

# Rook-drawing for planar graphs

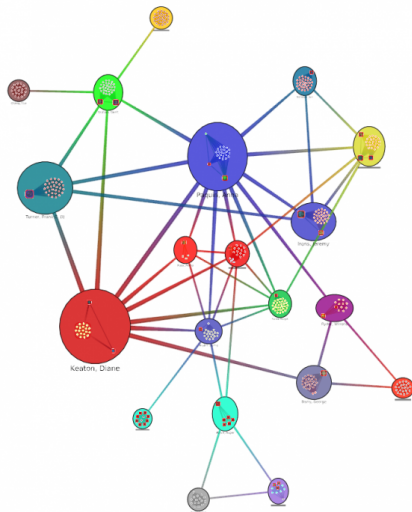
Claire Pennarun

From joint work with David Auber, Nicolas Bonichon and Paul Dorbec

LaBRI, Bordeaux

November 13<sup>th</sup>, 2014

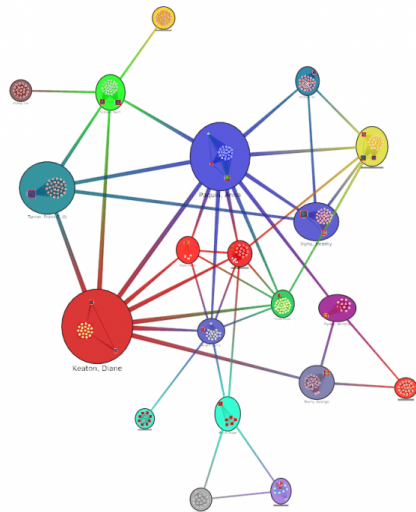
# Drawing graphs



# Drawing graphs

Find a drawing such that the following operations on a graph are easy to implement and to visualize :

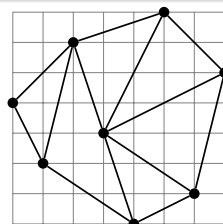
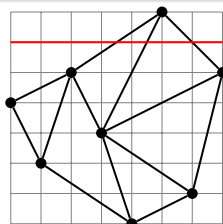
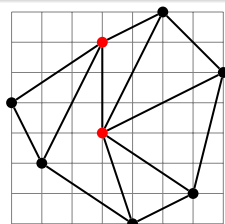
- deletion/addition of nodes in the drawing
- node "expansion", i.e. zoom in a hierarchical structure



## Definition

A **rook-drawing** of a graph of  $n$  vertices :

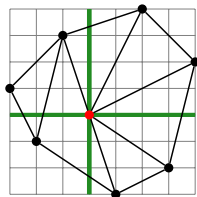
- Straight-lines
- Regular grid  $n \times n$
- One vertex per line and column exactly



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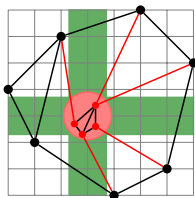
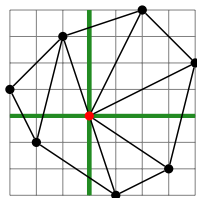
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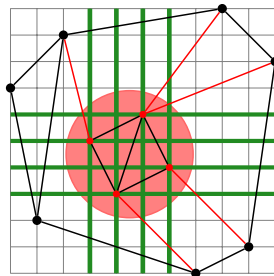
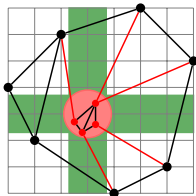
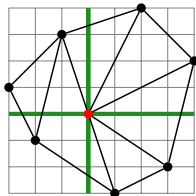


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# Planar rook-drawing

What we already know :

- Straight-lines drawing ([Fáry, 1948] : every planar graph)
- Grid drawing ([de Fraysseix, 1988], [Schnyder, 1990] : every planar graph on an  $n \times n$  grid)



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Is there a planar rook-drawing for every **planar** graph with  $n$  vertices on a  $n \times n$  grid ?

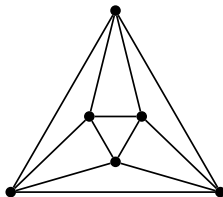
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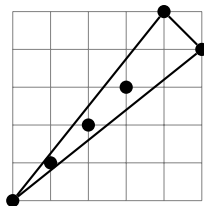
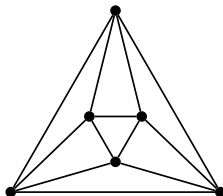
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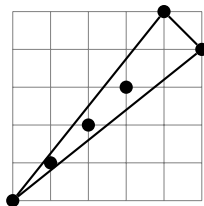
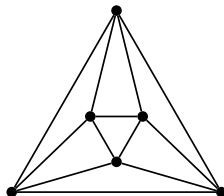
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Is there a planar rook-drawing for every **planar** graph with  $n$  vertices on a  $n \times n$  grid?

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- Planar straight-lines rook-drawing for outerplanar graphs
- Planar *polyline* rook-drawing with a few bends for planar graphs

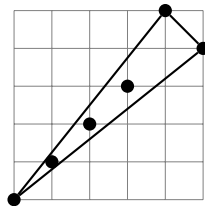
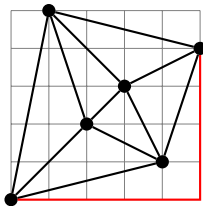
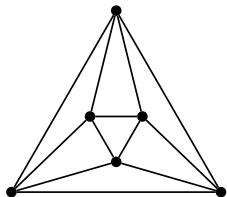
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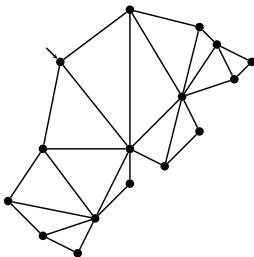
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# Rook-drawing for outerplanar graphs

A graph is **outerplanar** if it has a planar drawing such that all its vertices are on the outer face.



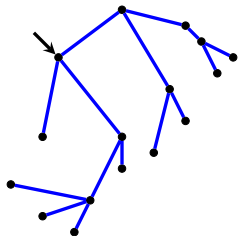
## Result

Every outerplanar graph with  $n$  vertices has a rook-drawing computed in linear time on a grid  $n \times n$ .

# Rook-drawing for outerplanar graphs

[Bonichon, Gavaille, Hanusse, 2005]

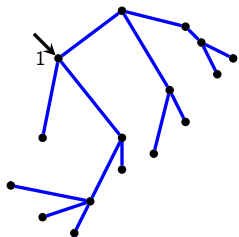
- edges of  $G \rightarrow T_r, T_b$
- edges of  $T_r$  : between  $u$  and its first non-descendant found after  $u$  in a clockwise pre-ordering of  $T_b$ .



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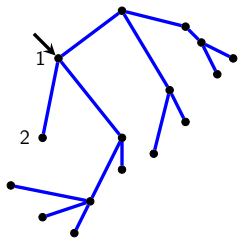
- $x$  : ccw pre-order depth-first
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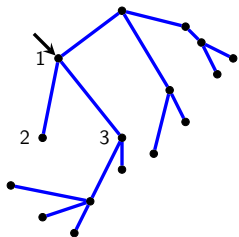


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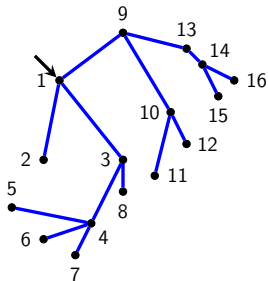


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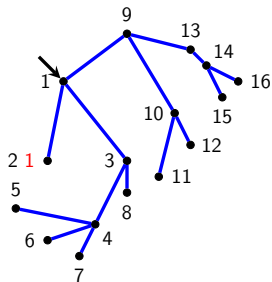


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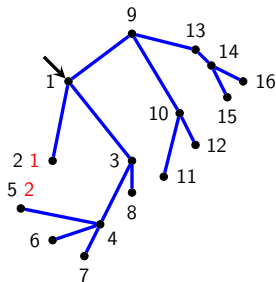


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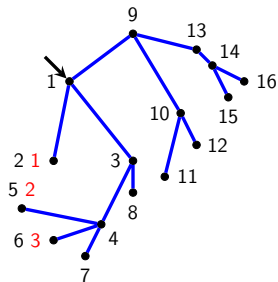


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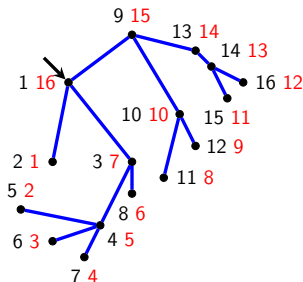


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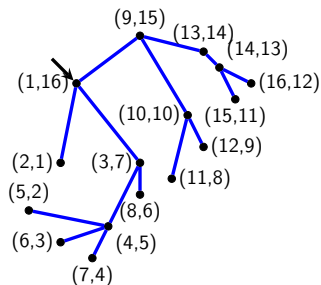


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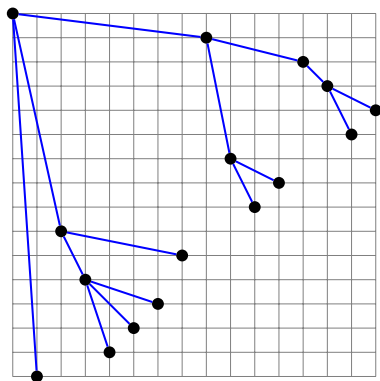
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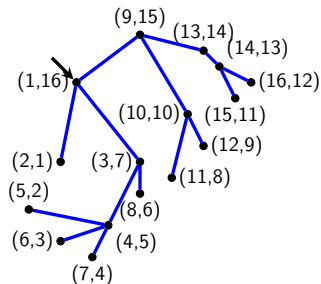




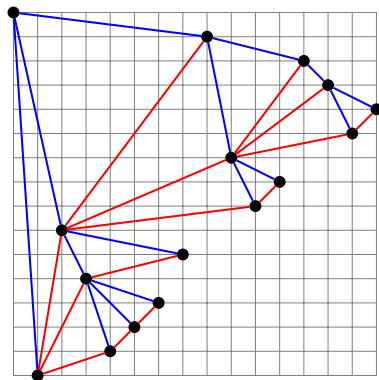
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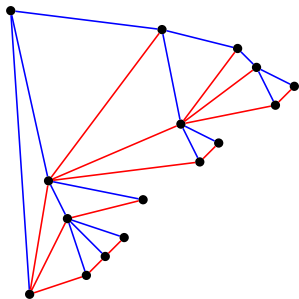
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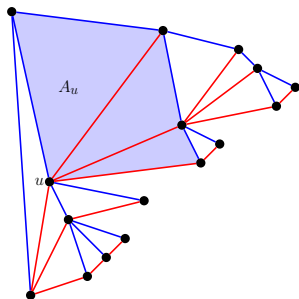
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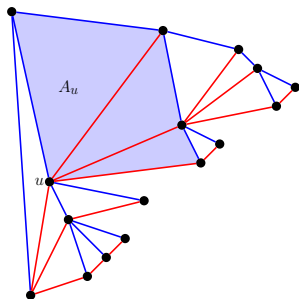


# Rook-drawing for outerplanar graphs



For each vertex  $u$  not a leaf of  $T_r$  :  
define an area  $A_u$  with only red edges  
leading to  $u$  (the areas  $A_i$  are  
disjoint).

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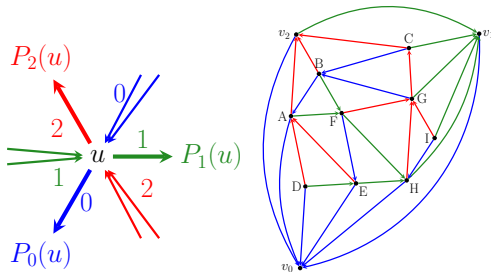


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The drawing is planar within  $A_u$  and  
blue and red edges can not cross.

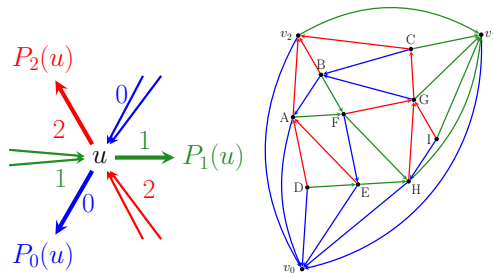
# Schnyder woods

A **Schnyder wood** is a partition of the interior edges of  $G$  in three trees  $T_0$ ,  $T_1$  and  $T_2$  (directed toward the root) and with a particular configuration around each node :



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[Schnyder 1989]

Every plane triangulation admits at least one Schnyder wood, and it can be computed in linear time.

We consider  $G$  a plane triangulation (with exterior nodes  $v_0$ ,  $v_1$  and  $v_2$ ).

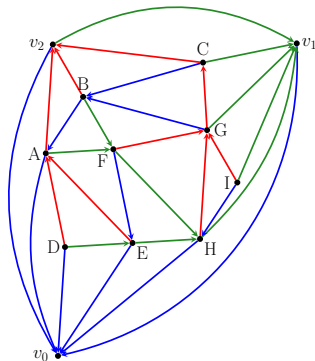
## Main result

Every planar graph with  $n$  nodes admits a planar polyline rook drawing on a  $n \times n$  grid, with at most  $n - 2$  bends (at most one per edge). Such a drawing is computed in linear time.

Proof : based on an algorithm of [Bonichon, Mosbah, Le Saëc, 2002] optimizing the area of a polyline drawing.

# Planar polyline rook-drawing

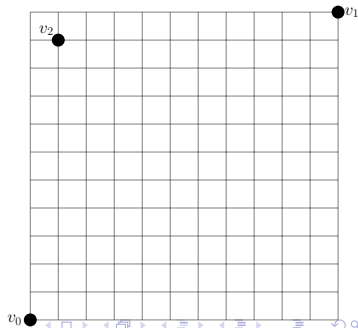
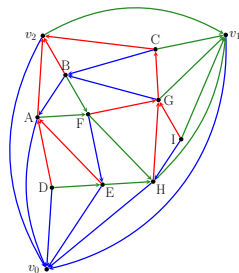
- Compute a realizer  $(T_0, T_1, T_2)$  of the graph  $G$ .





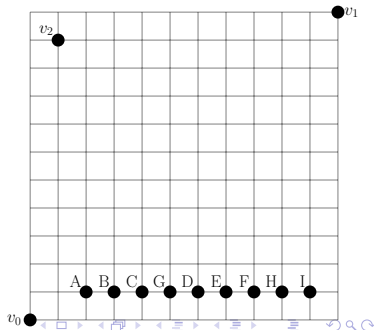
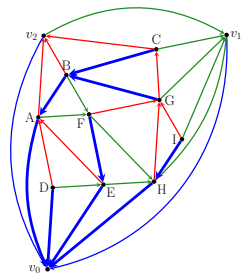
# Planar polyline rook-drawing

- Compute a realizer  $(T_0, T_1, T_2)$  of the graph  $G$ .
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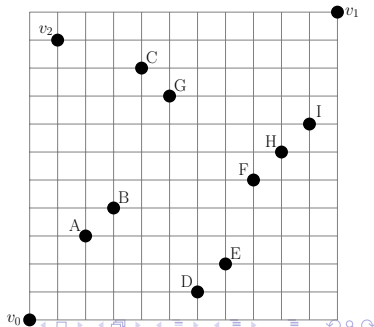
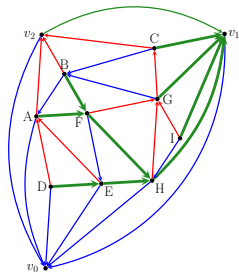
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- Column order : clockwise preordering of  $T_0 = \{ABCGDEFHI\}$ .



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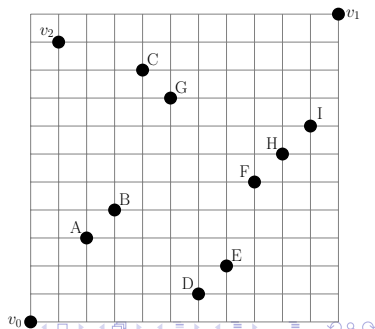
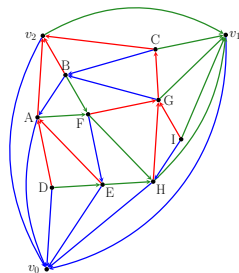
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- Line order : clockwise postordering of  $T_1 = \{DEABFHIGC\}$ .



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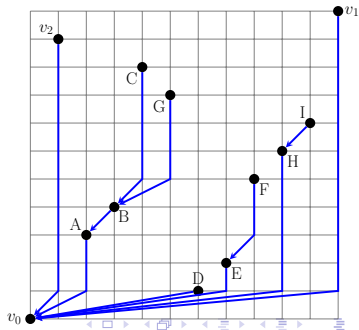
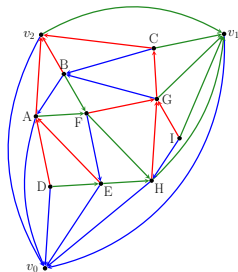
Column order + Line order  
= 1 vertex for each line and column !



# Edges bends

If we bend all edges :

- The edges  $(u, P_0(u))$  are bent at  $(x(u), y(P_0(u)) + 1)$
- The edges  $(u, P_1(u))$  are bent at  $(x(P_1(u)) - 1, y(u))$
- The edges  $(u, P_2(u))$  are bent at  $(x(P_2(u)) + 1, y(u))$

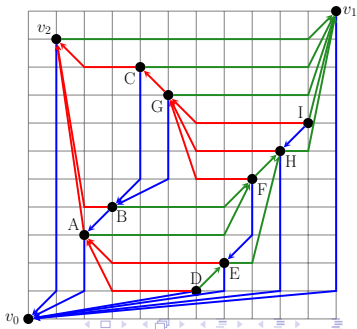
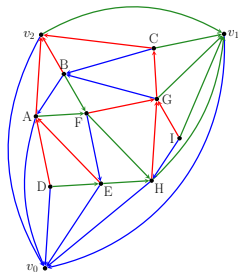


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- The edges  $(u, P_2(u))$  are bent at  $(x(P_2(u)) + 1, y(u))$

In fact, we can straighten many of them ! (All red edges, left-most blue edges and green ones from blue leafs)



# Edges bends

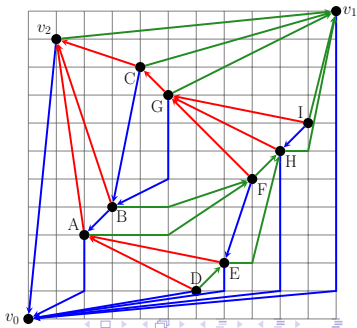
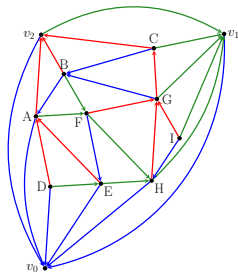
If we bend all edges :

- The edges  $(u, P_0(u))$  are bent at  $(x(u), y(P_0(u)) + 1)$
- The edges  $(u, P_1(u))$  are bent at  $(x(P_1(u)) - 1, y(u))$
- The edges  $(u, P_2(u))$  are bent at  $(x(P_2(u)) + 1, y(u))$

In fact, we can straighten many of them ! (All red edges, left-most blue edges and green ones from blue leafs)

Around a node :

$\#bends = \#\{\text{blue children}\}$   
 $\rightarrow n - 2$  bends in the drawing



Open questions :

- Reduce the number of bends necessary to draw a given planar graph ?
- Characterization of planar graphs for which a straight-lines rook-drawing is not possible / is possible
- What is the minimum grid size requested to draw a straight-lines rook-drawing ?



Thank you for your attention !