Rook-drawing for planar graphs

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From joint work with David Auber, Nicolas Bonichon and Paul Dorbec

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Find a drawing such that the following operations on a graph are easy to implement and to visualize:

- deletion/addition of nodes in the drawing
- node "expansion", i.e. zoom in a hierarchical structure
Rook-drawing

Definition

A rook-drawing of a graph of $n$ vertices:

- Straight-lines
- Regular grid $n \times n$
- One vertex per line and column exactly
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Planar rook-drawing

What we already know:

- Straight-lines drawing ([Fáry, 1948] : every planar graph)
- Grid drawing ([de Fraysseix, 1988], [Schnyder, 1990] : every planar graph on an $n \times n$ grid)

Is there a planar rook-drawing for every planar graph with $n$ vertices on a $n \times n$ grid?

No!
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A graph is **outerplanar** if it has a planar drawing such that all its vertices are on the outer face.

**Result**

Every outerplanar graph with $n$ vertices has a rook-drawing computed in linear time on a grid $n \times n$. 
[Bonichon, Gavoille, Hanusse, 2005]

- edges of $G \rightarrow T_r, T_b$
- edges of $T_r$: between $u$ and its first non-descendant found after $u$ in a clockwise pre-ordering of $T_b$. 

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Rook-drawing for outerplanar graphs

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$y$: ccw post-order depth-first
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For each vertex $u$ not a leaf of $T$:

- Define an area $A_u$ with only red edges leading to $u$ (the areas $A_i$ are disjoint).

The drawing is planar within $A_u$ and blue and red edges can not cross.
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A **Schnyder wood** is a partition of the interior edges of $G$ in three trees $T_0$, $T_1$ and $T_2$ (directed toward the root) and with a particular configuration around each node:

![Diagram](image)

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$$P_2(u) \quad P_1(u) \quad P_0(u)$$

Every plane triangulation admits at least one Schnyder wood, and it can be computed in linear time.

[Schnyder 1989]
We consider $G$ a plane triangulation (with exterior nodes $v_0$, $v_1$ and $v_2$).

**Main result**

Every planar graph with $n$ nodes admits a planar polyline rook drawing on a $n \times n$ grid, with at most $n - 2$ bends (at most one per edge). Such a drawing is computed in linear time.

Proof: based on an algorithm of [Bonichon, Mosbah, Le Saëc, 2002] optimizing the area of a polyline drawing.
Compute a realizer $(T_0, T_1, T_2)$ of the graph $G$. 

- Column order: clockwise preordering of $T_0 = \{ABCGDEFHI\}$.
- Line order: clockwise postordering of $T_1 = \{DEABFHIGC\}$.

Column order + Line order = 1 vertex for each line and column!
Compute a realizer \((T_0, T_1, T_2)\) of the graph \(G\).

Place \(v_0\) at \((0, 0)\), place \(v_2\) at \((1, n - 2)\), place \(v_1\) at \((n - 1, n - 1)\).
Planar polyline rook-drawing

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\[\text{Column order + Line order} = 1 \text{ vertex for each line and column!}\]
Edges bends

If we bend all edges:

- The edges \((u, P_0(u))\) are bent at \((x(u), y(P_0(u)) + 1)\)
- The edges \((u, P_1(u))\) are bent at \((x(P_1(u)) - 1, y(u))\)
- The edges \((u, P_2(u))\) are bent at \((x(P_2(u)) + 1, y(u))\)
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In fact, we can straighten many of them! (All red edges, left-most blue edges and green ones from blue leafs)
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Around a node:
- \(\#bends = \#\{\text{blue children}\}\)
- \(\rightarrow n - 2\) bends in the drawing
Conclusion

Open questions:

- Reduce the number of bends necessary to draw a given planar graph?
- Caracterization of planar graphs for which a straight-lines rook-drawing is not possible / is possible
- What is the minimum grid size requested to draw a straight-lines rook-drawing?
Thank you for your attention!