in f(x) \cap f(z) \), this vertex also belongs to \( f(y) \) by the definition of a tree-decomposition. Hence \( v \notin g(x) = f(x) - f(y) \). Contradiction with the assumption that \( v \in g(x) \cap X_i \).

**Case 3:** \( z \neq x \) and \( z \) is a descendent of \( x \).

Let \( x = y_0 \to y_1 \to y_2 \to \cdots \to y_{n-1} \to z = y_n \) be the path in \( T \) from \( x \) to \( z \). Let \( m \) be the least integer such that \( v' \in f(y_m) \). If \( m = 0 \), then \( v' \in f(x) \) as desired. Otherwise, \( v' \in g(y_m) \) by Claim 1 (because otherwise, if \( v' \in f(z') \) for some ancestor \( z' \) of \( y_m \), then \( v' \in f(y_{m-1}) \) by the definition of a tree-decomposition, contradicting the definition of \( m \)). Let the color of \( y_m \) in \( T \) be \( j \in p(i) \). By the definition of \( C, \gamma \), and \( p \), the colors of \( y_1, \ldots , y_{k+1} \) must be pairwise distinct and outside of the set \( p(i) \). Hence \( m > k + 1 \).

But we get a contradiction because the cliques \( f(y_0), \ldots , f(y_{k+1}), \ldots , f(y_m) \) are pairwise distinct and all contain \( v \) (since \( v \in f(z) \)), so that \( G \) has clique-degree larger than \( k \).

Hence finally we have the desired result in the first and third case. The second case cannot happen. \( \square \)

We now show how to construct the hypergraph of cliques \( H = \text{cl}(G) \). For every node \( x \) in \( T \), we pick a vertex in \( g(x) \) and we form in this way a set \( U \) of vertices of \( G \) in bijection with \( \mathbb{N}_T \) and \( E_H \). For every \( u \in U \), we denote by \( e(u) \) the clique \( f(x) \) where \( x \) is the unique node of \( T \) such that \( u \) is in \( g(x) \). It follows from Claim 2 that for every \( u \in U \)
and every \( w \in V_G \):
\[
w \in e(u) \quad \text{if and only if} \quad w \text{ is adjacent to } u \text{ and } w \in X_j \quad \text{for some } j \text{ in } p(i), \text{ where } i \text{ is such that } u \in X_i.
\]

If \( X_0, \ldots, X_{2k}, U \) are as above, we can thus construct \(|H|_2\) as follows:

(i) A pair \((v, 1)\) where \(v\) is in \(V_G\) represents the vertex \(v\) of \(V_H\).

(ii) A pair \((u, 2)\) where \(u\) is in \(U\) represents the clique \(e(u)\).

It follows that \(\text{inc}_H\) is defined in such a way that:

(iii) \(((w, 1), (u, 2)) \in \text{inc}_H\) if and only if \(u \in U\), \(w\) is adjacent to \(u\) and \(w \in X_j\) for some \(j\) in \(p(i)\), where \(i\) is such that \(u \in X_i\).

Hence, \(|H|_2\) can be defined from \(|G|_1, X_0, \ldots, X_{2k}, U\) by a definable transduction.

It remains only to write the formula expressing that sets \(X_0, \ldots, X_{2k}, U\) are indeed as required so that the structure defined by conditions (i)–(iii) is actually \(|\text{cl}(G)|_2\). For any tuple \((X_0, \ldots, X_{2k}, U)\) of subsets of \(V_G\), there exists a structure \(S\) with domain \(V_G \times \{1\} \cup U \times \{2\}\), and the binary relation \(\text{inc}_S\) defined by (iii). This structure is indeed \(|\text{cl}(G)|_2\) if and only if:

(a) a subset \(K\) of \(V_G\) is a clique if and only if there exists a vertex \(u\) in \(U\) such that \(K\) is the set of vertices \(w\) of \(G\) such that \(w\) is adjacent to \(u\) and \(w \in X_j\) for some \(j\) in \(p(i)\), where \(i\) is such that \(u \in X_i\);

(b) distinct vertices in \(U\) correspond in this way to distinct cliques.

These two conditions can be expressed by \(\text{MS}_1\)-formulas. By the first part of the proof, sets \(X_0, \ldots, X_{2k}, U\) of vertices satisfying them do exist whenever \(G\) is chordal of clique-degree at most \(k\). Hence, the transduction \(\text{cl}\) is \((1,2)\)-definable on the class of chordal graphs of clique-degree at most \(k\). \qed

**Proof of Theorem 4.8.** Let \(L\) be a set of chordal graphs of clique degree at most \(k\) having a decidable \(\text{MS}_1\)-theory. Then it follows from Lemma 4.9 and Corollary 1.3 that \(\text{cl}(L)\) has a decidable \(\text{MS}_2\)-theory. Since \(L\) has clique-degree at most \(k\), the hypergraphs in \(\text{cl}(L)\) have degree at most \(k\). Hence, the set of graphs \(\text{bpg}(\text{cl}(L))\) has a decidable \(\text{MS}_3\)-theory and its graphs have quasi-degree at most \(k\). Hence \(\text{bpg}(\text{cl}(L))\) has bounded tree-width by Theorem 4.4 and this set is contained in \(\alpha(\mathbb{B})\) for some \((1,2)\)-definable transduction \(\alpha\). (See Corollary A.2 in the appendix). Hence \(L \subseteq \text{cl}^{-1} \circ \text{bpg}^{-1} \circ \alpha(\mathbb{B})\) and the transduction \(\text{cl}^{-1} \circ \text{bpg}^{-1} \circ \alpha\) is \((1,1)\)-definable since it is the composition of \(\alpha, \text{bpg}^{-1}\) and \(\text{cl}^{-1}\) which are respectively \((1,2)\)-, \((2,2)\)-, and \((2,1)\)-definable transductions. \qed

5. Conclusion

We summarize the known answers to Question 2 of the introduction in Tables 1 and 2. Whenever we write "graphs" or "hypergraphs", we understand that the result
concerns directed as well as undirected graphs or hypergraphs. Question marks indicate open problems or conjectures.

For proving Conjecture 4.7, we lack a theorem analogous to Proposition 1.4 (proved in [23]) that would say that graphs of large clique-width have some well-defined structure. Let us recall that Proposition 1.4 says that every square grid is contained as a minor in every graph of tree-width at least \( k \), for some large enough \( k \) (depending on the considered grid). All results mentioned in these two tables depend on this result.

<table>
<thead>
<tr>
<th>Type of graphs or hypergraphs</th>
<th>Property of a set ( L ) having a decidable ( \text{MS}_2 )-theory</th>
<th>References and remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphs</td>
<td>Bounded tree-width</td>
<td>[24, 25]</td>
</tr>
<tr>
<td>Hypergraphs of bounded rank</td>
<td>Bounded tree-width</td>
<td>[9] for directed hypergraphs Theorem 4.2 for undirected hypergraphs</td>
</tr>
<tr>
<td>Undirected hypergraphs</td>
<td>( ? ) bounded clique-width of the set ( \text{bpg}(L) )( ? ? )</td>
<td>Consequence of Conjecture 4.7 by Proposition 4.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type of graphs (all simple)</th>
<th>Property of a set ( L ) having a decidable ( \text{MS}_2 )-theory</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphs of bounded degree</td>
<td>Bounded tree-width</td>
<td>[7, 14]</td>
</tr>
<tr>
<td>Planar graphs; graphs that do not contain a fixed graph as a minor</td>
<td>Bounded tree-width</td>
<td>[24, 25] for planar graphs [7, 14]</td>
</tr>
<tr>
<td>Directed graphs</td>
<td>Bounded tree-width (if ( L ) is closed under reorientation)</td>
<td>Theorem 2.1</td>
</tr>
<tr>
<td>Graphs of bounded quasi-degree</td>
<td>Bounded tree-width</td>
<td>Theorem 4.4</td>
</tr>
<tr>
<td>Chordal graphs of bounded clique degree</td>
<td>Bounded clique-width</td>
<td>Theorem 4.8</td>
</tr>
</tbody>
</table>
Appendix: Context-free graph grammars, complexity measures on graphs and definable transductions

In this appendix we define clique-width, a complexity measure on graphs and we review the relationships between context-free graph grammars, tree-width, clique-width, and definable transductions of graphs.

Clique-width will be defined in terms of operations on graphs that are equipped with distinguished vertices. A graph with ports is a pair $H$ consisting of a graph $G$ and a family of pairwise disjoint sets of distinguished vertices. It is of the form $H = \langle G, (pt_u)_{u \in \{1, \ldots, m\}} \rangle$, where each $pt_u$ is a subset of $V_G$. We say that $x$ is an $i$-port if and only if $x$ belongs to $pt_i$.

We review three operations on simple graphs with ports. The first one is the unary operation $\eta_{i,j}$ that takes as input a simple directed graph with ports and augments it by adding directed edges from every $i$-port to every $j$-port. Such edges are added only between distinct vertices (so that they do not create loops) and only if they do not create multiple edges. When dealing with undirected graphs, we shall use the similar operation, denoted in the same way, that adds undirected edges. Hence, for undirected graphs, $\eta_{i,j}$ and $\eta_{j,i}$ denote the same operation.

The second operation is the disjoint union. We denote by $H \oplus H'$ the union of two disjoint copies of $H$ and $H'$. (Strictly speaking the result depends on the chosen copies of $H$ and $H'$. However, any two graphs obtained in this way are isomorphic, and we aim here at defining graphs up to isomorphism.)

The third operation is the renaming of ports: if $z$ is a finite subset of $\mathbb{N}_+ \times \mathbb{N}_+$, then $\pi_z(H)$ is the graph with ports $K = \langle H, (pt_{i,j})_{i,j \in \{1, \ldots, m\}} \rangle$ where $pt_{i,j}(x) :\iff pt_{j,i}(x)$ for some $(i,j) \in z$ and $m = \max \{i/i(i,j) \in z \text{ for some } j\}$. We shall also use a constant 1, denoting the graph with a single vertex that is the 1-port.

By combining these operations and the constant 1, one can form terms denoting simple graphs with ports. The width of a term $t$ is the maximum value of a port number in a graph denoted by a subterm of $t$. The clique-width of $H$ be the minimum width of a term denoting a graph isomorphic to $H$. We call clique-width this value because the operations upon which it is based build graphs by gluing together cliques and complete bipartite graphs. (See [12] for examples.) The tree-width of a hypergraph can be characterized similarly as the minimum width of a term denoting it, where these alternative terms are built with different operations (see [12]).

From operations on a class of graphs or hypergraphs, one can define context-free grammars as systems of recursive set equations built with set union and the considered operations. A system of equations is a tuple of the form $S = \langle u_1 = p_1, \ldots, u_n = p_n \rangle$ where:
- $u_i$ is an unknown, intended to denote a set of graphs or hypergraphs;
- $p_i$ is a polynomial, i.e., a sum of the form $t_1 \cup t_2 \cup \cdots \cup t_m$ where each $t_j$ is a finite term written with the unknowns and the considered operations; each term $t_i$ is called a monomial.

A grammar is a pair $\Gamma = (S, u_i)$ consisting of a system $S$ and an unknown $u_i$ of $S$ playing the role of the initial nonterminal in standard grammars. Every system $S$ as
above has a least solution in \( \mathcal{P}(\mathcal{G}) \times \cdots \times \mathcal{P}(\mathcal{G}) \), denoted by \((L(S, u_1), \ldots, L(S, u_n))\). The set defined by \( F \) is \( L(F) = L(S, u_k) \).

By using the operations on simple graphs with ports introduced in the definition of clique-width, one obtains the VR graph grammars (VR stands for Vertex Replacement and refers to an alternative definition in terms of rewriting sequences, see [13] for this alternative definition and the equivalence with the present one). By using the operations appropriate for the characterization of tree-width of hypergraphs, one obtains the HR hypergraph grammars (HR stands for Hyperedge Replacement and refers also to a definition in terms of rewriting sequences, see [3]). We have the following remarkably parallel characterizations and properties of the corresponding sets of graphs and hypergraphs.

A.1. Theorem. (1) A set of graphs is VR if and only if it is of the form \( \tau(\mathcal{B}) \) for some \((1,1)\)-definable transduction from trees to simple graphs.

(2) The graphs in a VR set have uniformly bounded clique-width, and, for each integer \( k \), the set of simple graphs of clique-width at most \( k \) is VR.

(3) The MS1-theory of a VR set is decidable.

(4) A set of hypergraphs is HR if and only if it is of the form \( \tau(\mathcal{B}) \) for some \((1,2)\)-definable transduction \( \tau \) from trees to hypergraphs.

(5) The hypergraphs in a HR set have uniformly bounded tree-width, and, for each integer \( k \), the set of hypergraphs of tree-width at most \( k \) is HR.

(6) The MS2-theory of a HR set is decidable.

Proof. (1) is proved in a slightly weaker form in [18]; see [14] for the present formulation; (2) is proved in [13]; (3) follows from (1), Corollary 1.3 and decidability of the MS1-theory of \( \mathcal{B} \); another proof is sketched in [13]; (4) is proved in [14]; (5) is proved in [9, 12]; (6) follows from (4), Corollary 1.3 and decidability of the MS1-theory of \( \mathcal{B} \); another proof is given in [6]. \( \square \)

The following corollary gives alternative formulations to conjectures 4.6 and 4.7.

A.2. Corollary. (1) Let \( L \) be a set of simple graphs. The following conditions are equivalent:

(i) \( L \subseteq L' \) for some VR set \( L' \),

(ii) the graphs in \( L \) have uniformly bounded clique-width,

(iii) \( L \subseteq \tau(\mathcal{B}) \) for some \((1,1)\)-definable transduction \( \tau \) from trees to simple graphs.

(2) Let \( L \) be a set of hypergraphs. The following conditions are equivalent:

(i) \( L \subseteq L' \) for some HR set \( L' \),

(ii) the hypergraphs in \( L \) have uniformly bounded tree-width,

(iii) \( L \subseteq \tau(\mathcal{B}) \) for some \((1,2)\)-definable transduction \( \tau \) from trees to hypergraphs.

With the operations on simple hypergraphs with ports introduced in [10, Theorem 4.6], one could also define the clique-width of hypergraphs and obtain results similar
to the assertions of (A.1) and (A.2) concerning VR sets. However, we differ developing these aspects to the time when the theory will be richer.

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References