Rook-drawing for planar graphs

Claire Pennarun
From joint work with David Auber, Nicolas Bonichon and Paul Dorbec

LaBRI, Bordeaux

Labyrinth Day
April 3rd, 2015
We want to draw large graphs representing dynamic data:

- addition/deletion of nodes
- a metanode = a group of nodes
- preserve the relative positions of nodes
- low complexity of algorithms (linear, if possible)

→ new type of drawing with constraints: rook-drawing
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Rook-drawing

Definition

A rook-drawing of a graph of $n$ vertices:

- Straight-lines
- Regular grid $n \times n$
- One vertex per line and column exactly

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A **rook-drawing** of a graph of \( n \) vertices \( \rightarrow \) two orders on the nodes

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Rook-drawing for planar graphs

Labyrinth Day
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What we already know:

- Straight-lines drawing ([Fáry, 1948]: every planar graph)
- Grid drawing ([de Fraysseix, 1988], [Schnyder, 1990]: every planar graph on an \((n - 2) \times (n - 2)\) grid)
Planar rook-drawing

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Is there a **planar** rook-drawing for every planar graph? **No!**

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A counter-example

Three exterior nodes $a$, $b$ and $c$. Inner nodes: inside the area delimited by $(ab)$, $(bc)$ and $(ca)$. 
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$\alpha \geq 45^\circ$, $\beta \geq 45^\circ$. $Mbc$ right-angled
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$\alpha = \beta = 45^\circ \rightarrow x(b) = y(c)$. 
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Fill these line and column with $c$ and $b$!
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$\alpha = \beta = 45^\circ \rightarrow x(b) = y(c)$.
$(bc)$ prevents any node to fill the line under $b$ or the column at the left of $c$.
Fill these line and column with $c$ and $b$!
Inner nodes: along a diagonal $\rightarrow$ problem with the edges...
A graph is **outerplanar** if it has a planar drawing such that all its vertices are on the outer face.

**Result**

Every outerplanar graph has a rook-drawing which can be computed in linear time.
[Bonichon, Gavoille, Hanusse, 2005]

- edges of $G$ outerplanar map $\rightarrow T_r, T_b$
- edges of $T_r$: between $u$ and its first non-descendant found after $u$ in a clockwise pre-ordering of $T_b$. 
Rook-drawing for outerplanar graphs

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$x$: ccw pre-order depth-first

$y$: ccw post-order depth-first

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- edges of $G$ outerplanar map $\rightarrow T_r, T_b$
- edges of $T_r$: between $u$ and the first node below $u$, unrelated in $T_b$.

$x$: ccw pre-order depth-first

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Hypothesis: $T_v$ of depth $k$ (+ red edges) admits a planar rook-drawing in the grid $[x(v), x(v) + |T_v| - 1] \times [y(v) - |T_v| + 1, y(v)]$.

Proof by induction on depth:
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Proof by induction on depth:
The children subtrees are placed in distinct areas and are ”well” drawn.
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Edges from $v$ to its children: no crossings!
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Additional red edges are between children subtrees: no crossings!
Main result

Every planar graph with $n$ nodes admits a planar polyline rook-drawing, with at most $n - 3$ bends (at most one per edge). Such a drawing can be computed in linear time.

$G$ a triangulation (else, make it triangulated and remove the edges at the end) with exterior nodes $v_0$, $v_1$ and $v_2$

Proof : based on an algorithm of [Bonichon, Mosbah, Le Saëc, 2002] optimizing the area of a polyline drawing.
A **Schnyder wood** is a partition of the internal edges of a triangulation in three trees $T_0$, $T_1$ and $T_2$ (directed toward the root) and with a particular configuration around each inner node:

$$P_2(u) \quad P_1(u) \quad P_0(u)$$
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\[ P_2(u) \]
\[ P_1(u) \]
\[ P_0(u) \]

Every plane triangulation admits at least one Schnyder wood, and it can be computed in linear time.
\((T_0, T_1, T_2)\) : Schnyder wood of \(G\).
(T₀, T₁, T₂) : Schnyder wood of G.

(v₁v₀), (v₂v₀), (v₂v₁)
Planar polyline rook-drawing - Nodes

- \((T_0, T_1, T_2)\) : Schnyder wood of \(G\).
- \((v_1v_0), (v_2v_0), (v_2v_1)\)
- \(x\) : clockwise preordering of \(T_0\)
  \[= \{v_0v_2ABCDEFGHIv_1\}\]
Planar polyline rook-drawing - Nodes

- \((T_0, T_1, T_2)\) : Schnyder wood of \(G\).
- \((v_1 v_0), (v_2 v_0), (v_2 v_1)\)
- \(x\) : clockwise preordering of \(T_0\) \(= \{v_0 v_2 ABCGDEFHIv_1\}\).
- \(y\) : clockwise postordering of \(T_1\) \(= \{DEABFHIGCV_2 v_1\}\) \((v_0 = 0)\).
The edges $(u, P_0(u))$ are bent at $(x(u), y(P_0(u)) + 1)$.

First blue child: directly right to its father → straighten

The edges $(u, P_1(u))$ are bent at $(x(\text{last descendant}_0(u)), y(u))$ (no bend if $u$ is a leaf of $T_0$).

Edges of $T_2$: not bent

$k$ = number of leaves in $T_0$

$T_0$ = $n - 1$ edges

$T_1$ = $n - 2 - k$ bends.

$T_2$ = 0 bends.

→ $n - 3$ bends in the drawing of $G$. 
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Planar polyline rook-drawing - Edges

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\(T_0 = n - 1 - (n - k) \) bends.
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Planar polyline rook-drawing - Edges

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\(\rightarrow n - 3 \text{ bends in the drawing of } G.\)
Proof of planarity (some ideas)

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Edges direction

For each inner node $v$:

- $P_0(v)$ is left and below $v$.
- $P_1(v)$ is right and above $v$.
- $P_2(v)$ is left and above $v$. 
Every node $v$ with $x(u) < x(v) < x(P_1(u))$ has $y(v) < y(u)$ if $v$ is not a descendant of $u$ in $T_0$. 

$v$ : between $u$ and $P_1(u)$. 

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$v$: between $u$ and $P_1(u)$. $\rightarrow v$ in area $(v_0, u), (u, P_1(u)), (P_1(u), v_0)$.

Path $v \rightarrow v_2$ has to “leave” the area!
Properties of green/red edges

Every node $v$ with $x(u) < x(v) < x(P_1(u))$ has $y(v) < y(u)$ if $v$ is not a descendant of $u$ in $T_0$.

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Intersection $t$ on path $(v_0, u)$.

Paths $v \rightarrow t$ and $u \rightarrow t$. 
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\[
\begin{align*}
\text{v : between } u \text{ and } P_1(u). & \quad \rightarrow v \text{ in area } (v_0, u), (u, P_1(u)), (P_1(u), v_0). \\
\text{Path } v \rightarrow v_2 \text{ has to “leave” the area!} \\
\text{Intersection } t \text{ on path } (v_0, u). \\
\text{Paths } v \rightarrow t \text{ and } u \rightarrow t. \\
\rightarrow \text{path from } v \text{ to } u \text{ is going upwards} \\
& = y(v) < y(u).
\end{align*}
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Path $v \rightarrow v_2$ has to “leave” the area!
Intersection $t$ on path $(v_0, u)$.

Paths $v \rightarrow t$ and $u \rightarrow t$.

$\rightarrow$ path from $v$ to $u$ is going upwards $\Rightarrow y(v) < y(u)$.

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The edges to the children cannot cross each other.
The edges of $T_1$ do not cross each other.

Subtrees live in different areas (by construction). The bends: $y$-decreasing (by construction); $x$-increasing:
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$u_{i+1}$ not descendant of $u_i$ in $T_0$

$x(u_{i+1}) > x(u_i)$

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The last descendant of $u_i$ in $T_0$ is on the right to the one of $u_{i+1}$. 
The edges of $T_1$ do not cross each other.

Subtrees live in different areas (by construction). The bends: $y$-decreasing (by construction); $x$-increasing:

$u_{i+1}$ descendant of $u_i$ in $T_0$

$u_{i+1}$ not descendant of $u_i$ in $T_0$

$x(u_{i+1}) > x(u_i)$

The last descendant of $u_i$ in $T_0$ is on the right to the one of $u_{i+1}$.

$u_{i+1}$ should be below the edge $(u_i, u)$
Non-crossing - green

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$x(u_{i+1}) > x(u_i)$  $x(u_{i+1}) < x(u_i)$

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Descendants of $u_{i+1}$ are between $x(u_{i+1})$ and $x(u_i)$. 
Open questions:

- Reduce the number of bends necessary to draw a given planar graph?
- Characterization of planar graphs for which a straight-lines rook-drawing is (not) possible
- What is the minimum grid size requested to draw a planar straight-lines rook-drawing for a given planar graph?
Thank you for your attention!