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Theoretical Computer Science 299 (2003) 1–36

Theoretical Computer Science

www.elsevier.com/locate/tcs

Fundamental Study

# The monadic second-order logic of graphs XIV: uniformly sparse graphs and edge set quantifications ☆

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Received 7 January 2000; received in revised form 18 December 2000; accepted 1 May 2002
Communicated by M. Ito

**Abstract**

We consider the class $\mathbf{US_k}$ of *uniformly k-sparse simple graphs*, i.e., the class of finite or countable simple graphs, every finite subgraph of which has a number of edges bounded by $k$ times the number of vertices. We prove that for each $k$, every monadic second-order formula (intended to express a graph property) that uses variables denoting sets of edges can be effectively translated into a monadic second-order formula where all set variables denote sets of vertices and that expresses the same property of the graphs in $\mathbf{US_k}$. This result extends to the class of *uniformly k-sparse simple hypergraphs* of rank at most $m$ (for any $k$ and $m$).

It follows that every subclass of $\mathbf{US_k}$ consisting of finite graphs of bounded clique-width has bounded tree-width. Clique-width is a graph complexity measure similar to tree-width and relevant to the construction of polynomial algorithms for **NP**-complete problems on special classes of graphs.


*Keywords:* Monadic second-order logic; Graphs; Hypergraphs; Edge set quantifications; Sparse; Tree-width; Clique-width

## 0. Introduction

*Descriptive complexity* is the study of logical languages that characterize complexity classes. For example, a graph problem (the input of which is a finite graph, without

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PII: S0304-3975(02)00578-9

auxiliary integer or real data) is in the class **NP** (resp. in the polynomial hierarchy) if and only if it is expressible by an existential second-order formula (resp. by a second-order formula). This assumes that we consider the input graph as a relational structure, consisting of the set of vertices and a binary "edge" relation on this set. For linearly ordered graphs, a problem is in the class **P** if and only if it is expressible by a first-order formula using certain least-fixed point operations. Yet other classes have similar logical characterizations. We refer the reader to the survey by Immerman [20] or to the book by Ebbinghaus and Flum [17].

*Monadic second-order logic* is the fragment of second-order logic such that quantified relation symbols are monadic (i.e., unary) hence denote sets. We will refer to them as *set variables*. This language does not characterize any specific complexity class. It contains **NP**-complete problems like the 3-colorability problem, but defines linear time testable properties on special classes of graphs, in particular on the classes of graphs of tree-width at most $k$ for each $k$, or on other classes of hierarchically constructed graphs like those of *clique-width* at most $k$, for any $k$ (this notion, recalled in Section 1.5 is investigated in [15]).

There are actually two variants of MS logic (*MS* will abbreviate monadic second-order), denoted by $\mathrm{MS}_1$ and $\mathrm{MS}_2$. In $\mathrm{MS}_1$, set variables only denote sets of vertices. In $\mathrm{MS}_2$, set variables can also denote sets of edges of the considered graph. The linear time complexity result holds for $\mathrm{MS}_2$ (resp. $\mathrm{MS}_1$) definable properties and for graphs of bounded tree-width (resp. bounded clique-width). The language $\mathrm{MS}_2$ is stronger but the classes of graphs for which the linear time algorithms exist are more restricted. For example the class of all finite cliques has unbounded tree-width but clique-width 2. It is thus useful to understand the border line between $\mathrm{MS}_1$ and $\mathrm{MS}_2$, as well as the one between bounded tree-width and bounded clique-width.

In this article, we continue the investigation of the expressive power of monadic second-order logic in this respect. We improve previous results in the following ways.

1. It is proved in [8] that $\mathrm{MS}_1$ and $\mathrm{MS}_2$ are equally powerful for expressing properties of finite simple graphs belonging to several classes:

   (i) the class of graphs of degree at most $d$,

   (ii) the class of planar graphs, that of graphs of tree-width at most $t$, and more generally, any class of graphs without some fixed finite graph as a minor.

   These classes have in common that their graphs are *uniformly* $k$*-sparse* (for some $k$) which means that all subgraphs of their graphs have a number of edges at most $k$ times the number of vertices. We prove that this condition, which subsumes and generalizes properly the above cited ones, is actually sufficient to insure the result of [8].
2. We extend this result to uniformly $k$-sparse simple *hypergraphs* of rank at most $m$ (for $m=2$ we have the case of graphs),
3. We also extend it to *countably infinite* uniformly $k$-sparse simple hypergraphs of rank at most $m$ (hence also to countably infinite uniformly $k$-sparse simple graphs).
4. In order to deal with finite graphs, we use a result of [9] which consists in defining by $\mathrm{MS}_2$ formulas an *orientation* of the hyperedges of rank at most $m$ of any finite undirected hypergraph. An orientation is a linear order on the sets of

vertices of the considered hyperedges. (We need this even for graphs because Theorem 2.3 concerning hypergraphs is used for the proofs of Theorems 2.5 and 4.1 about graphs).

For the extension to *countably infinite* graphs, we generalize this technique to countably infinite hypergraphs: we use certain *depth-first spanning trees and* "*hypertrees*" in countably infinite graphs and hypergraphs. We also correct a mistake in the proof of [9, Proposition 3.9] (see Theorem 2.3 and Remark 2.4).

The two main theorems (Theorems 4.1 and 5.2) can be stated informally as follows:

**Main Theorem.** *For each integer $k$, one can effectively transform a given monadic second-order formula using edge set quantifications into one that uses only vertex set quantifications and is equivalent to the given one on finite or countable, uniformly $k$-sparse, simple, directed or undirected graphs. More generally this result holds for finite or countable, uniformly $k$-sparse, simple, directed or undirected hypergraphs of rank at most $m$, for any fixed $m$.*

The main corollary (Corollary 4.2) is the following:

**Corollary.** *A class of finite, uniformly $k$-sparse, simple, directed or undirected graphs has bounded tree-width if and only if it has bounded clique-width.*

Hence, the same constraint of uniform $k$-sparseness collapses simultaneously $MS_2$ onto $MS_1$ and bounded clique-width onto bounded tree-width. This result has some consequences on the complexity of verifying the graph properties and of computing the optimization and counting graph functions that are specified by monadic second-order formulas with or without edge set quantifications. It is also relevant to the decidability of the monadic (second-order) theory of certain classes of graphs.

The paper is organized as follows. Definitions and notation concerning graphs and hypergraphs are in Section 1, together with the theorems on depth-first spanning trees and hypertrees in countable graphs and hypergraphs. We also review notation on MS logic in this section.

The main result of Section 2 is Theorem 2.5 stating that in directed graphs of indegree at most $k$, one can define by $MS_1$ formulas a binary relation on the set of vertices which is a linear order on the set of predecessors of each vertex. It uses a definition by $MS_2$ formulas of orientations of hypergraphs of bounded rank.

Section 3 deals with uniformly $k$-sparse graphs and hypergraphs; it is devoted to graph theoretical lemmas on orientations and colorings of these graphs and hypergraphs. Section 4 contains the proof of the first main theorem (Theorem 4.1), concerning graphs. This theorem is then extended to hypergraphs in Section 5.

This paper demonstrates the use of nontrivial graph properties (particular orientations, particular colorings, spanning trees) for quite involved constructions of logical formulas. It is fair to observe that the sizes of the constructed formulas are quite large.

## 1. Graphs, hypergraphs and logic

### 1.1. Graphs and hypergraphs

A *hypergraph* is a tuple $H = \langle V_H, E_H, Vert_H \rangle$ consisting of a set of *vertices* $V_H$, a set of *hyperedges* $E_H$ (disjoint with $V_H$) and a mapping $Vert_H$ with domain $E_H$ describing the hyperedges.

There are two cases. If $H$ is *undirected* then for every $e \in E_H$ the object $Vert_H(e)$ is a finite nonempty subset of $V_H$ called *the set of vertices of* $e$. If $H$ is *directed* then $Vert_H(e)$ is a finite nonempty sequence of elements of $V_H$ where no vertex occurs twice. In both cases we say that $H$ is *simple* if the mapping $Vert_H$ is one-to-one. If $Vert_H(e) = Vert_H(e')$ and $e \neq e'$, we say that $e$ and $e'$ form a pair of *multiple hyperedges*. The *rank of* $e \in E_H$ is the cardinality of $Vert_H(e)$ in the first case and its length in the second. The *rank of* $H$ is the maximal rank of its hyperedges. (Hyperedges always have finite rank.)

We will denote by $\mathbf{UH}$ the class of finite or countable undirected hypergraphs, by $\mathbf{UH}_k$ the subclass of those of rank at most $k$, and by $\mathbf{H}$ and $\mathbf{H}_k$ the corresponding classes of directed hypergraphs.

We let $und : \mathbf{H} \rightarrow \mathbf{UH}$ be the mapping such that for every $H \in \mathbf{H}$, $und(H)$ is the hypergraph $H'$ such that $V_{H'} = V_H$, $E_{H'} = E_H$, and $Vert_{H'}(e)$ is the set of vertices occurring in the sequence $Vert_H(e)$. A hyperedge has the same rank in $H$ and in $und(H)$ since its sequence of vertices in $H$ has no repetitions. The mapping $und$ transforms a directed hypergraph into its underlying undirected hypergraph. An *orientation* of an undirected hypergraph $H$ is a directed hypergraph $H'$ such that $und(H') = H$.

A *graph* is a hypergraph all edges of which are of rank 2. Hence, in this paper, graphs may be directed or undirected, they may have multiple edges but they will have no loops. Graphs and hypergraphs are always finite or countably infinite. Edges and hyperedges are "by default" undirected, or the considered property is independent of orientations. We specify "undirected" only for emphasis.

For every hypergraph $H$, directed or not, we let $K(H)$ be the simple undirected graph with set of vertices $V_H$ and an edge between $x$ and $y$ if and only if $x \neq y$ and $x$, $y$ are vertices of some $e \in E_H$. We say that $H$ is *connected* if $K(H)$ is connected. The notion of connected component of a hypergraph follows immediately.

For a directed hypergraph $H$ we let $\vec{K}(H)$ be the directed graph $G$ such that $V_G = V_H$, $E_G = \{(e,i,j)/e \in E_H,\ 1 \leqslant i < j \leqslant rank(e)\}$ and $(e,i,j)$ links the $i$th vertex of the sequence $Vert_H(e)$ to the $j$th one. We say that $H$ is *acyclic* if the graph $\vec{K}(H)$ is. It is clear that every undirected hypergraph $H$ is $und(H')$ for some acyclic hypergraph $H'$: for finding $H'$, it suffices to take any linear order on $V_H$ and to define $Vert_{H'}(e)$ as the enumeration of $Vert_H(e)$ in increasing order with respect to this order.

Let $H$ be a hypergraph, directed or not. A *subhypergraph* $K$ of $H$ (denoted by $K \subseteq H$) is a hypergraph having vertices among those of $H$, hyperedges among those of $H$, with the same sequence or set of vertices as in $H$. If $N \subseteq E_H$ we denote by $H[N]$ the subhypergraph of $H$ with $N$ as set of hyperedges, and such that the vertices are those of the hyperedges in $N$. If $X \subseteq V_H$ we denote by $H[X]$ the *induced subhypergraph* $K$ of $H$ with $V_K = X$ and $E_K$ defined as the set of hyperedges of $H$ having all their vertices in $X$. If $G \subseteq H$, we let $H - G$ denote the subhypergraph $H[V_H - V_G]$. It consists of the hyperedges of $H$ with no vertex in $V_G$ and possibly some isolated vertices.

In a directed graph $G$ we say that an edge *links* $x$, its *source* to $y$, its *target* if it is directed from $x$ to $y$; then $x$ is a *predecessor* of $y$, and $y$ a successor of $x$. We say that it *links* $x$ *and* $y$ if it links $x$ to $y$ or $y$ to $x$. We denote by $indeg_G(x)$, $x \in V_G$, the *indegree* of $x$, i.e., the number of edges with target $x$. If $G$ is directed or undirected, we denote by $deg_G(x)$ the degree of a vertex $x$, i.e., the number of edges incident with $x$ (edge directions do not matter.)

### 1.2. Trees

A *tree* is a simple directed acyclic graph $T$ such that there exists a unique vertex of indegree 0, called the *root* of $T$, and every vertex is reachable from the root by a unique directed path. The vertices of a tree will be called *nodes*. A tree $T$ is represented by the relational structure (see Theorem 1.4) $\langle N_T, Suc_T \rangle$ the domain of which, $N_T$, is the set of *nodes* of $T$ and where $Suc_T$ is a binary relation on $N_T$ called the *successor relation* and representing edges ($(x, y)$ belongs to $Suc_T$ if and only if $y$ is a successor of $x$). A tree is *binary* if every node has at most two successors.

We define a partial order on $N_T$ by $x \leqslant_T y$ if and only if $y$ is on the unique directed path from the root to $x$. In a drawing of the tree with the root on the top, $x$ is below $y$ if and only if $x <_T y$. The root is thus the unique maximum element, the leaves are the minimal ones.

### 1.3. Spanning trees and hypertrees

A *depth-first tree* in an undirected graph $G$ is a tree $T$ such that $und(T) \subseteq G$ and any two nodes of the tree that are adjacent in $G$ are comparable under $\leqslant_T$. We say that a tree $T$ such that $und(T) \subseteq G$ is a *spanning tree* of $G$ if $N_T = V_G$.

It is well known that every finite connected graph has a depth-first spanning tree. The classical depth-first traversal algorithm produces such spanning trees. We use the term depth-first to qualify the distinguished property of these trees. We will extend this result to countably infinite graphs. However, we cannot rest on the depth-first traversal algorithm, which can enter an infinite branch of a depth-first tree and miss a part of the graph. We will use another proof. We will later extend it to hypergraphs, with appropriately tuned definitions.

**Proposition 1.1.** *Let $G$ be a countable connected graph and $s \in V_G$. There exists in $G$ a depth-first spanning tree with root $s$.*

**Proof.** If $T$ is a depth-first tree in $G$ and $X \subseteq V_G - N_T$, we let $Att(X, T)$ be the set of vertices $u \in N_T$ such that some edge $e$ links $u$ and a vertex of $X$, and we call it the set of *attachment vertices* of $X$ to $T$. We let $V_G = \{v_1, v_2, \ldots, v_n, \ldots\}$ with $v_1 = s$. We define a sequence $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_n \subseteq \cdots$ such that:

(H1): each $T_i$ is a finite tree with root $s$ such that $und(T_i) \subseteq G$ and $v_i \in N_{T_i}$,

(H2): each tree $T_i$ is depth-first,

(H3): for each connected component $C$ of $G - T_i$, the set $Att(V_C, T_i)$ is linearly ordered under $\leqslant_{T_i}$.

We construct such a sequence as follows:
(1) $T_1$ is reduced to the root $s = v_1$. Conditions (H1)–(H3) hold in a trivial way.
(2) We define $T_{i+1}$ from $T_i$ as follows.

If $v_{i+1} \in N_{T_i}$ we let $T_{i+1} = T_i$. Otherwise we let $C$ be the connected component of $G - T_i$ containing $v_{i+1}$. Then $Att(V_C, T_i) \neq \emptyset$ since $G$ is connected. We let $x$ be the unique $\leqslant_{T_i}$-minimal vertex in $Att(V_C, T_i)$ (it is unique by (H3) for $T_i$). We let $y$ in $V_C$ be a neighbor of $x$. Let $P$ be a (possibly empty) path in $C$ between $y$ and $v_{i+1}$. We let $T_{i+1}$ consist of $T_i$ augmented with the edge $x \to y$ and the edges of $P$ directed from $y$ towards $v_{i+1}$. Hence $T_i \subseteq T_{i+1}$ and $T_{i+1}$ satisfies condition (H1).

We now check that $T_{i+1}$ is depth-first. Since $T_i$ is depth-first and $P$ is a path, the only possibility for $T_{i+1}$ not to be depth-first is the existence of an edge linking a vertex $u$ of $P$ and a vertex $w \in N_{T_i}$ that are incomparable with respect to $\leqslant_{T_{i+1}}$. But in this case $w \in Att(V_C, T_i)$ hence $u \leqslant_{T_{i+1}} x \leqslant_{T_i} w$, and $u \leqslant_{T_{i+1}} w$. Contradiction. Hence $T_{i+1}$ is depth-first.

We now check condition (H3) for $T_{i+1}$. Let $D$ be a connected component of $G - T_{i+1}$. If $V_D \cap V_C \neq \emptyset$ then $D \subseteq C$ and

$$Att(V_D, T_{i+1}) \subseteq Att(V_C, T_i) \cup V_P.$$

It follows that $Att(V_D, T_{i+1})$ is linearly ordered with respect to $\leqslant_{T_{i+1}}$ since $Att(V_C, T_i)$ and $V_P$ are both linearly ordered, and since $u \leqslant_{T_{i+1}} x$ for every $u \in V_P$ and $x \leqslant_{T_i} v$ for every $v \in Att(V_C, T_i)$, whence $x \leqslant_{T_{i+1}} v$.

If $V_D \cap V_C = \emptyset$ then clearly, $D$ is a connected component of $G - T_i$, $D \neq C$ and $Att(V_D, T_{i+1}) = Att(V_D, T_i)$, because if an edge links $D$ to some vertex in $P$, then $D$ would be included in $C$, the connected component of $G - T_i$ containing $P$. Hence $Att(V_D, T_{i+1})$ is linearly ordered under $\leqslant_{T_i}$ (since (H3) holds for $T_i$), hence also under $\leqslant_{T_{i+1}}$.

We let $T$ be the union of the trees $T_i$. It is a directed tree with root $s$; furthermore $und(T) \subseteq G$, $V_G = N_T$ since every vertex of $G$ belongs to some $T_i$, and it is depth-first since each $T_i$ is. Hence $T$ is a depth-first spanning tree of $G$. $\square$

This result does not hold for uncountable graphs: consider a complete uncountable graph having a depth-first spanning tree; all its vertices must be on a directed path in this tree, but every path is at most countable, hence we get a contradiction.

In order to state an analogous result for countable hypergraphs, extending what was done in [9] for finite hypergraphs, we recall some definitions from that article.

A *hyperpath* in a hypergraph $H$ is a sequence $(e_1, e_2, \ldots, e_n)$ such that $e_1, \ldots, e_n$ are pairwise distinct hyperedges, $n \geqslant 1$ and

$$Vert_H(e_i) \cap Vert_H(e_j) = \emptyset \quad \text{if } 1 \leqslant i < i+1 < j \leqslant n,$$

$$Vert_H(e_i) \cap Vert_H(e_{i+1}) \neq \emptyset \text{ if } 1 \leqslant i < n.$$

We say that this hyperpath *links* $e_1$ to $e_n$.

**Lemma 1.2.** *Let $H$ be a connected hypergraph, let $e, f \in E_H$, $e \neq f$. There exists in $H$ a hyperpath of the form $P = (e, e_1, e_2, \ldots, e_n, f)$, with $n \geqslant 0$.*

**Proof.** If $Vert_H(e) \cap Vert_H(f) \neq \emptyset$ we let $P = (e, f)$. Otherwise we consider a shortest path $(u_0, u_1, u_2, \ldots, u_{n-1}, u_n)$ in $K(H)$ from a vertex $u_0$ of $Vert_H(e)$ to a vertex $u_n$ of $Vert_H(f)$.

For each $i = 1, \ldots, n$, we let $e_i$ in $E_H$ be such that $u_{i-1}, u_i \in Vert_H(e_i)$. We let $e_0 = e$ and $e_{n+1} = f$.

If $e_i = e_j$ for $0 \leqslant i < j \leqslant n+1$, then $(u_0, u_1, \ldots, u_n)$ could be replaced by the shorter path $(u_0, \ldots, u_{i-1}, u_j, \ldots, u_n)$. Hence the hyperedges in $P = (e_0, e_1, \ldots, e_{n+1})$ are pairwise distinct.

If $w \in Vert_H(e_i) \cap Vert_H(e_j)$ for $0 \leqslant i < i+1 < j \leqslant n+1$ then $(u_0, u_1, \ldots, u_n)$ can be replaced by the shorter path $(u_0, \ldots, u_{i-1}, w, u_j, \ldots, u_n)$. Hence $Vert_H(e_i) \cap Vert_H(e_j) = \emptyset$ and $P$ is a hyperpath as desired. $\square$

**Lemma 1.3.** *Let $H$ be a hypergraph, let $N \subseteq E_H$, $e, f \in N$, such that $Vert_H(e) \cap Vert_H(f) = \emptyset$. The set $N$ is the set of hyperedges of a hyperpath in $H$ linking $e$ to $f$ if and only if $H[N]$ is connected and, for every proper subset $N'$ of $N$, containing $e$ and $f$, the subhypergraph $H[N']$ is not connected.*

**Proof.** The "only if" direction is clear from the definitions.

"If". Let $H[N]$ be connected and satisfy the minimality condition. There exists by Lemma 1.2 a hyperpath linking $e$ to $f$ with set of hyperedges $N' \subseteq N$. By the first part $H[N']$ is connected. By the minimality condition, $N = N'$. $\square$

Let $H$ be a hypergraph. A *hypertree* in $H$ is a tree $T = \langle N_T, Suc_T \rangle$ such that $N_T \subseteq E_H, Vert_H(e) \cap Vert_H(e') \neq \emptyset$ for every $e \in N_T$ and $e' \in Suc_T(e)$, and for every $e, e' \in N_T$ if $e \neq e'$ and $U = Vert_H(e) \cap Vert_H(e') \neq \emptyset$ then either $e$ and $e'$ are adjacent in $T$ or they are successors of some $e'' \in N_T$ such that $U \subseteq Vert_H(e'')$. Every directed path in $T$ is thus a hyperpath in $H$.

We define $V(T, e)$ for $e \in N_T$, by $V(T, e) := Vert_H(e)$ if $e$ is the root of $T$, and $V(T, e) = Vert_H(e) - Vert_H(e')$ if $e \in Suc_T(e')$.

We let $V(T) := \bigcup \{Vert_H(e)/e \in N_T\}$. The sets $V(T, e)$ form a partition of $V(T)$. We let $<_T$ be the strict partial order on $V(T)$ defined by: $x <_T y$ if and only if $x \in V(T, e)$, $y \in V(T, e')$ for some $e, e' \in N_T$ such that $e <_T e'$.

We let $\sim_T$ be the equivalence relation on $V(T)$ defined by

$$x \sim_T y \quad \text{if and only if} \quad x, y \in V(T, e) \quad \text{for some } e \in N_T.$$

We let $\leqslant_T$ be the quasi-order on $V(T)$ (a *quasi-order* is a transitive and reflexive binary relation) defined by

$$\begin{aligned} x \leqslant_T y \quad & \text{if and only if } x <_T y \text{ or } x \sim_T y, \\ & \text{if and only if } x \in V(T, e), y \in V(T, e'), \text{ for some} \\ & \qquad e, e' \text{ in } N_T \text{ such that } e \leqslant_T e'. \end{aligned}$$

(We denote by the same symbol $\leqslant_T$ a quasi-order on $V(T)$ and a partial order on $N_T$. Since $V(T) \subseteq V_H$ and $N_T \subseteq E_H$ no confusion should arise.)

A hypertree $T$ in $H$ is *depth-first* if for every $e \in E_H - N_T$:

(D1) either $Card(Vert_H(e) \cap V(T)) \leqslant 1$ or

(D2) $e$ has two distinct vertices $x, y$ in $V(T)$, such that $x \leqslant_T y$.

We say that $T$, depth-first, is *spanning* if $V(T) = V_H$, which implies that for every $e \in E_H - N_T$ of rank at least 2 we have (D2).

The existence of $T$ depth-first and spanning in $H$ implies that $H$ is connected. In [9] we defined directly the notion of a depth-first spanning hypertree. Here, we consider depth-first hypertrees that are possibly not spanning, in order to extend to hypergraphs the proof of Proposition 1.1, and in particular, the inductive assertion (H2). A hypertree in a connected hypergraph is depth-first and spanning if and only if it is depth-first spanning in the sense of [9].

The connected hypergraph $H$ with set of vertices $\{1, \ldots, 9\}$, and hyperedges $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{1, 4, 5\}$, $\{2, 4, 6\}$, $\{3, 5, 6\}$ has no depth-first spanning hypertree, because any such hypertree should contain all hyperedges, but $H$ is not a hypertree. This motivates the restriction of Theorem 1.4 to *special hypergraphs*. A hypergraph is special if some vertex $s$, called a *special vertex*, is adjacent to each other vertex by a hyperedge of rank 2, i.e., by an edge. An edge incident with $s$ is called a *special edge*.

**Theorem 1.4.** *Every special hypergraph with special edge $r$ has a depth-first and spanning hypertree with root $r$.*

We need the notion of *attachment vertex*, similar to the one used in the proof of Proposition 1.1. If $T$ is a hypertree in a hypergraph $H$, we denote by $H - T$ the subhypergraph $K = H[E_K]$ of $H$ with set of hyperedges $E_K = \{e \in E_H / Vert_H(e) \cap V(T) = \emptyset\}$. If $C$ is a subhypergraph of $H - T$ we let:

$$Att(C, T) = \{u \in V(T) \,/\, \text{for some } e \in E_H,\ Vert_H(e) \cap V_C \neq \emptyset, \text{ and } \{u\} = Vert_H(e) \cap V(T)\}.$$

We call $Att(C, T)$ the set of *attachment vertices of $C$ to $T$*.

**Proof of Theorem 1.4.** Let $H$ be a special hypergraph with special hyperedge $r$. We enumerate $V_H$ as $\{v_1, v_2, v_3, \ldots, v_n, \ldots\}$ with $v_1 = s$, the special vertex. We will construct an increasing sequence of hypertrees in $H$:

$$T_1 \subseteq T_2 \subseteq \cdots \subseteq T_i \subseteq \cdots$$

such that, for every $i$:

(C1) $T_i$ is a depth-first hypertree in $H$, its root is $r$ and $v_i \in V(T_i)$,

(C2) for every connected subhypergraph $K$ of $H - T_i$ the set $Att(K, T_i)$ is linearly quasi-ordered by $\leqslant_{T_i}$ (*linearly* quasi-ordered means that any two elements are comparable with respect to $\leqslant_{T_i}$.)

We let $T_1$ consist just of the root $r$, so that (C1) and (C2) hold trivially.

We construct $T_{i+1}$ from $T_i$ as follows. If $v_{i+1} \in V(T_i)$ we let $T_{i+1} = T_i$. Otherwise, we let $C$ be the connected component of $H - T_i$ containing $v_{i+1}$. The set $A = Att(C, T_i)$ is nonempty (it contains at least $v_1$) and linearly quasi-ordered under $\leqslant_{T_i}$ since $T_i$ satisfies (C2). By this linearity property, we can choose $x \in A$ such that $x \leqslant_{T_i} x'$ for all $x' \in A$. We let $e \in N_{T_i}$ be such that $x \in V(T_i, e)$; this $e$ is unique, it does not depend on the choice of $x$.

We now choose $f \in E_H - N_{T_i}$ such that $Vert_H(f) \cap V(T_i) = \{x\}$ and $Vert_H(f) \cap V_C \neq \emptyset$. Let $P$ be a hyperpath in $C \cup \{f\}$ (namely the subhypergraph $C$ augmented with the hyperedge $f$ and its vertices not in $C$) linking $f$ to some hyperedge $g$ such that $v_{i+1} \in Vert_H(g)$. This path is empty if $v_{i+1} \in Vert_H(f)$ (in this case $g = f$). There is at least one such hyperpath by Lemma 1.2, since $C \cup \{f\}$ is connected.

We extend $T_i$ into $T_{i+1}$ by adding $f$ as new successor of $e$, and the hyperedges of $V_P$ in such a way that $g$ is a leaf of $T_{i+1}$. Hence $T_{i+1}$ is a hypertree in $H$, its root is $r, T_i \subseteq T_{i+1}$ and $v_{i+1} \in V(T_{i+1})$.

We check (C1), i.e., that $T_{i+1}$ is depth-first. Let $h \in E_H - N_{T_{i+1}}$ having at least two vertices in $V(T_{i+1})$. We want to prove that it has two vertices that are comparable under $\leqslant_{T_{i+1}}$.

*Case* 1: $h$ has two vertices in $V(T_i)$. Then, it has two vertices comparable under $\leqslant_{T_i}$ because $T_i$ is depth-first, and these two are comparable under $\leqslant_{T_{i+1}}$.

*Case* 2: $h$ has two vertices in $V_P$. They are comparable under $\leqslant_{T_{i+1}}$ by construction of $T_{i+1}$.

*Case* 3: $h$ has exactly two vertices in $V(T_{i+1})$, one of them, say $u$, in $V_P - \{x\}$, and the other, $w$, in $V(T_i) - \{x\}$. Hence $w \in Att(C, T_i)$ and so $x \leqslant_{T_i} w$.

It follows that $u \leqslant_{T_{i+1}} x \leqslant_{T_{i+1}} w$. Hence $u, w$ are comparable, as desired, and (C1) holds.

It remains to check condition (C2). Let $K$ be a connected subhypergraph of $H - T_{i+1}$. We must prove that $Att(K, T_{i+1})$ is linearly quasi-ordered under $\leqslant_{T_{i+1}}$. Observe that $H - T_{i+1} \subseteq H - T_i$. Hence $K$ is contained in $C$ or is disjoint from it. We have

$$Att(K, T_{i+1}) \subseteq V_P \cup Att(K, T_i). \tag{$*$}$$

*Case* 1: $K \subseteq C$. Then $Att(K, T_i) \subseteq Att(C, T_i)$. The result holds, using $(*)$ because the vertices in $V_P$ are all $\leqslant_{T_{i+1}}$-smaller than $x$, and linearly quasi-ordered, and those in $Att(C, T_i)$ are all $\leqslant_{T_{i+1}}$-larger than $x$ and linearly quasi-ordered under $\leqslant_{T_i}$ by (C2) applied to $C$ in $T_i$, whence also under $\leqslant_{T_{i+1}}$.

*Case* 2: *$K$ and $C$ are disjoint*. We first observe that if $u$ belongs to $Vert_H(f)$ then $u \leqslant_{T_{i+1}} x$.

Let us assume that $Att(K, T_{i+1})$ has an element $u$ in $V_P - Vert_H(f)$. Consider $h$, a hyperedge of $E_H$ that has vertices in $K$ and $u$ as single vertex in $V(T_{i+1})$. This hyperedge has no vertex in $V(T_i)$, hence it belongs to the connected component $C$ (because $u$ is in $C$) and then $K \subseteq C$ contradicting the hypothesis.

Hence by $(*)$, $Att(K, T_{i+1}) \subseteq Att(K, T_i) \cup Vert_H(f)$. Consider any two vertices $u$ and $v$ in $Att(K, T_{i+1})$. If they are both in $Att(K, T_i)$ they are comparable with respect to $\leqslant_{T_i}$ since (C2) holds for $K$ and $T_i$, hence they are comparable with respect to $\leqslant_{T_{i+1}}$. If $u$ is in $Vert_H(f)$ and $v$ is in $Att(K, T_i)$ then we have $u \leqslant_{T_{i+1}} x \leqslant_{T_i} v$ by the initial remark and the choice of $x$. If they are both in $Vert_H(f)$ they are comparable with respect to $\leqslant_{T_{i+1}}$ by the definition of $\leqslant_{T_{i+1}}$. Hence in all cases, they are comparable with respect to $\leqslant_{T_{i+1}}$ which establishes (C2).

We now define $T$ as the union of the hypertrees $T_i$, $i \geqslant 1$. It is a hypertree in $H$ with root $r$ and $V(T) = V_H$ (since each vertex belongs to some $V(T_i)$). It is depth-first because for every hyperedge $h \in E_H$, there is $i$ such that $Vert_H(h) \subseteq V(T_i)$ and, since $h$ has at least two vertices and $T_i$ is depth-first, $h$ has two vertices, say $x, y$, such that $x \leqslant_{T_i} y$, whence $x \leqslant_T y$. The hypertree $T$ is spanning since $V(T) = V_H$. □

Let us consider what this theorem means for a hypergraph $H$ with all hyperedges of rank 2, i.e., for a graph with a vertex $s$ adjacent to all other vertices. We know that by Proposition 1.1 there is in $H$ a depth-first spanning tree $T$ with root $s$. Its edges form a depth-first spanning hypertree in $H$ with root any edge of the tree $T$ incident with $s$. Hence, Proposition 1.1 yields the result of Theorem 1.4.

The reader may ask why we do not use a depth-first spanning tree of the graph $K(H)$ associated with a connected hypergraph $H$. The reason is that if $H$ is undirected, we are unable to construct $K(H)$ from $H$ by monadic second-order formulas. See the comments following Lemma 1.5. (To be precise, the mapping $K$ is not (2,2)-definable.) Hence, we *need to define first an orientation* of $H$ by MS formulas, and we do not know how to do that without depth-first and spanning hypertrees (used in Proposition 2.2).

### 1.4. Relational structures and monadic second-order logic

Let $R$ be a finite set of symbols where each element $r$ in $R$ has a *rank* $\rho(r)$ in $\mathbf{N}_+$. A symbol $r$ in $R$ is a $\rho(r)$-ary relation symbol. An $R$-(*relational*) *structure* is a tuple $S = \langle D_s, (r_s)_{r \in R} \rangle$ where $D_s$ is a finite or countable set, called the *domain* of $S$, and $r_s$ is a subset of $D_s^{\rho(r)}$ for each $r$ in $R$. We will denote by $\mathscr{S}(R)$ the class of $R$-structures. Two isomorphic structures will be considered as equal. We will not discuss this point in proofs. The context will make clear when we need concrete structures or structures up to isomorphism.

The *MS formulas*, intended to describe properties of $R$-structures $S$ (for fixed $R$), are written with variables of two types, namely lower case letters $x, x', y, \ldots$ denoting elements of $D_s$, and upper case letters $X, Y, Y', \ldots$ denoting subsets of $D_s$. The atomic formulas are of the forms $x = y, x \in X, r(x_1, \ldots, x_n)$ (where $r$ is in $R$ and $n = \rho(r)$), and formulas are formed with propositional connectives and quantifications over the two kinds of variables. For every finite set $W$ of object and set variables, we denote by $\mathscr{L}(R, W)$ the set of all formulas that are written with relational symbols from $R$ and have their free variables in $W$. If $S$ is an $R$-structure, $\varphi \in \mathscr{L}(R, W)$, and $\gamma$ is a $W$-assignment in $S$ (i.e., $\gamma(X)$ is a subset of $D_s$ for a set variable $X$, and $\gamma(x) \in D_s$ for an object variable $x$; we write this $\gamma : W \to S$ to be short), we write $(S, \gamma) \models \varphi$ if and only if $\varphi$ holds in $S$ with the values of the free variables of $\varphi$ being defined by $\gamma$. We write $S \models \varphi$ in the case where $\varphi$ has no free variable.

Graphs and hypergraphs can be represented in several ways by relational structures. Our purpose is to use MS formulas to write some of their properties through their various representations.

For a directed graph $G$, we let $|G|_1 = \langle V_G, edg_G \rangle$ and $|G|_2 = \langle D_G, inc_G \rangle$ where $D_G := V_G \cup E_G, edg_G$ is the set of pairs $(x, y)$ such that some edge links $x$ to $y$, and $inc_G$ is the set of triples $(e, x, y)$ such that the edge $e$ links $x$ to $y$.

If $G$ is undirected, the definitions are similar with "$x$ and $y$", instead of "$x$ to $y$" . Thus $edg_G$ is symmetric (because edges have no direction).

For representing a hypergraph $H$ where all hyperedges are of rank $k$, we do similarly with a $k$-ary relation symbol *edg* and a $(k+1)$-ary relation *inc*. Hence, in particular, if $H$ is undirected, $edg_H(x_1,\dots,x_k)$ holds if and only if $\{x_1,\dots,x_k\} = Vert_H(e)$ for some hyperedge $e$, and $inc_H(e,x_1,\dots,x_k)$ holds if and only if $e \in E_H$ and $\{x_1,\dots,x_k\} = Vert_H(e)$. For a hypergraph with hyperedges of various ranks bounded by some integer, we overload the symbols *edg* and *inc* and use them for sequences of arguments of various lengths.

An $\mathrm{MS}_1$ formula ($\mathrm{MS}_2$ formula) is an MS formula written with the relation symbol *edg* (the relation symbol *inc*). It is intended to express a property of a structure of the form $|H|_1$ (resp. $|H|_2$), where $H$ is a graph or a hypergraph.

We will say that a property $P$ of the hypergraphs $H$ of a class $\mathscr{C}$ is *expressed by a logical formula* $\varphi$ *via the representation* $|H|$ if, for every $H$ in $\mathscr{C}$, the property $P(H)$ holds if and only if $|H| \models \varphi$. In particular, we will say that a property of hypergraphs is $\mathrm{MS}_i$-*definable* (where $i$ is 1 or 2), if it is expressible by an $\mathrm{MS}_i$-formula, via the representation $|-|_i$.

The structure $|H|_1$ is less expressive than $|H|_2$ for representing properties of a hypergraph $H$ by MS formulas for the obvious reason that one cannot express "in" $|H|_1$ properties dealing with multiple edges. However, this is also the case if $H$ is assumed to be simple. For instance, the existence of a Hamiltonian cycle, or of a spanning tree of out-degree at most 2 in a simple graph are $\mathrm{MS}_2$-definable properties that are not $\mathrm{MS}_1$-definable. (See [10, Proposition 5.2.9].)

We are interested in classes of graphs and hypergraphs $\mathscr{C}$ for which there exists an algorithm $f$ transforming an $\mathrm{MS}_2$-formula $\psi$ into an $\mathrm{MS}_1$-formula $f(\psi)$, such that for every $H$ in $\mathscr{C}$, $f(\psi)$ holds in $|H|_1$ if and only if $\psi$ holds in $|H|_2$. For such a class, $\mathrm{MS}_2$ and $\mathrm{MS}_1$ are equally expressive.

However, the drawback of this formulation is that it says nothing on formulas with free variables. And these formulas are useful for algorithmic applications [12–14]. We will use an alternate formulation, based on transformations of relational structures, called *monadic second-order definable transductions of relational structures*. (This notion is an adaptation of that of *interpretation* used in first-order logic for defining interreductions between theories. See [7].)

Let $R$ and $Q$ be two finite ranked sets of relation symbols.

The idea is to define from $S \in \mathscr{S}(R)$, a $Q$-structure $T$ with domain included in $D_s \times \{1,\dots,k\}$. Taking a subset of the product with $\{1,\dots,k\}$ is important because it makes possible to define $T$ with a larger domain than $S$ (but $k$ is fixed). This subset is defined by (fixed) MS formulas and so are the relations of $T$. Furthermore, $T$ can be defined (in a unique way) from $S$ *and some auxiliary subsets of* $D_s$ specified by parameters.

Let $W$ be a finite set of set variables, called the *parameters*. A $(Q,R)$-*definition scheme* is a tuple of formulas of the form

$$\Delta = (\varphi, \psi_1, \dots, \psi_k, (\theta_w)_{w \in Q^*k})$$

where

$$k > 0, Q^*k := \{(q,j)/q \in Q, j \in \{1,\dots,k\}^{\rho(q)}\},$$

$$\varphi \in \mathscr{L}(R,W),$$

$$\psi_i \in \mathscr{L}(R, W \cup \{x_1\}) \text{ for } i = 1,\dots,k,$$

$$\theta_w \in \mathscr{L}(R, W \cup \{x_1,\dots,x_{\rho(q)}\}), \text{ for } w = (q,j) \in Q^*k.$$

We now explain how these formulas are used.

Let $S \in \mathscr{S}(R)$, let $\gamma$ be a $W$-assignment in $S$. A $Q$-structure $T$ with domain $D_T \subseteq D_s \times \{1,\dots,k\}$ is *defined by $\Delta$ in* $(S,\gamma)$ if:

(i) $(S,\gamma) \models \varphi$,

(ii) $D_T = \{(d,i)/d \in D_s, i \in \{1,\dots,k\},\ (S,\gamma,d) \models \psi_i\}$,

(iii) for each $q$ in $Q$:

$$q_T = \{((d_1,i_1),\dots,(d_t,i_t)) \in (D_T)^t/(S,\gamma,d_1,\dots,d_t) \models \theta_{(q,j)}\},$$

where $j = (i_1,\dots,i_t)$ and $t = \rho(q)$.

(By $(S,\gamma,d_1,\dots,d_t) \models \theta_{(q,j)}$, we mean $(S,\gamma') \models \theta_{(q,j)}$, where $\gamma'$ is the assignment extending $\gamma$, such that $\gamma'(x_i) = d_i$ for all $i = 1,\dots,t$ and similarly for $(S,\gamma,d) \models \psi_i$.) Since $T$ is associated in a unique way with $S$, $\gamma$ and $\Delta$ whenever it is defined, i.e., whenever $(S,\gamma) \models \varphi$, we can use the functional notation $def_\Delta(S,\gamma)$ for $T$.

The *transduction defined by $\Delta$* is the relation:

$$def_\Delta := \{(S,T)/T = def_\Delta(S,\gamma) \text{ for some } W\text{-assignment } \gamma \text{ in } S\} \subseteq \mathscr{S}(R) \times \mathscr{S}(Q).$$

A transduction $f \subseteq \mathscr{S}(R) \times \mathscr{S}(Q)$ is *MS-definable* (or is an *MS-transduction*) if it is equal to $def_\Delta$ for some $(Q,R)$-definition scheme $\Delta$. We also consider $def_\Delta$ as a mapping from $\mathscr{S}(R)$ to the power set of $\mathscr{S}(Q)$ by letting $def_\Delta(S) = \{T/(S,T) \in def_\Delta\}$.

These definitions apply to graphs and hypergraphs via their representations by relational structures of two types, as explained above.

We say that a binary relation $\tau$ on hypergraphs is an *$(i,j)$-definable MS-transduction*, where $i$ and $j$ belong to $\{1,2\}$ if the relation $\{(|H|_i, |H'|_j)/(H,H') \in \tau\}$ is an MS-transduction.

A special case of interest is when the identity is an $(1,2)$-definable MS-transduction on a class of graphs (or hypergraphs). This means that by means of MS formulas, one can specify the edges (or hyperedges) as pairs $(x,i)$ of vertices $x$ and numbers $i$ in a fixed finite set, and in such a way that the incidences between vertices and edges coded so are definable by MS-formulas on vertices.

An essential tool is the *Backwards Translation Lemma*, Lemma 1.5. It says that if $T = def_\Delta(S,\gamma)$ then the monadic second-order properties of $T$ can be expressed as monadic second-order properties of $(S,\gamma)$.

Let $\Delta = (\varphi, \psi_1,\dots,\psi_k, (\theta_w)_{w \in Q^*k})$ be a $(Q,R)$-definition scheme, written with a set of parameters $W$. Let $V$ be a set of set variables disjoint from $W$. For every variable $X$ in $V$, for every $i = 1,\dots,k$, we let $X_i$ be a new variable. We let $V' := \{X_i/X \in V,\ i = 1,\dots,k\}$. For every mapping $\eta : V' \to \mathscr{P}(D_S)$, we let $\eta_k : V \to \mathscr{P}(D_S \times \{1,\dots,k\})$ be defined by

$\eta_k(X) = \eta(X_1) \times \{1\} \cup \cdots \cup \eta(X_k) \times \{k\}$. (Note that every mapping from $V$ to $\mathscr{P}(D_S \times \{1, \ldots, k\})$ is of this form.) With these notations we can state [7,9,10]:

**Lemma 1.5.** *For every formula $\beta$ in $\mathscr{L}(Q, V)$, one can construct a formula $\beta'$ in $\mathscr{L}(R, V' \cup W)$ satisfying the following*:

*For every $S$ in $\mathscr{S}(R)$, for every assignment $\gamma : W \to S$, for every assignment $\eta : V' \to S$, we have*:

(i) $def_\Delta(S, \gamma)$ *is defined*, $\eta_k$ *is a V-assignment in* $def_\Delta(S, \gamma)$, *and* $(def_\Delta(S, \gamma), \eta_k) \models \beta$ *if and only if*

(ii) $(S, \eta \cup \gamma) \models \beta'$.

In particular, if the identity is an $(1,2)$-definable MS-transduction on a class of graphs (or hypergraphs), then every $MS_2$ formula can be translated into an $MS_1$ formula expressing the same property of the graphs or hypergraphs of this class.

From this lemma, we get also that the composition of two MS-transductions is an MS-transduction, and that, if a class $L$ of relational structures has a *decidable MS-theory* (which means that given any MS formula, one can decide whether it is satisfied in some structure of $L$) and $\tau$ is an MS-transduction, then $\tau(L)$ has also a decidable MS-theory. See [9] for more details.

We illustrate these definitions and Lemma 1.5 with an example.

It is not hard to see that the mapping $K$ from hypergraphs of rank at most $m$ to graphs (defined in Section 1.1) is a (1,1)-definable MS-transduction. Since the connectivity of a graph is $MS_1$-definable, the connectivity of a hypergraph of rank at most $m$ is $MS_1$-definable. This follows from Lemma 1.5.

We now consider the mapping $\vec{K}$ from directed hypergraphs of rank at most $m$ to directed graphs. It is also (1,1)-definable. It follows in particular from Lemma 1.5 that the acyclicity of a hypergraph of rank at most $m$ is $MS_1$-definable, since the acyclicity of a directed graph is [10, Lemma 5.2.8].

This mapping is also $(2,2)$-definable. The edges of $G = \vec{K}(H)$ are defined as triples $(e, i, j)$ where $e$ is a hyperedge and $1 \leqslant i < j \leqslant rank(e)$. A pair $(i, j)$ as above can be coded by an integer between 1 and $m(m-1)$. Hence, the edges of $G$ can be represented as pairs $(e, n)$ where $e$ is a hyperedge of $H$ and $1 \leqslant n \leqslant m(m-1)$. The source and the target of such an edge can be determined by MS-formulas, thanks to the orientation of $e$. Hence, we obtain that $\vec{K}$ is a $(2,2)$-definable MS-transduction (we omit further details). So is $K$, on directed hypergraphs.

This construction does not apply to the mapping $K$ on *undirected* hypergraphs, because we miss the availability of the ordering on the vertices of hyperedges. However, it is $(2,2)$-definable since, as we will see in Theorem 2.3, there exists a $(2,2)$-definable MS-transduction that orients hypergraphs of rank at most $m$, and since the composition of two MS-transductions is an MS-transduction.

### *1.5. Tree-width and clique-width*

*Tree-width* is a graph complexity measure which is the paradigm of parametrized complexity. We refer to the book by Downey and Fellows [16] or to the survey by

Bodlaender [2]. We only recall that for each $k$, every graph property expressible in $MS_2$ is testable in linear time on graphs of tree-width at most $k$. Some **NP** complete properties like Hamiltonicity, fall in this category.

*Clique-width* is somewhat similar. It is studied in [12,15]. We review the definition.

We consider graph operations dealing with simple $k$-graphs, i.e., simple graphs (directed or not) where each vertex is given one and only one color among $\{1,\ldots,k\}$. One binary operation is disjoint union. The unary operations are $\rho_{i\to j}$ which colors by $j$ every vertex originally colored by $i$, $\alpha_{i,j}$ for $i\neq j$ which adds to the graph directed edges from every vertex colored by $i$ to every vertex colored by $j$, $\eta_{i,j}$ for $i\neq j$ which adds similarly undirected edges. Basic graphs are vertices colored by 1. Every finite simple graph can be defined by a $k$-expression, i.e., an algebraic expression built with these operations (and colors limited to $1,\ldots,k$) for some $k$. Its *clique-width* is the minimal such $k$. Cliques have clique-width 2. Every graph of tree-width $k$ has clique-width $2^{O(k)}$.

For each $k$, each $MS_1$-definable graph property of a graph of clique-width $k$ is testable in linear time from the $k$-expression (3-colorability is $MS_1$ but Hamiltonicity is not). The complexity of constructing a $k$-expression from the graph is presently not known.

Hence to summarize, $MS_2$ is more powerful as a language than $MS_1$ but linear algorithms are derivable from $MS_2$ formulas for smaller classes of graphs.

We recall results from Courcelle and Engelfriet [11,18] showing the close connections between MS logic and graph complexity measures like tree-width and clique-width.

**Theorem 1.6.** *A set of finite simple graphs L is contained in the image of the set of finite binary trees under a* $(1,1)$-*definable* (*resp. a* $(1,2)$-*definable*) *MS-transduction if and only if it has bounded clique-width* (*resp. bounded tree-width*).

## 2. Definition of orientation in countable hypergraphs by $MS_2$ formulas

The main result of this section is Theorem 2.5 stating that in directed graphs of indegree at most $k$, one can define by $MS_1$ formulas a binary relation on the set of vertices which is a linear order on the set of predecessors of each vertex. The proof uses auxiliary results formulated in terms of hypergraphs. In particular, we prove that MS formulas can define orientations of countable undirected hypergraphs of bounded rank. We will use an induction on the rank, hence we will start by orienting countable graphs.

From the graph theoretical point of view, all these definitions of orderings and orientations are straightforward. But the difficulty is to formalize them by MS formulas.

Let $\mathscr{C}$ be a class of undirected graphs (or hypergraphs), and $\theta(X_1,\ldots,X_n)$ and $\omega(X_1,\ldots,X_n,x,y)$ be two $MS_2$ formulas. We say that $(\theta,\omega)$ *orients* the graphs (or hypergraphs) in $\mathscr{C}$ if:

(i) For every $H\in\mathscr{C}$ there exists an $n$-tuple $(X_1,\ldots,X_n)$ of subsets of $V_H\cup E_H$ such that $(|H|_2,X_1,\ldots,X_n)\models\theta$.

(ii) For every $H \in \mathscr{C}$, for every such tuple $X_1, \ldots, X_n$ the binary relation:

$$\{(x, y) \in V_H \cup E_H / (|H|_2, X_1, \ldots, X_n, x, y) \models \omega\}$$

is a linear order on each set $Vert_H(e)$, $e \in E_H$.

These linear orders make thus $H$ into a directed graph or hypergraph. The orientations defined in this way have a special property: if two hyperedges $e$ and $f$ share two vertices $x$ and $y$, then these vertices are in the same relative order in $e$ and in $f$. This order is determined from a single binary relation on vertices. In particular, a graph oriented in this way has no pair of opposite edges.

We say that $(\theta, \omega)$ as above *orients acyclically* the graphs (or hypergraphs) in $\mathscr{C}$ if the resulting directed graphs or hypergraphs are acyclic, for every tuple $X_1, \ldots, X_n$ satisfying $\theta$.

**Theorem 2.1.** *There exists a pair of* $\mathrm{MS}_2$*-formulas that orients acyclically all graphs. There exists a* $(2, 2)$*-definable MS-transduction that associates with every graph at least one* (*acyclic*) *orientation of this graph.*

**Proof.** This result is proved in [9, Theorem 3.2] for finite graphs. The proof is based on the existence of a depth-first spanning tree in every finite connected graph. It extends immediately to countable connected graphs by means of Proposition 1.1.

The proof given in [9] for finite (not necessarily connected) graphs defines a formula $\theta(U, X)$ expressing that $X$ is a set of vertices, such that $X$ has one and only one vertex in each connected component, and $U$ is the union of the sets of edges of depth-first spanning trees, one for each connected component, with roots in $X$. The formula $\omega$ defines a partial order $\leqslant$ such that $x \leqslant y$ if and only if $y$ is on the (necessarily unique) path having all its edges in $U$ that links $x$ and a vertex of $X$. Any two adjacent vertices in the graph are comparable under $\leqslant$ since the trees are depth-first.

From the partial order $\leqslant$ on $V_G$ defined by $(\theta, \omega)$ and a pair $(U, X)$ satisfying $\theta$, one defines an orientation $H$ of $G$ by deciding that an edge $e$ linking $x$ and $y$ in $G$ will link $x$ to $y$ in $H$ if $x \leqslant y$, and will link $y$ to $x$ in $H$ if $y \leqslant x$. (Note that $x$ and $y$ are comparable as observed above.) This orientation is acyclic.

This definition can be put in the form of a $(2, 2)$-definable MS-transduction taking $U$ and $X$ as parameters. □

We now consider hypergraphs. Let $\mathscr{C}$ be a class of undirected hypergraphs. We say that a pair of $\mathrm{MS}_2$ formulas of the form $(\theta(X_1, \ldots, X_n), \sigma(X_1, \ldots, X_n, e, x))$ *splits* the hyperedges of the hypergraphs in $\mathscr{C}$ if

(i) for every $H \in \mathscr{C}$, there exists an $n$-tuple $(X_1, \ldots, X_n)$ of subsets of $V_H \cup E_H$ such that $(|H|_2, X_1, \ldots, X_n) \models \theta$,

(ii) for every $H \in \mathscr{C}$, for every such $n$-tuple, for every $e \in E_H$ of rank at least 2, the set of elements $x \in V_H \cup E_H$ such that $(|H|_2, X_1, \ldots, X_n, e, x) \models \sigma$ is a proper nonempty subset of $Vert_H(e)$, that we will denote by $V_1(e)$ (the mapping $V_1$ depends actually on $\theta, \sigma, X_1, \ldots, X_n$ assumed to be known from the context).

Our objective is to orient hypergraphs. The hyperedge splitting is an intermediate step making it possible to perform an induction on the rank. The base case is that of a graph. Splitting an edge is nothing but distinguishing a source and a target, hence defining a direction for this edge.

**Proposition 2.2.** *For every $k \geqslant 2$, we can construct a pair of* $\mathrm{MS}_2$*-formulas that splits the hyperedges of hypergraphs in* $\mathbf{UH_k}$.

**Proof.** We first explain how to construct $(\theta, \sigma)$ that splits the hyperedges of every special hypergraph in $\mathbf{UH_k}$. Our main tool is Theorem 1.4 that concerns only special hypergraphs. Later we will extend the result to the full class $\mathbf{UH_k}$.

We will use parameters $S, N, X_1, \ldots, X_k$ subject to the following conditions expressed by an $\mathrm{MS}_2$ formula $\theta(S, N, X_1, \ldots, X_k)$, and relative to some $H \in \mathbf{UH_k}$:

(T1) $S = \{\mathsf{s}, r\}$, $\mathsf{s} \in V_H$, $r \in E_H$, $r$ is an edge incident with $\mathsf{s}$, and $H$ is a special hypergraph with special vertex $\mathsf{s}$,

(T2) $N \subseteq E_H$, $r \in N$, and $N$ is the set of nodes of a depth-first and spanning hypertree $T$ of $H$ with root $r$,

(T3) $X_i \subseteq V_H$ for each $i = 1, \ldots, k$, and the sets $X_1, \ldots, X_k$ define a partition of $V(T)$ ($= V_H$ because $T$ is spanning) such that $Card(X_i \cap V(T, e)) \leqslant 1$ for each $e \in N$. (The notation is as in Section 1.3).

One can build $\mathrm{MS}_2$-formulas (similar to those constructed in [9, Lemma 3.4] such that:

- $\pi(N, e, e')$ expresses that $N \subseteq E_H$ is the set of nodes of a hyperpath from $e$ to $e'$ (it can be constructed by Lemma 1.3),
- $\varphi_1(N, r)$ expresses that $N \subseteq E_H$ is the set of nodes of a hypertree $T$ in $H$ with root $r$ (see [9, Lemma 3.4]),
- $\varphi_2(N, r, e, e')$ expresses that $\varphi_1(N, r)$ holds, $e, e' \in N$ and $e' \in Suc_T(e)$, where $T$ is the hypertree defined by $N$ and $r$ (see [9, Lemma 3.4]),
- $\varphi_3(N, r, e, x)$ expresses that $\varphi_1(N, r)$ holds and that $x \in V(T, e)$ where $T$ is defined by $N$ and $r$,
- $\varphi_4(N, r, x, y)$ expresses that $\varphi_1(N, r)$ holds, $x, y \in V_H$ and $x <_T y$ where $T$ is the hypertree defined by $N$ and $r$ (see [9, p. 125]),
- $\varphi_5(N, r)$ expresses that $\varphi_1(N, r)$ holds and that $T$ defined by $N$ and $r$ is depth-first and spanning (i.e., T2 holds),
- $\varphi_6(N, r, X_1, \ldots, X_k)$ expresses that $\varphi_1(N, r)$ holds and that condition T3 holds.

Hence an MS formula $\theta(S, N, X_1, \ldots, X_k)$ can be constructed from $\varphi_1, \ldots, \varphi_6$ to express conditions T1–T3. It remains to construct $\sigma$.

For every $e$ in $E_H$ of rank at least two, we let $Min(e) = \{u \in Vert_H(e)/$there is no $w$ in $Vert_H(e)$ with $w <_T u\}$ where $T$ is the depth-first spanning hypertree of $H$ defined by $N, r$ (see condition T2).

It is clear that $Min(e)$ is nonempty. Hence, there is a smallest integer $i$ such that $Min(e) \cap X_i \neq \emptyset$ since $X_1, \ldots, X_k$ define a partition of $V(T)$. We want to prove that $Min(e) \cap X_i$ is a proper subset of $Vert_H(e)$.

We have one of the following 3 cases:

(1) $e = r$, $Min(e) = Vert_H(r)$ and thus $Min(e) \cap X_i \neq Min(e)$ by condition T3 and since $r$ has two vertices,

(2) $e \in N - \{r\}, Min(e) = V(T,e)$. Hence $Min(e)$ is a proper subset of $Vert_H(e)$, and so is $Min(e) \cap X_i$ (in these two cases, $Min(e) \cap X_i$ is a singleton by T3).

(3) $e \in E_H - N$. We first consider the case where $e$ has two vertices $x$ and $y$ such that $x <_T y$: thus $y \notin Min(e)$, hence $Min(e)$ is a proper subset of $Vert_H(e)$, and so is $Min(e) \cap X_i$; if no such pair exists, then, since $T$ is depth-first, there are two distinct vertices $x$ and $y$ in $Vert_H(e)$ such that $x \sim_T y$, hence which are both in $V(T,e')$ for some $e' \in N$. They are both in $Min(e)$ (since there is nothing below them with respect to $<_T$) but at least one of them is not in $Min(e) \cap X_i$ (since, by condition T3, $V(T,e') \cap X_i$ has at most one element). Hence $Min(e) \cap X_i$ is a proper subset of $Vert_H(e)$.

An MS-formula $\sigma(S,N,X_1,\dots,X_k,e,x)$ can be written so as to hold for $e \in E_H$ and $x \in V_H \cup E_H$ if and only if $Min(e) \cap X_j \neq \emptyset$ for some $j$, and $x \in Min(e) \cap X_i$ where $i$ is the smallest such integer $j$.

Hence, $\{x \in V_H / (|H|_2, S, N, X_1, \dots, X_k, e, x) \models \sigma\}$, which is equal to $Min(e) \cap X_i$, is a proper nonempty subset of $Vert_H(e)$.

The existence of $S,N,X_1,\dots,X_k$ satisfying $\theta$ for every special hypergraph $H$ of rank at most $k$ follows from Theorem 1.4 and the fact that the sets $V(T,e)$ have cardinality at most $k$. Hence we have the desired pair of MS-formulas, but working only for special hypergraphs.

We now consider the general case where $H$ is not necessarily special. We make $H$ into a special hypergraph $H^+$ by adding a new vertex $s$, and for each $x \in V_H$, a new edge between $s$ and $x$. The transformation $\tau$ of $|H|_2$ into $|H^+|_2$ is an MS-transduction. The pair $(\theta,\sigma)$ constructed in the first part of the proof, which defines in $|H^+|_2$ a nonempty proper subset $V_1(e)$ of $Vert_{H^+}(e)$ for each $e \in E_{H^+}$, can be translated back (by Lemma 1.5) via $\tau$ into a pair $(\theta',\sigma')$ that defines in $|H|_2$, for each $e \in E_H$, the nonempty proper subset $V_1(e)$ of $Vert_H(e)$ defined first by $(\theta,\sigma)$. Hence $(\theta',\sigma')$ is the desired pair of formulas, working for hypergraphs in $\mathbf{UH_k}$. $\square$

**Theorem 2.3.** *For every $k \geqslant 2$, we can construct a pair of* $\mathrm{MS}_2$*-formulas that orients acyclically the hypergraphs in* $\mathbf{UH_k}$.

**Proof.** The proof is by induction on $k$. The case of hypergraphs with all hyperedges of rank 2 is proved in Theorem 2.1, because hyperedges of rank 1 are already (trivially) oriented. We consider the general case, $k > 2$.

Let $(\theta,\sigma)$ be the pair of formulas obtained by Proposition 2.2, that splits each hyperedge $e$ of rank at least 2 of a hypergraph in $\mathbf{UH_k}$. We let $X_1,\dots,X_n$ be the free variables of $\theta$, and we fix an $n$-tuple of subsets of $V_H \cup E_H$ also denoted by $(X_1,\dots,X_n)$ satisfying $\theta$. For each hyperedge $e$ of $H$ of rank at least 2, we obtain a nonempty proper subset $V_1(e)$ of $Vert_H(e)$. (The notation $V_1(e)$ refers to the definition of a pair of formulas that split hyperedges.) We obtain in this way two hypergraphs $H_1$ and $H_2$ in $\mathbf{UH_{k-1}}$ such that:

$$V_{H_1} = V_{H_2} = V_H,$$

$$E_{H_1} = E_{H_2} = \{e/e \in E_H,\ e \text{ has rank at least } 2\}$$

and for every $e$ in this set:

$$Vert_{H_1}(e) = V_1(e) \quad \text{and}$$

$$Vert_{H_2}(e) = Vert_H(e) - V_1(e).$$

Hence, we have two MS-transductions mapping $|H|_2$ (for $H$ in $\mathbf{UH_k}$) to $|H_1|_2$ and to $|H_2|_2$. (The hypergraphs $H_1$ and $H_2$ depend on the $n$-tuple $(X_1,\ldots,X_n)$ satisfying $\theta$).

By using the induction hypothesis, we have a pair of $\mathrm{MS}_2$ formulas $(\theta',\omega)$ that orients acyclically the hypergraphs in $\mathbf{UH_{k-1}}$. We let $Y_1,\ldots,Y_p$ be the free variables of $\theta'$. We now use this pair for $H_1$ and for $H_2$. It defines in $|H_1|_2$ and in $|H_2|_2$ two binary relations $r_1$ and $r_2$ on $V_H$ $(= V_{H_1} = V_{H_2})$ which are linear orders on $Vert_{H_1}(e)$ and on $Vert_{H_2}(e)$ for each hyperedge $e$ of $H_1$ and $H_2$, respectively. Using the Backwards Translation Lemma (Lemma 1.5), the pair $(\theta',\omega)$ can be translated into two pairs $(\theta_1,\omega_1)$ and $(\theta_2,\omega_2)$ which define these binary relations in $|H|_2$. The free variables of $\theta_1$ are in $\{X_1,\ldots,X_n,Z_1,\ldots,Z_p\}$ and those of $\theta_2$ are in $\{X_1,\ldots,X_n,U_1,\ldots,U_p\}$, where $Z_1,\ldots,Z_p$ and $U_1,\ldots,U_p$ correspond to $Y_1,\ldots,Y_p$ via the two backwards translations.

For each $e\in E_H$, we define the linear order $\leqslant_e$ of $Vert_H(e)$ as follows:

$x \leqslant_e y$ if and only if $x = y$,

or $x, y \in V_1(e)$ and $r_1(x, y)$ holds,

or $x, y \in Vert_H(e) - V_1(e)$ and $r_2(x, y)$ holds,

or $x \in V_1(e)$, $y \in Vert_H(e) - V_1(e)$.

The linear orders $\leqslant_e$ are thus MS-definable in $|H|_2$ in terms of subsets $X_1,\ldots,X_n$, $Z_1,\ldots,Z_p,U_1,\ldots,U_p$ of $V_H \cup E_H$ satisfying $\theta \wedge \theta_1 \wedge \theta_2$ (and these sets do exist by Proposition 2.2 and the induction hypothesis). Hence we have a (2,2)-definable MS-transduction $\tau$ that associates with every $H$ in $\mathbf{UH_k}$ (and by using well-chosen sets $X_1,\ldots,X_n$, $Z_1,\ldots,Z_p,U_1,\ldots,U_p$) an orientation $H'$ of $H$.

However, $H'$ is not necessarily acyclic. The linear orders $\leqslant_e$ are not necessarily the restriction of a single partial order, or even of a single binary relation on $V_H$. But we want formulas that define such a relation.

An additional MS-transduction $\alpha$ can transform $H'$ into an acyclic orientation of $H$, from which a partial order will be easy to obtain. We explain this final step.

We recall from [9, Proposition 3.10] that for each $k$, there exists an MS-transduction $\beta$ that associates with $|H|_2$ for $H\in\mathbf{H_k}$ the set:

$$\{|H'|_2/H' \in \mathbf{H_k},\ und(H') = und(H)\}.$$

(The proof is given in [9] for finite hypergraphs but works for infinite ones as well). The transduction $\beta$ uses parameters $W_1,\ldots,W_q$. By Lemma 1.5 and since acyclicity is $\mathrm{MS}_1$-definable, (see Sections 1.1 and 1.4), one can construct an MS formula $\psi$ expressing that the orientation $H'$ obtained from a $q$-tuple $(W_1,\ldots,W_q)$ is acyclic. Hence, one obtains an MS-transduction $\alpha$ that associates with $|H|_2$ for $H\in\mathbf{H_k}$ the set of structures $|H'|_2$ such that $H'$ is an acyclic orientation of $und(H)$: it suffices to replace in the definition scheme of $\beta$ the first formula $\varphi$ by $\varphi \wedge \psi$. Since every hypergraph has an acyclic orientation, the transduction $\alpha$ produces nonempty sets. Hence by applying $\alpha$ after the MS-transduction $\tau$ obtained in the first part of the proof, one obtains a $(2,2)$-definable MS-transduction $\tau'$ that transforms $H$ in $\mathbf{UH_k}$ into a nonempty set of acyclic orientations of $H$. This transduction uses parameters $X_1,\dots,X_n,Z_1,\dots,Z_p,U_1,\dots,U_p,W'_1,\dots,W'_q$ (where $W'_1,\dots,W'_q$ correspond to the parameters $W_1,\dots,W_q$ of $\alpha$ by the backwards translation associated with $\tau$).

The desired binary relation, call it $r$, can be defined as the reflexive and transitive closure of the edge relation of the graph $\vec{K}(H'')$ for any hypergraph $H''$ in $\tau'(H)$. This relation is a partial order because the graph $\vec{K}(H'')$ is acyclic, and is $\mathrm{MS}_1$ definable in $\vec{K}(\tau''(H))$.

Each of these hypergraphs $H''$ is defined from a suitable choice of parameters $X_1,\dots,X_n,Z_1,\dots,Z_p,U_1,\dots,U_p,W'_1,\dots,W'_q$. One obtains the desired pair $(\theta'',\omega'')$ of MS formulas as follows: the formula $\theta''$ is the conjunction of $\theta \wedge \theta_1 \wedge \theta_2$ and of the backwards translation (under $\tau$) of $\varphi \wedge \psi$, which is the condition on the parameters of $\alpha$. The formula $\omega''$ is the backwards translation of the formula defining the relation $r$ relative to the transduction $\tau'$. $\square$

**Remark 2.4.** The proof [9, Proposition 3.9, p. 128] which says that for a finite hypergraph $H$ of rank at most $m$ (where $m$ is fixed), one can define by MS-formulas a partial order on vertices that is linear on each hyperedge, is incorrect because if $e \in E_H$ and $Vert_H(e) \subseteq V(T)$, it may happen that the partial order $\leqslant$ (we refer to the proof in [9, p. 129]) is not linear on $Vert_H(e)$. This is actually the case of hyperedge 2 of the hypergraph $H$ of Fig. 5, p. 123 of that article. However two vertices at least are comparable under $\leqslant$ hence one can handle this case by induction on $m$ as done here in the proof of Theorem 2.3. The proof of Theorem 2.3, restricted to finite hypergraphs, gives a correct proof [9, Proposition 3.9]. $\square$

Here is now the main theorem of this section. For stating it, we denote by $Pred_G(x)$ the set of predecessors of a vertex $x$ in a simple directed graph $G$, i.e., of vertices which are the source of an edge with target $x$. We denote by $\mathbf{Indeg}(\leqslant k)$ the set of simple directed graphs of indegree at most $k$, i.e., such that every set $Pred_G(x)$ has cardinality at most $k$.

**Theorem 2.5.** *For every $k$, there exists a pair $(\delta(X_1,\dots,X_n),\ \pi(X_1,\dots,X_n,x,y))$ of $\mathrm{MS}_1$ formulas, such that, for every simple directed graph $G$ of indegree at most $k$:*

(i) *There exists an $n$-tuple $(X_1,\dots,X_n)$ of subsets of $V_G$ such that $(|G|_1,X_1,\dots,X_n) \models \delta$.*

(ii) *For every such tuple $X_1,\dots,X_n$ the binary relation:*

$$\{(x,y) \in V_G/(|G|_1,X_1,\dots,X_n,x,y) \models \pi\}$$

*is a linear order on each set $Pred_G(x)$, $x \in V_G$.*

**Proof.** With every $G \in \mathbf{Indeg}(\leqslant k)$, we associate an undirected hypergraph $H = H(G) \in \mathbf{UH_k}$ defined as follows:

$$V_H = V_G,$$

$$E_H = \{\bar{v}/v \in V_G, Card(Pred_G(v)) \geqslant 2\},$$

$$Vert_H(\bar{v}) = Pred_G(v) \text{ for every } \bar{v} \in E_H.$$

It is clear that we have an MS-transduction: $|G|_1 \mapsto |H(G)|_2$.

The pair of $MS_2$-formulas $(\theta, \omega)$ constructed in Theorem 2.3 that defines a binary relation (actually even a partial order) on $V_H$, for $H \in \mathbf{UH_k}$, which is a linear order on the hyperedges of $H$ can thus be translated (by Lemma 1.5) into a pair $(\delta, \pi)$ of $\mathbf{MS}_1$-formulas that defines for every $G \in \mathbf{Indeg}(\leqslant k)$ a binary relation (even a partial order) on $V_G$, which is a linear order on each set $Pred_G(x)$. □

## 3. Uniformly *k*-sparse graphs and hypergraphs

We now introduce the central notion of the paper, which subsumes, in the case of graphs, the conditions of bounded degree, of planarity and of bounded tree-width.

A finite hypergraph $G$ is *k-sparse* if $Card(E_G) \leqslant k.Card(V_G)$. A finite or countable hypergraph is *uniformly k-sparse* if every finite subhypergraph is $k$-sparse. A set of hypergraphs $\mathscr{C}$ is *k-sparse* (*uniformly k-sparse*) if all its elements are so.

Since the $n$-clique $K_n$ has $n$ vertices and $n(n-1)/2$ edges, the set $K = \{K_n/n \geqslant 1\}$ is not $k$-sparse for any $k$. Consider the set of graphs $\hat{K} = \{\hat{K}_m/m \geqslant 1\}$ where each graph $\hat{K}_n$ consists of $K_n$ augmented with a vertex $\bar{e}$ linked to $x$ and $y$, for every edge $e$ of $K_n$ linking $x$ and $y$. Since $\hat{K}_n$ has $n(n+1)/2$ vertices and $3n(n-1)/2$ edges, it is 3-sparse. Since $K_n$ is a subgraph of $\hat{K}_n$, the set $\hat{K}$ is not uniformly $k$-sparse for any $k$.

We recall that an orientation of an undirected graph is a directed graph whose underlying undirected graph is the given graph.

**Lemma 3.1.** (1) *A graph of degree at most* $2d$ *is uniformly d-sparse.*

(2) *A graph is uniformly k-sparse if and only if it has an orientation of indegree at most k.*

**Proof.** (1) Follows from the fact that for every finite graph $G$ we have:

$$2.Card(E_G) = \sum\{deg_G(v)/v \in V_G\}.$$

(2) “If” is clear since for every finite directed graph $G$:

$$Card(E_G) = \sum\{indeg_G(v)/v \in V_G\}.$$

“Only if” For finite graphs this is proved as a lemma [19, Theorem 6.13]. (We will extend this proof to hypergraphs in Lemma 3.3.)

We now extend this result to countable graphs by means of Koenig's lemma. Let $G$ be countably infinite and uniformly $k$-sparse. Consider an increasing sequence of finite induced subgraphs of $G$ : $\emptyset = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_i \subseteq \cdots$ such that $G = \cup\{G_i/i \geqslant 0\}$. We let $A$ be the set of pairs $(i,H)$ such that $i \geqslant 0, H$ is an orientation of $G_i$ of indegree at most $k$. For each $i$, there is at least one by the lemma of [19] for finite graphs.

We let $(i,H) \to (i+1,H')$ if and only if $H = H'[V_{G_i}]$. If $H'$ has indegree at most $k$, so has each of its subgraphs. It follows that $(A,\to)$ is a tree with root $(0,\emptyset)$ where $(i+1,H')$ is a successor of $(i,H)$ if $(i,H) \to (i+1,H')$. Each node has finitely many successors. Hence this tree has an infinite branch, say:
$(0,\emptyset) \to (1,H_1) \to (2,H_2) \to \cdots \to (i,H_i) \to \cdots$ . The graph $H = \cup\{H_i/i \geqslant 1\}$ is an orientation of $G$ of indegree at most $k$ as one checks easily. □

A finite graph of tree-width at most $k$ is $k$-sparse (by the second part of this lemma, because it is a subgraph of a $k$-tree, and $k$-trees are constructed from a given clique $K_k$ by iterated addition of new vertices linked to $k$ existing ones; it suffices to orient arbitrarily the edges of the base $k$-clique, and to orient the other ones towards the new vertices; see [2] for $k$-trees and graphs of tree-width $k$). This also holds for countable graphs since tree-width is monotone for subgraph inclusion. A result of Mader [3, Theorem 1.14, p. 375]) says that a finite graph without $K_p$ as a minor is $2^{p-3}$-sparse. Hence countable graphs without $K_p$ as a minor (or without any simple graph with $p$ vertices as a minor) are also uniformly $2^{p-3}$-sparse.

Our objective is now to define orientations of bounded indegree by means of colors given to the vertices. We first recall a result of [22] saying that if a finite simple directed graph without pairs of opposite edges has indegree at most $k$, then the directions of edges can be determined from an appropriate $m$-coloring of the vertices where $m = m(k) = 2^{2k(k+1)+1} - 1$. Formally the authors construct (in Theorem 10) a directed graph $T = T(k)$ with $m(k)$ vertices, such that $edg_T \cap (edg_T)^{-1} = \emptyset$ (i.e., no edge has an opposite edge) and for every finite directed graph $G$ of indegree at most $k$ without pairs of opposite edges, there is a *homomorphism*: $G \to T$, i.e., a mapping $\varphi : V_G \to V_T$ such that, for every edge linking $x$ to $y$ in $G$, there is an edge linking $\varphi(x)$ to $\varphi(y)$ in $T$. The existence of such a homomorphism when $G$ is directed of indegree at most $k$, without pairs of opposite edges and countable can be proved from the finite case by Koenig's Lemma (as in Lemma 3.1.2).

**Proposition 3.2.** (1) *Let $k \geqslant 1$. The property that a simple graph is uniformly $k$-sparse is expressible by an* $\mathrm{MS}_1$ *formula.*

(2) *For each $k$, the mapping that associates with a simple undirected graph its orientations of indegree at most $k$ is a* (1,1)*-definable MS-transduction.*

**Proof.** We use the construction of [22] recalled above. Let $V_T = \{1,\ldots,m\}$, $m = m(k)$. Let $\theta(X_1,\ldots,X_m)$ be the $\mathrm{MS}_1$ formula expressing the following conditions about a simple undirected graph $G$ given by $|G|_1$:

(i) $X_1,\ldots,X_m$ form a partition of $V_G$,

(ii) if $x \in X_i$ and $y \in X_j$ are adjacent in $G$, then $i$ and $j$ are different and adjacent in $T$.

We let then $G(X_1,\dots,X_m)$ be the orientation $G'$ of $G$ defined as follows:
(iii) $(x,y)\in edg_{G'}$ if and only if $(x,y)\in edg_G$, $x\in X_i, y\in X_j$ and $(i,j)\in edg_T$.

It is now easy to write a first-order formula $\theta(X_1,\dots,X_m)$ saying that, for a simple graph $G$ given by $|G|_1$, the tuple $(X_1,\dots,X_m)$ satisfies (i)–(ii) and $G(X_1,\dots,X_m)$ has indegree at most $k$. Since, by the result of [22], every orientation of $G$ of indegree at most $k$ can be defined by such a partition, we get that the mapping that associates with every such graph $G$ given by $|G|_1$ the set of structures

$$\{|H|_1/H \text{ is an orientation of } G \text{ of indegree at most } k\}$$

is an MS-transduction. This establishes assertion (2).

For assertion (1), it follows from Lemma 3.1.2 that the $MS_1$ formula $\exists X_1,\dots,X_m$ $\theta(X_1,\dots,X_m)$ defines the uniformly $k$-sparse simple graphs. □

**Remark.** The property that a finite graph $G$ is 1-sparse is not $MS_2$ expressible. For a counter-example consider $G_{n,m}=K_n \oplus I_m$ where $\oplus$ denotes disjoint union and $I_m$ the graph consisting of $m$ isolated vertices. Then $G_{n,m}$ is 1-sparse if and only if $n(n-1)/2\leqslant n+m$, if and only if $m\geqslant n(n-3)/2$. If the property "$G$ is 1-sparse" would be $MS_2$-definable, the set $\mathrm{L}=\{G_{n,m}/n\geqslant 4,\ m\geqslant n(n-3)/2\}$ would be $MS_2$-definable. By the results of [6,10] saying that, roughly speaking, the syntactic $\{\oplus\}$-congruence $\approx$ of an $MS_2$-definable set of graphs has finitely many classes, we get that for every $m$ and $n$ such that $G_{n,m}\in L$, we also have $G_{n,m'}\in L$ for some $m'$ in a finite set of integers (independent of $n$). The idea is to replace $I_m$ in $G_{n,m}$ by a smaller $\approx$-equivalent graph $I_{m'}$, so that $G_{n,m'}\in L$. But this is not possible for all $n$ by the definition of $L$. □

We now extend these results to hypergraphs (for use in Section 5). An orientation of an undirected hypergraph $H$ is a family of linear orders, one on each set of vertices of a hyperedge. We introduce a weaker notion.

A *semi-orientation* of an undirected hypergraph $H$ is a pair $S=(H,tgt)$ where $tgt$ is a mapping: $E_H\to V_H$ that associates with each hyperedge $e$ one of its vertices. We will call this vertex the *target* of $e$. We say that $S$ is a *semi-directed* hypergraph. We let $indeg_S(v)$, the *indegree* of $v$ in $S$, be the number of hyperedges $e$ such that $tgt(e)=v$.

The following lemma is an extension of Lemma 3.1.2.

**Lemma 3.3.** *Let $k\geqslant 1$. A hypergraph $H$ is uniformly $k$-sparse if and only if it has a semi-orientation $S$ of indegree at most $k$.*

**Proof.** The "if" part is clear since for every finite hypergraph $H$ with a semi-orientation $S$:

$$Card(E_H)=\sum\{indeg_S(v)/v\in V_H\}. \qquad (*')$$

"Only if". In the special case of graphs, the result is the lemma of [19] used in Lemma 3.1.2. We prove the result for finite hypergraphs. The extension to countable ones is by Koenig's lemma as in the proof of Lemma 3.1.2. We first observe using $(*')$ above that if a finite hypergraph $H$ is $k$-sparse and semi-directed (by $tgt$), then

$$\sum\{indeg_S(v)/v\in V_H\}\leqslant k.Card(V_H),$$

where $S=(H,tgt)$. Hence, we must have $indeg_S(w)<k$ for some $w$ if, for some $v$, we have $indeg_S(v)>k$.

Consider now $H$ finite and uniformly $k$-sparse. Let $tgt$ be any semi-orientation of $H$, and $S=(H,tgt)$. We say that a vertex $v$ is *bad* if $indeg_S(v)>k$. We let the badness of $tgt$ be $\sum\{indeg_S(v)-k/v \text{ is bad}\}$. We are looking for a semi-orientation of badness 0.

Let the chosen semi-orientation $tgt$ have positive badness: we will transform it into $tgt'$ of smaller badness. Let $v$ be a bad vertex. Let $X$ be the smallest subset of $E_H$ containing all the hyperedges with target, either $v$ or a vertex of some hyperedge in $X$. We let $V(X)$ denote the union of the sets of vertices $Vert_H(e)$, $e\in X$. Then $H[X]$ is $k$-sparse, hence by the initial observation, it has a vertex $w$ such that $indeg_S(w)<k$. Since $X$ is defined as a transitive closure, there exists a sequence of hyperedges $e_1,\dots,e_n$ in $X$ and a sequence of pairwise distinct vertices $v_1,v_2,\dots,v_n=v$ in $V(X)$ such that $w\in Vert_H(e_1)$, $tgt(e_i)=v_i$ for each $i=1,\dots,n$. We now define $tgt'$ on $E_H$ from $tgt$ as follows:

$$tgt'(e_1)=w,$$

$$tgt'(e_i)=v_{i-1} \quad \text{for } i=2,\dots,n,$$

$$tgt'(e)=tgt(e) \quad \text{for } e\in E_H-\{e_1,\dots,e_n\}.$$

It is clear that in $S':=(H,tgt')$ we have

$$indeg_{S'}(v)=indeg_S(v)-1,$$

$$indeg_{S'}(x)=indeg_S(x) \quad \text{for } x\in V_H-\{v,w\},$$

$$indeg_{S'}(w)=indeg_S(w)+1\leqslant k.$$

Hence the badness of $tgt'$ is equal to the badness of $tgt$ minus 1. By repeating this step one obtains a function $tgt$ of badness 0, hence a semi-orientation of $H$ as desired. □

The following technical lemma will be used in Section 5 in order to define semi-orientations of hypergraphs by MS formulas, as we did in Proposition 3.2 for graphs, and again with the help of the colorings obtained from [22].

With a semi-directed hypergraph $S=(H,tgt)$, we associate a simple directed graph $D=Dir(S)$ defined as follows:

$$V_D=V_H,$$

$$E_D=\{[x,y]/x,y\in V_D, x\neq y, x\in Vert_H(e), y=tgt(e) \text{ for some } e\in E_H\},$$

$$Vert_D([x,y])=(x,y) \text{ for } [x,y]\in E_D.$$

**Lemma 3.4.** *Let $k\geqslant 1$ and $m\geqslant 2$. Let $H$ be a hypergraph of rank at most $m$ having a semi-orientation of indegree at most $k$. It has a semi-orientation $S$ of indegree at most $mk^2$ such that the directed graph $Dir(S)$ has no pair of opposite edges.*

**Proof.** Again we prove this for $H$ finite and the case where $H$ is countable follows easily from Koenig's lemma, as in the proof of Lemma 3.1.2. Let us consider $H$ of rank at most $m$, and $S=(H,tgt)$ a semi-orientation of $H$. A vertex $v$ is *bad for* $S$ if $v\to w$ and $w\to v$ in $Dir(S)$ for some $w$. We assume that

(i) every bad vertex has indegree at most $k$ in $S$,

(ii) the other vertices have indegree at most $mk^2$.

We will modify $tgt$ into $tgt'$ such that (i) and (ii) still hold for $S'=(H,tgt')$ and $S'$ has less bad vertices than $S$. By repeating this step finitely many times, we will obtain a semi-orientation of $H$ without bad vertices and with indegree at most $mk^2$ as desired.

We do that as follows. Let $v$ be a bad vertex. Let $e_1,\dots,e_l$, $l\leqslant k$ be the hyperedges $e$ in $S$ such that $tgt(e)=v$. Let $X=V(\{e_1,\dots,e_l\})-\{v\}$. ($V(\{e_1,\dots,e_l\})$ denotes the union of the sets of vertices of $e_1,\dots,e_l$). Hence $Card(X)\leqslant k(m-1)$.

Let $Y$ be the set of hyperedges $e$ such that $v\in Vert_H(e)$ and $tgt(e)\in X$. The cardinality of $Y$ is at most $k^2(m-1)$ since every such $tgt(e)$ is bad, hence of indegree at most $k$. We transform $tgt$ into $tgt'$ by letting $tgt'(e)=v$ for each $e\in Y$ and $tgt'(e)=tgt(e)$ for $e\in E_H-Y$. For $S'=(H,tgt')$ we have:

$$indeg_{S'}(v)\leqslant k+k^2(m-1)\leqslant mk^2$$

and

$$indeg_{S'}(x)\leqslant indeg_S(x)\quad \text{for } x\in V_H-\{v\}.$$

In $Dir(S')$ there are no two opposite edges $v\to x$ and $x\to v$ for any $x\in V_H-\{v\}$, i.e., $v$ is not bad in $S'$. Furthermore, if $x,y\in V_H-\{v\}$ and $x\to y$ in $Dir(S')$, then this edge "comes from" a hyperedge not in $\{e_1,\dots,e_l\}\cup Y$, hence was present in $Dir(S)$. It follows that bad vertices in $S'$ were already bad in $S$, hence the number of bad vertices has decreased by at least one. Properties (i) and (ii) still hold in $S'$. □

## 4. $\mathbf{MS_2}$ versus $\mathbf{MS_1}$ for sparse graphs

### 4.1. The first main theorem

Theorem 4.1 is our first main theorem. An informal statement is given in the introduction.

We want to have classes $\mathscr{C}$ of directed or undirected simple graphs such that there exists an MS-transduction that defines $|G|_2$ from $|G|_1$ for each $G\in\mathscr{C}$. As in [9] we say in this case that *the identity is a* (1,2)-*definable* MS transduction on $\mathscr{C}$.

Since an MS-transduction transforms a structure $S$ into a structure $T$ with $Card(D_T)\leqslant k.Card(D_S)$ for some fixed $k$, the finite graphs in a class $\mathscr{C}$ as above are necessarily $(k-1)$-sparse. Our main result is a kind of converse.

**Theorem 4.1.** (i) *For each $k$, the identity is a* $(1,2)$-*definable* MS *transduction on the class* $\mathbf{US_k}$ *of uniformly $k$-sparse, finite or countable, simple, directed or undirected graphs.*

(ii) *The same properties of the graphs in* $\mathbf{US_k}$ *are expressible by* $\mathrm{MS}_1$ *and by* $\mathrm{MS}_2$ *formulas.*

**Proof.** (i) We first prove the case of undirected simple graphs. We define the transformation $|G|_1 \to |G|_2$ as the composition of several MS-transductions. The first one $\alpha$ maps $|G|_1$ to $|G'|_1$ where $G'$ is an orientation of $G$ of indegree at most $k$ (there exists one by Lemma 3.1.2). The transduction $\alpha$ exists by Proposition 3.2.2.

The second one $\beta$ maps $|G'|_1$ to $(|G'|_1, R)$ where $R$ is a binary relation on $V_{G'}$ which is linear on each set $Pred_{G'}(x)$, $x \in V_{G'}$. We know its existence by Theorem 2.5. A third one $\gamma$ maps $(|G'|_1, R)$ to $|G'|_2$. We can construct it such that the domain $D$ of $|G'|_2$ is as follows:

$$D = \{(v,i)/v \in V_{G'}, \quad 0 \leqslant i \leqslant Card(Pred_{G'}(v))\}.$$

An element $(v,0)$ of $D$ represents the vertex $v$ of $G'$. An element $(v,i)$ of $D, i \geqslant 1$ represents the unique edge from $u$ to $v$ where $u$ is the $i$th element of $Pred_{G'}(v)$ with respect to the linear order induced by the relation $R$. (It is unique since $G$ is simple.) This transduction uses no parameter. For each $i$, a first-order formula $\mu_i(u,v)$ written in terms of $R$ can express that $u$ is the $i$th element of $Pred_{G'}(v)$. The incidence relation $inc_{G'}$ is definable by an *MS* formula, hence $\gamma$ is an MS-transduction.

Finally, we let $\delta$, that maps $|G'|_2$ to $|G|_2$, be the MS-transduction representing *und*. Its effect is to define $inc_G(e,u,v)$ as: $inc_{G'}(e,u,v) \vee inc_{G'}(e,v,u)$. The desired transduction is thus: $\delta \circ \gamma \circ \beta \circ \alpha$. It is an MS-transduction since MS-transductions are closed under composition, and it defines $|G|_2$ from $|G|_1$ for every uniformly $k$-sparse undirected simple graph $G$.

We now consider the case of simple directed graphs. We let $H$ be directed, simple and uniformly $k$-sparse. The graph $und(H)$ is not necessarily simple, because $H$ may have opposite edges, say $e$ linking $x$ to $y$ and $e'$ linking $y$ to $x$. We let $G$ be the simple graph obtained from $und(H)$ by fusing any two edges with same sets of ends. It is clearly simple and uniformly $k$-sparse.

We let $\varepsilon$ map $|H|_1$ to $(|G|_1, edg_H)$. The transduction $\gamma \circ \beta \circ \alpha$ defined from the above $\alpha$, $\beta$, $\gamma$ maps $(|G|_1, edg_H)$ to $(|G'|_2, edg_H)$. We need an additional transduction $\eta$ mapping $(|G'|_2, edg_H)$ to $|H|_2$, that we can describe as follows. It defines:

$$V_H \text{ as } V_{G'} \times \{0\}, \quad \text{and} \quad E_H \subseteq E_{G'} \times \{1,2\} \text{ such that:}$$

$$E_H = \{(e,1)/e \in E_{G'},\ e \text{ links } x \text{ to } y \text{ in } G' \text{ and } edg_H(x,y) \text{ holds}\}$$
$$\cup \{(e,2)/e \in E_{G'},\ e \text{ links } x \text{ to } y \text{ in } G' \text{ and } edg_H(y,x) \text{ holds}\}.$$

We define $inc_H$ in $|H|_2$ such that:

$inc_H((e,1),x,y)$ holds if and only if $e$ links $x$ to $y$ in $G'$;

$inc_H((e,2),x,y)$ holds if and only if $e$ links $y$ to $x$ in $G'$.

Finally the desired transduction is $\eta \circ \gamma \circ \beta \circ \alpha \circ \varepsilon$ which maps $|H|_1$ to $|H|_2$ for every simple directed graph $H$ that is uniformly $k$-sparse.

(ii) That every $MS_1$ formula can be translated into an equivalent $MS_2$ formula is true for all graphs. For each class $\mathbf{US_k}$ we have an opposite translation, by (i) and Lemma 1.5. □

**Corollary 4.2.** *For every class of finite graphs* $\mathscr{C}$ *included in* $\mathbf{US_k}$, *we have the following properties*:
(i) $\mathscr{C}$ *is* $MS_1$-*definable if and only if it is* $MS_2$-*definable*.
(ii) $\mathscr{C}$ *has bounded tree-width if and only if it has bounded clique-width.*

**Proof.** (i) A special case of Theorem 4.1(ii).
(ii) "If". Let $\mathscr{C}$ have bounded clique-width. By Theorem 1.6, it is contained in the image of the set of finite binary trees under a $(1,1)$-definable MS-transduction. By composition with the $(1,2)$-definable identity (Theorem 4.1), it is contained in the image of the same set of trees under a $(1,2)$-definable MS-transduction. Hence it has bounded tree-width by Theorem 1.6.
The "Only if" direction holds in general, for finite simple directed or undirected graphs by Courcelle and Olariu [15]. □

Hence, the same condition of uniform $k$-sparseness collapses two otherwise proper inclusions, that of the family of $MS_2$-definable sets of simple graphs onto that of $MS_1$-definable ones, and that of the family of sets of graphs of bounded clique-width onto that of sets of bounded tree-width.

### *4.2. Consequences for the theory of algorithms*

It is well known that graph properties expressible by MS formulas are decidable in linear time over families of tree-structured graphs, like those of tree-width at most $k$ for each $k$. See [12,13,16] (where *optimization* and *counting* problems are also discussed; here we only consider *verification* of graph properties).

This general statement covers actually two cases.

The $MS_2$ expressible verification problems can be solved in linear time on graphs and hypergraphs of tree-width at most $k$. The algorithms process in linear time the tree representing a tree decomposition witnessing the upper bound on tree-width, but such a tree can be obtained in linear time [2], hence the algorithms are linear in the total numbers of vertices and edges of the input graphs (although the constants are huge).

A fully parallel result is [12, Theorem 4]: the $MS_1$-expressible verification (and actually also optimization and counting) problems can be solved in linear time on graphs of clique-width at most $k$, provided the tree representing a hierarchical decomposition witnessing the upper bound on clique-width is given. (The complexity of finding this tree is yet unknown. It is polynomial for $k=3$ [5], at most NP for $k \geqslant 4$; see [15] on basic properties of clique-width).

It follows from Theorem 4.1 that for each class $\mathscr{C}$ of finite uniformly $k$-sparse simple graphs, $MS_1$ and $MS_2$ are equally powerful. By Corollary 4.2, "bounded tree-width" is equivalent to "bounded clique-width" on these classes. Hence the two theorems cover exactly the same problems and subclasses of $\mathscr{C}$. Every $MS_2$ verification problem can be solved in linear time on every class $\mathscr{C}$ as above of bounded clique-width. Since the constructions underlying Corollary 4.2 are effective, one need only know $k$ and a bound on clique-width. However the resulting bound on tree-width (by 4.2(ii) is extremely large, so these results are more interesting for the theory of graph algorithms than for actual implementations.

### 4.3. Decidable monadic theories

We now discuss a conjecture made by Seese [23] and already considered in [9] saying that if a set of finite simple graphs $L$ has a decidable $MS_1$-theory then $L \subseteq \tau(\mathbf{B})$ for some $(1,1)$-definable MS-transduction $\tau$ (where $\mathbf{B}$ denotes the set of finite binary trees), hence has bounded clique-width by Theorem 1.6. Since conversely, every set of graphs $\tau(\mathbf{B})$ for $\tau$ as above, has a decidable $MS_1$-theory (see [10]), this means that, roughly speaking, only sets of trees, and sets of graphs which are "tree-structured" can have a decidable $MS_1$-theory. Sets of graphs having a decidable $MS_2$-theory have bounded tree-width, see [23] or [9].

For every simple undirected graph $G$, we denote by $cpl(G)$ its *edge-complement*, i.e., the simple undirected graph (it is loop-free as are all our graphs) such that $V_{cpl(G)} = V_G$ and two vertices are adjacent if and only if they are not adjacent in $G$. If $G$ is simple and directed, we define a directed edge-complement also denoted by $cpl(G)$ having an edge from $x$ to $y(\neq x)$ if and only if there is no edge in $G$ linking $x$ to $y$. In both cases $cpl(cpl(G)) = G$. If $L$ is a set of graphs, then $cpl(L)$ denotes $\{cpl(G)/G \in L\}$.

**Proposition 4.3.** *Let $L$ be a set of finite simple directed or undirected graphs having a decidable* $MS_1$*-theory. If $L$ or $cpl(L)$ is uniformly $k$-sparse for some $k$ then $L$ has bounded clique width.*

By Theorem 1.6, this result means that Seese's conjecture holds for sets of finite simple graphs that have either few edges, or on the contrary, are "dense". It is still open for intermediate cases.

**Proof.** The proof is the same for directed and undirected graphs. Let $L$ be uniformly $k$-sparse. Since, by Theorem 4.1, the identity on $L$ is $(1,2)$-definable and $L$ has a decidable $MS_1$-theory, it also has a decidable $MS_2$-theory. Hence $L$ has bounded tree-width [9,23]. Hence it has bounded clique-width in [15, Theorem 5.5].

If $cpl(L)$ is uniformly $k$-sparse and $L$ has a decidable $MS_1$-theory, then $cpl(L)$ also has a decidable $MS_1$-theory (because the transformation $\gamma : |G|_1 \to |cpl(G)|_1$ is an MS-transduction) hence $cpl(L)$ has bounded clique-width by the first part. So has $L$ because for every directed or undirected graph, $cpl(G)$ has clique-width at most twice that of $G$ [15, Theorem 4.1] and $cpl(cpl(L)) = L$. $\square$

### 4.4. Two counterexamples

We first give an example of a class $\mathscr{C}$ of finite directed simple graphs that is 3-sparse, not uniformly $k$-sparse for any $k$ and on which the identity is nevertheless

$(1,2)$-definable. Hence, Theorem 4.1(i) does not cover all classes of graphs having a $(1,2)$-definable identity.

We take for $\mathscr{C}$ the set of graphs $G$ formed with a tournament $K$ (i.e., any orientation of a clique) with at least 4 vertices and for each edge $e$ of $K$ linking $u$ to $v$, we add a vertex $\bar{e}$ and two edges $e_1$ linking $\bar{e}$ to $u$ and $e_2$ linking $\bar{e}$ to $v$. (The set $und(\mathscr{C})$ has already been used as an example in the beginning of Section 3). For $G$ as above, and from $|G|_1$ with domain $D = V_K \cup W$ where $W := \{\bar{e}/e \in E_K\}$, we construct $T = |G|_2$ with domain

$$D' = D \times \{0\} \cup W \times \{1,2,3\},$$

where

$$(v,0) \text{ represents } \quad v \text{ in } V_G,$$

$$(\bar{e},3) \text{ represents } e \text{ in } E_G,$$

$$(\bar{e},i) \text{ represents } e_i \text{ in } E_G, \text{ for } i = 1,2.$$

We observe that $W$ can be characterized as the set of vertices of $G$ of degree 2. Hence $W$ and $V_K$ can be defined in the structure $|G|_1$ by first-order formulas. Thus, we can define $inc_T$ in such a way that:

$inc_T(f,x,y)$ holds if and only if:

(1) $f = (w,i)$ for some $w \in W$, some $i \in \{1,2,3\}$,
(2) there is a unique pair $(u,v)$, $u,v \in V_K$ such that $edg_G(w,u)$, $edg_G(w,v)$ and $edg_G(u,v)$ hold, (this means that $w = \bar{e}$, $e \in E_K$ and $e$ links $u$ to $v$),
(3) either $i = 1$, $x = (w,0)$, $y = (u,0)$,
or $i = 2$, $x = (w,0)$, $y = (v,0)$,
or $i = 3$, $x = (u,0)$, $y = (v,0)$.

Then the structure $T$ is isomorphic to $|G|_2$. This example shows that, in some very special cases, the mapping that transforms $|G|_2$ into $|G|_1$ can be an MS-transduction for certain graphs that are $k$-sparse but not uniformly $k$-sparse. However, this transduction does not work for their subgraphs.

Our second example shows that the technique of [8] cannot be used to prove Theorem 4.1(i), even for the class $\mathscr{C}$ of finite uniformly 2-sparse undirected simple graphs.

In that paper, it is proved that the identity on a class $\mathscr{C}$ of finite undirected simple graphs is $(1,2)$-definable if for some fixed integer $k$, every graph $G \in \mathscr{C}$ has an orientation $H$ and a coloring $\gamma : V_G \to \{1,\dots,k\}$ with $k$ colors satisfying the following properties:

(1) $\gamma$ is *good* which means that if $H$ has an edge $x \to y$, it has no edge $y' \to x'$ with $\gamma(x) = \gamma(x')$ and $\gamma(y) = \gamma(y')$;
(2) $\gamma$ is *semi-strong* which means that if $H$ has two edges $y \to x$ and $y' \to x$ (with $y \neq y'$) then $\gamma(y) \neq \gamma(y')$ (and of course $\gamma(y) \neq \gamma(x)$, $\gamma(y') \neq \gamma(x)$); this implies that $H \in \mathbf{Indeg}(\leqslant k)$.

We consider the set $\mathscr{C}$ of graphs $K'_n$, $n \geqslant 2$ where $K'_n$ is obtained from the $n$-clique $K_n$ by the insertion of a vertex $\bar{e}$ on each edge $e$. Hence $K'_n$ has an orientation of indegree 2 and is thus uniformly 2-sparse, by Lemma 3.1.2.

Assume now that for some $k$, each $K'_n$ has an orientation $H_n$ and a semi-strong coloring $\gamma_n$ with $k$ colors (to obtain a contradiction, we need not assume that $\gamma_n$ is good). If this orientation is such that $x \leftarrow \bar{e} \rightarrow y$ where $e$ links $x$ and $y$ in $K_n$, then we can reverse $x \leftarrow \bar{e}$ into $x \rightarrow \bar{e}$ and $\gamma_n$ is still semi-strong for this new orientation. Hence we can assume that for every edge $e$ of $K_n$ we have in $H_n$:

$$\text{either } x \rightarrow \bar{e} \rightarrow y, \tag{1}$$

$$\text{or } x \rightarrow \bar{e} \leftarrow y. \tag{2}$$

We fix $n = 2k(k+1)$, we consider the undirected graph $G$ such that $V_G = V_{K_n}$ and two vertices $x$ and $y$ are linked if and only if (2) holds. Since no vertex $x$ in $H_n$ has more than $k$ incoming edges, $G$ is the edge complement of a uniformly $k$-sparse graph with $n$ vertices, by Lemma 3.1.2.

A theorem by Turan (see [4, Theorem 1.1.1, p. 1234]) says that if an undirected simple graph has $ps$ vertices ($p, s \geqslant 2$) and more than $p^2 s(s-1)/2$ edges (i.e., more than the number of edges of the complete $s$-partite graph with $s$ stable sets having each $p$ vertices), then it must have $K_{s+1}$ as a subgraph.

Let us choose $p = 2k+2$, $s = k$. Hence $n = ps$ since $n$ is fixed as $2k(k+1)$. Then, the edge complement of $G$ has at most $kps$ edges, hence $G$ has at least $n(n-1)/2 - kps$ edges, and this number is equal to $ps((ps-1)/2 - k)$ which is more than $p^2 s(s-1)/2$ since $ps - 1 - 2k = 2k^2 + 2k - 1 - 2k = 2k^2 - 1 > p(s-1) = (2k+2)(k-1) = 2k^2 - 2$.

Hence $G$ contains $K_{k+1}$ as a subgraph. There exist in $G$ two adjacent vertices $x, y$ having the same color $\gamma_n(x) = \gamma_n(y)$. Hence $\gamma_n$ is not a coloring of $K'_n$ and we get a contradiction. □

### 4.5. The closed monadic second-order hierarchy

A consequence of Theorem 4.1 is that for every $k$ and every closed $MS_2$ formula $\psi$ there exists a closed $MS_1$-formula $\phi$ which is equivalent to $\psi$ on all uniformly $k$-sparse graphs. How are these formulas related?

We will consider the closed monadic second-order hierarchy defined in [1] and considered by Matz [21]. Let us call $\Gamma_0$ the set of first-order formulas. For every $n$, we let $\Gamma_{n+1}$ be the least set of formulas containing $\Gamma_n$, their negations and closed under first-order quantifications and existential monadic second-order quantifications.

In most of the MS-transductions we use, the relations of the constructed structures are defined by quantifier-free formulas over the given structures. It follows that each class $\Gamma_n$ is invariant under the Backwards Translations (recalled in Lemma 1.5) associated with these transformations.

The most complicated transductions are those based on Theorem 2.3. The relations of the constructed structures are defined by formulas in $\Gamma_p$ where $p$ depends linearly on $k$ (because of the induction on $k$). It follows that if an $MS_2$ formula $\psi$ belongs to $\Gamma_n$, then its translation as an $MS_1$-formula belongs to $\Gamma_{n+p}$.

## 5. $MS_2$ versus $MS_1$ for hypergraphs

Our second main theorem, Theorem 5.2, is the extension of Theorem 4.1 to uniformly $k$-sparse simple hypergraphs of rank at most $m$, for every $m \geqslant 3$ and $k \geqslant 1$.

We first present the general structure of the proof, the main part of which is Proposition 5.1. First we reduce the general case to that of *m-hypergraphs*, i.e., of undirected simple hypergraphs with all hyperedges of the same size $m$. The case of $m=2$ is Theorem 4.1. The proof is an induction on $m$.

Consider a uniformly $k$-sparse $m$-hypergraph $H$. In each hyperedge we select a vertex called its target. We obtain thus a semi-orientation $S$ of $H$ (as defined in Section 3), and the corresponding graph $Dir(S)$ with directed edges from each vertex of a hyperedge to its target. By Lemmas 3.3 and 3.4, we can do that in such a way that the graph $Dir(S)$ has bounded indegree and has no pair of opposite edges. Hence, its orientation (whence also $S$) can be defined by MS formulas from a vertex coloring of $H$, using the result of [22] discussed before the proof of Proposition 3.2.

A hyperedge of size $m$ of the given hypergraph $H$ can be identified with a set of $m-1$ edges of the associated graph $Dir(S)$, hence, with a hyperedge of an $(m-1)$-hypergraph called the *derived hypergraph* $\partial(S)$ of $S$, the vertices of which are the edges of $Dir(S)$. These vertices can be defined by MS formulas from the initial hypergraph $H$ by Theorem 4.1 that we can apply to $Dir(S)$ (since $Dir(S)$ has bounded indegree, it is uniformly sparse by Lemma 3.1.2). By using the induction hypothesis, one gets an MS-transduction mapping $|H|_1$ to $|\partial(S)|_2$ whence an MS-transduction mapping $|H|_1$ to $|H|_2$.

Fig. 1 shows a semi-directed 4-hypergraph $S$, the corresponding graph $Dir(S)$, and Fig. 2 shows the derived 3-hypergraph $\partial(S)$.

In order to have a representation of a semi-orientation $S=(H, tgt)$ of an $m$-hypergraph $H$ by a relational structure (we recall that $tgt$ is a mapping: $E_H \to V_H$ that associates

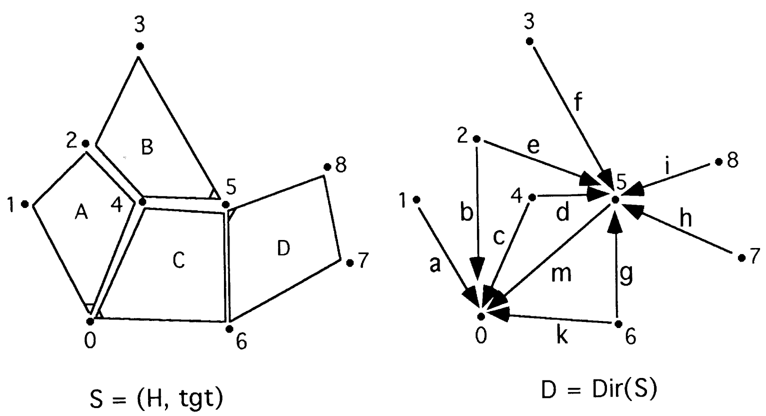


Fig. 1.

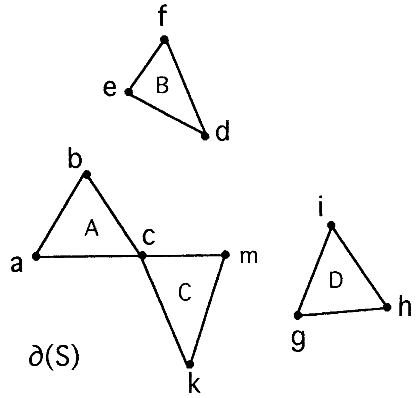


Fig. 2.

with each hyperedge one of its vertices, called its target), we define:

$$edg_S(x_1,\dots,x_m) \text{ if and only if } \quad edg_H(x_1,\dots,x_m) \quad \text{and} \quad x_m = tgt(e),$$

where $e$ is the hyperedge of $H$ with set of vertices $\{x_1,\dots,x_m\}$.

We let $|S|_1 = \langle V_H, edg_S\rangle$. We recall that the *indegree* (in $S$) of a vertex $v$ is the number of hyperedges $e$ such that $tgt(e) = v$.

For $m \geqslant 2$, $k \geqslant 1$ we let $\mathbf{USH_{k,m}}$ denote the class of undirected uniformly $k$-sparse simple $m$-hypergraphs.

**Proposition 5.1.** *Let* $m \geqslant 2$, $k \geqslant 1$.

(1) *There exists an MS-transduction that associates with* $|H|_1$, *for every* $H$ *in* $\mathbf{USH_{k,m}}$, *a structure* $|S|_1$ *where* $S$ *is a semi-orientation of* $H$ *of indegree at most* $mk^2$.

(2) *There exists an MS-transduction that associates with* $|H|_1$, *for every* $H$ *in* $\mathbf{USH_{k,m}}$, *a structure* $|H'|_2$ *where* $H'$ *is an orientation of* $H$.

**Proof.** (1) If $m = 2$, we have the desired transduction by Lemma 3.1.2 and Proposition 3.2.2, and with $k$ instead of $mk^2$.

We consider the general case. Let $H$ be in $\mathbf{USH_{k,m}}$. It has a semi-orientation of indegree at most $k$ by Lemma 3.3. By Lemma 3.4, we can transform it into one, say $S = (H, tgt)$ of indegree at most $mk^2$ such that $D = Dir(S)$ has no pair of opposite edges.

The graph $D$ has indegree at most $k' = (m-1)mk^2$. For applying [22, Theorem 10] (as in Proposition 3.2), we let $d = 2^{2k'(k'+1)+1} - 1$. By this theorem, there exists a homomorphism of $D$ into a certain directed graph $T(k')$ (with vertex set $\{1,\dots,d\}$ and no pair of opposite edges; this graph is constructed in [22]), i.e., a mapping $\gamma : V_H \to \{1,\dots,d\}$ such that, for every $x, y \in V_H$, if $x \to y$ in $D$, then $\gamma(x) \to \gamma(y)$ in the graph $T(k')$.

We let $X_1,\dots,X_d$ be the partition of $V_H$ such that $X_i=\gamma^{-1}(i)$ for each $i$.

We claim that the relation $edg_S$ is MS-definable from $edg_H$ and the sets $X_1,\dots,X_d$. This is true since, for all $x_1,\dots,x_m$ in $V_H$ we have:

(i) $edg_S(x_1,\dots,x_m)$
 if and only if
(ii) $edg_H(x_1,\dots,x_m)$ and $x_i\to x_m$ in $D$ for every $i=1,\dots,m-1$,
 if and only if
(iii) $edg_H(x_1,\dots,x_m)$ and $\gamma(x_i)\to\gamma(x_m)$ in $T(k')$ for every $i=1,\dots,m-1$.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow from the definitions. Assume that $x_1,\dots,x_m$ satisfies $edg_H(x_1,\dots,x_m)$ and $\gamma(x_i)\to\gamma(x_m)$ in $T(k')$ for every $i=1,\dots,m-1$. If the target of the corresponding hyperedge of $H$ is $x_p$ for $p<m$, then we have $x_m\to x_p$ in $D$ and $\gamma(x_m)\to\gamma(x_p)$ in $T(k')$ contradicting the fact that $T(k')$ has no pair of opposite edges. Hence, the target is $x_m$ and $edg_S(x_1,\dots,x_m)$ holds.

It follows that $edg_S$, whence also $edg_D$, are definable by MS formulas from the partition $X_1,\dots,X_d$ of $V_H$. Lemmas 3.3 and 3.4 insure the existence of such a partition, associated with a semi-orientation of $H$ such that $S$ and $Dir(S)$ satisfy the conditions of Lemma 3.4. It remains to prove that such a partition can be selected by an MS formula. Consider an arbitrary partition $X_1,\dots,X_d$. From it, and provided each set $Vert_H(e)$ has one and only vertex in some $X_i$, such that each other vertex is in some $X_j$ such that $j\to i$ in $T(k')$, one can define $edg_S$ by the equivalence of (i) and (iii).

It is now easy to express by an MS formula that the semi-orientation $S$ defined in this way has indegree at most $mk^2$. This can be formalized by a single MS formula taking $X_1,\dots,X_d$ as arguments. Hence, we have the desired MS-transduction.

(2) The proof is by induction on $m$, simultaneously for all $k$. The case $m=2$ concerns graphs and is known from the first part of the proof of Theorem 4.1.

We consider $m\geqslant 3$, and we let $H,\ S,\ D$ be as in the first part of this proof.

We let $\partial(S)$ be the simple undirected $(m-1)$-hypergraph such that:

$$V_{\partial(S)}=E_D,$$

$$E_{\partial(S)}=E_H,$$

$$Vert_{\partial(S)}(e)=\{[x_1,y],\dots,[x_{m-1},y]\}\subseteq E_D$$

where $Vert_H(e)=\{x_1,\dots,x_{m-1},y\}$ and $y=tgt(e)$ (i.e., $edg_S(x_1,\dots,x_{m-1},y)$ holds).

We show an example.

**Example.** Fig. 1 shows a semi-directed 4-hypergraph $S$ with vertices $0,1,\dots,8$, and hyperedges $A,B,C,D$, with respective targets: 0,5,0,5, together with the graph $D=Dir(S)$. The edge $c$ of $D$ from 4 to 0 comes from the two hyperedges $A$ and $C$. This fact is also clear on the 3-hypergraph $\partial(S)$: the edge $c$ is a vertex common to the hyperedges $A$ and $C$ of rank 3. See Fig. 2.

The vertices of $\partial(S)$ are the edges of $D$. Hyperedge $A$ of rank 4 in $S$ has rank 3 in $\partial(S)$, its vertices correspond to the pairs [1,0], [2,0], [4,0] of $S$.

We now consider the definability of these graphs and hypergraphs from $|H|_1$ by MS formulas.

We denote by $|S|_1+|D|_2$ the structure $\langle V_H \cup E_D, edg_S, inc_D\rangle$ which combines $|S|_1$ and $|D|_2$ (we recall that $V_D = V_H$) and by $|S|_1+|D|_2+|\partial(S)|_1$ the structure $\langle V_H \cup E_D, edg_S, inc_D, edg_{\partial(S)}\rangle$ which combines $|S|_1$, $|D|_2$ and $|\partial(S)|_1$ (we recall that $V_{\partial(S)} = E_D$).

**Claim 1.** *There exists an* MS *transduction associating* $|S|_1 + |D|_2 + |\partial(S)|_1$ *with* $|H|_1$.

**Proof.** We have MS transductions that define:

$|S|_1$ from $|H|_1$ (by (1), and in terms of suitable sets $X_1, \ldots, X_d$ that we know how to select by MS formulas),

$|D|_1$ from $|S|_1$ (clear from the definitions),

$|D|_2$ from $|D|_1$ (by Theorem 4.1 because $D$ is uniformly $k'$-sparse, cf. Lemma 3.1.2).

The definition schemes of these three MS transductions can be used to build one defining $|S|_1 + |D|_2 = \langle V_H \cup E_D, edg_S, inc_D\rangle$ from $|H|_1$.

Now the structure $|\partial(S)|_1 = \langle E_D, edg_{\partial(S)}\rangle$ is definable from the structure $|S|_1 + |D|_2$ by an MS-transduction because we have:

$edg_{\partial(S)}(e_1, \ldots, e_{m-1})$ holds if and only if there are vertices $x_1, \ldots, x_{m-1}, y$ such that $e_i$ links $x_i$ to $y$ in $D$ and $edg_S(x_1, \ldots, x_{m-1}, y)$ holds.

Finally, $|S|_1 + |D|_2 + |\partial(S)|_1$ is definable from $|H|_1$ by an MS-transduction. □

The next claim will allow us to use the induction hypothesis on $m$.

**Claim 2.** *The* $(m-1)$*-hypergraph* $\partial(S)$ *is uniformly* $2k$*-sparse.*

**Proof.** Let $Z \subseteq E_{\partial(S)}$. Let $U = \bigcup\{Vert_{\partial(S)}(e)/e \in Z\} \subseteq V_{\partial(S)}$. Since $V_{\partial(S)} = E_D$, the set $U$ can be enumerated as

$$\{[x_{1,1}, y_1], \ldots, [x_{1,n_1}, y_1], [x_{2,1}, y_2], \ldots, [x_{2,n_2}, y_2], \ldots, [x_{p,1}, y_p], \ldots, [x_{p,n_p}, y_p]\}$$

with $y_1, \ldots, y_p$ pairwise distinct. We let $X = \{x_{i,j}/1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant n_i\}$ and $Y = \{y_1, \ldots, y_p\}$ (they are subsets of $V_H$).

We have $E_{\partial(S)} = E_H$ but these hyperedges have different incidence relations in $H$ and in $\partial(S)$. We have $X \cup Y = \bigcup\{Vert_H(e)/e \in Z\}$.

Since $H$ is $k$-sparse:

$$Card(Z) \leqslant kCard(X \cup Y),$$

hence

$$Card(Z) \leqslant k(Card(X) + p) \leqslant 2kCard(U),$$

since obviously, $p = Card(Y) \leqslant Card(U)$ and $Card(X) \leqslant Card(U)$. Hence $\partial(S)$ is uniformly $2k$-sparse. □

By using the induction hypothesis, $\partial(S)$ belongs to $\mathbf{USH_{2k,m-1}}$ and we have an MS-transduction mapping $|\partial(S)|_1 \to |J|_2$ where $J$ is an orientation of $\partial(S)$.

We have also by Claim 1 an MS-transduction mapping $|H|_1$ to $|S|_1+|D|_2+|\partial(S)|_1$.

They can be combined into one transforming $|H|_1$ into $|S|_1+|D|_2+|J|_2=\langle V_H \cup V_J \cup E_J, edg_S, inc_D, inc_J\rangle$. Note that $V_J = E_D$.

Since $E_J = E_H$ we can define $H'$, the desired orientation of $H$, by taking:

$$V_{H'} = V_H,$$

$$E_{H'} = E_H = E_J,$$

$inc_{H'}(e,x_1,\ldots,x_m)$ holds if and only if $edg_S(x_1,\ldots,x_m)$ and $inc_J(e,d_1,\ldots,d_{m-1})$ hold for some edges $d_1,\ldots,d_{m-1}$ of $D$ with respective pairs of vertices $(x_1,x_m),\ldots,(x_{m-1},x_m)$.

Hence $H'$ is an orientation of $H$ and $|H'|_2$ is definable from $|H|_1$ by an MS-transduction. □

**Theorem 5.2.** *For every $m \geqslant 2$ and $k \geqslant 1$, we have the following:*

(i) *the identity is a $(1,2)$-definable* MS *transduction on the class of uniformly $k$-sparse, finite or countable, simple, directed or undirected hypergraphs of rank at most $m$;*

(ii) *the same properties of these hypergraphs are expressible by* $\mathrm{MS}_1$ *and by* $\mathrm{MS}_2$ *formulas.*

**Proof.** We first consider the case of undirected hypergraphs.

Since such a hypergraph $H$ of rank at most $m$ can be written as $H_1 \cup \cdots \cup H_m$ where $H_i = H[X_i]$ and $X_i$ is the set of hyperedges of $H$ of rank $i$, it is enough to prove the result for $m$-hypergraphs.

The proof is now quite similar to that of Theorem 4.1. We know by Proposition 5.1.2 that an MS-transduction can transform $|H|_1$ into $|H'|_2$ where $H'$ is an orientation of $H$. By composing it with the one mapping $|H'|_2$ to $|und(H')|_2 = |H|_2$ we get an MS-transduction defining the identity, as desired.

Let us now assume that $H$ is directed. The hypergraph $und(H)$ is not necessarily simple because different hyperedges of $H$ may have the same set of vertices. We let $K$ be the simple hypergraph obtained from $und(H)$ by fusing any two hyperedges with the same set of vertices. We can define from $|H|_1$ by an MS-transduction the structure $|K|_1$, and from it the structure $|K'|_2$ where $K'$ is some orientation of $K$, by Proposition 5.1.2 since $K$ is uniformly $k$-sparse. By an MS-transduction, we can define from $|H|_1$ the structure $\langle V_H \cup E_{K'}, inc_{K'}, edg_H\rangle$ (which is $|K'|_2$ augmented with $edg_H$), in which we can define $E_H$ as the set of pairs $(e,\pi)$ where $e \in E_{K'}$ and $\pi$ is a permutation of $\{1,\ldots,m\}$ such that $(x_{\pi(1)},\ldots,x_{\pi(m)})$ belongs to $edg_H$ and $(x_1,\ldots,x_m)$ is the sequence of vertices of $e$ in $K'$. The relation $inc_H$ is the set of tuples $((e,\pi),x_{\pi(1)},\ldots,x_{\pi(m)})$ such that $(e,\pi)$ is as above and $(e,x_1,\ldots,x_m)$ belongs to $inc_{K'}$. This yields a definition of $|H|_2$ in terms of $\langle V_H \cup E_{K'}, inc_{K'}, edg_H\rangle$, hence of $|H|_1$ by an MS-transduction.

That $\mathrm{MS}_1$ and $\mathrm{MS}_2$ are equally powerful on these classes of hypergraphs follows from Lemma 1.5. □

## 6. Conclusion

This paper contributes to the understanding of the relative expressive powers of the two dialects $MS_1$ and $MS_2$ of monadic second-order logic, and of the intimate relationships between this logic and hierarchical graph decompositions, which are essential in algorithmics. It unifies previously known proofs (from [8]). It shows that, in the case of finite simple graphs, the same constraint of uniform $k$-sparseness collapses simultaneously $MS_2$ onto $MS_1$ and bounded clique-width onto bounded tree-width. It demonstrates the use of nontrivial graph properties for quite involved constructions of logical formulas.

## Acknowledgements

I thank R. Diestel for the proof of Proposition 1.1, and the referees for many comments resulting in simplifications of proofs and improvements of the presentation.

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