

Counting planar Eulerian orientations

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Joint work with Nicolas Bonichon,
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LaBRI, Bordeaux, France

Kranjska Gora, June 25th 2015

SOME DEFINITIONS

- connected planar rooted maps (in a corner)
- with loops and multiple edges

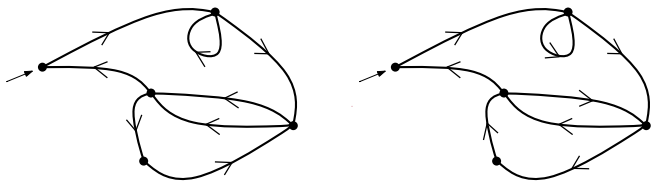
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- connected planar rooted maps (in a corner)
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A map G is called *Eulerian* if every vertex of G has an even degree.

A directed planar map G is called a *planar Eulerian orientation (PEO)* if every vertex of G has in-degree and out-degree equal.



MAIN QUESTION

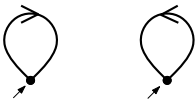
Given an integer m , how many PEOs are there with m edges?

FIRST PLANAR EULERIAN ORIENTATIONS

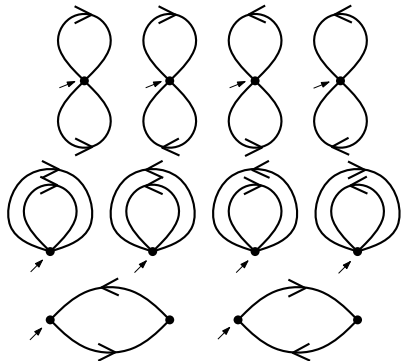
$m=0$



$m=1$



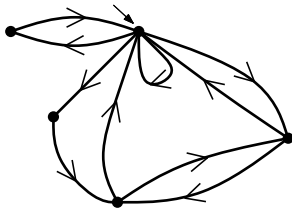
$m=2$



ITERATIVE GENERATION OF PEO

Given an integer m and the PEOs with $m - 1$ edges, how many are there with m edges ?

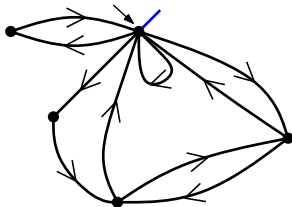
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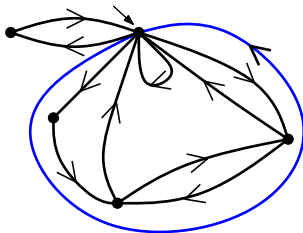


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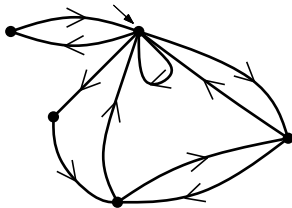


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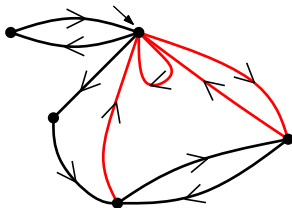


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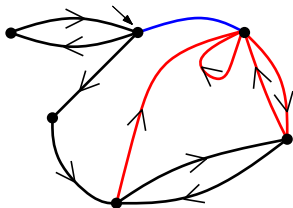


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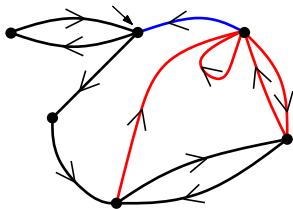


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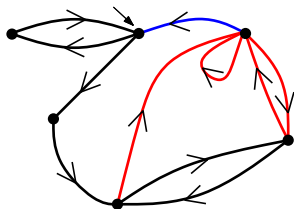


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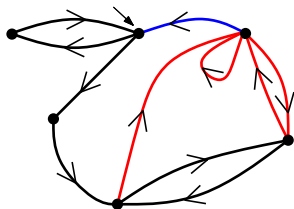
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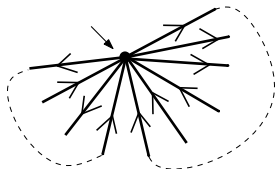


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In fact, just look at the root and forget the rest of the map!

ENCODING

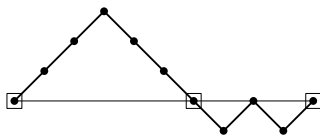
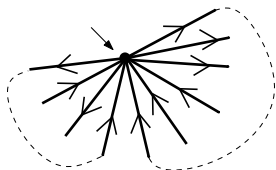
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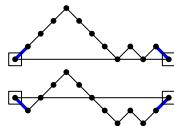
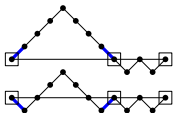
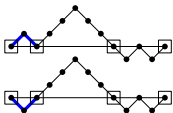
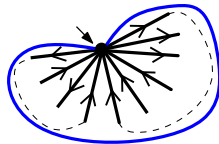
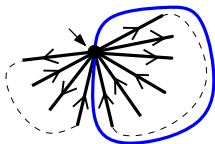
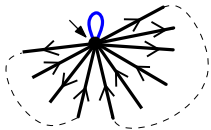
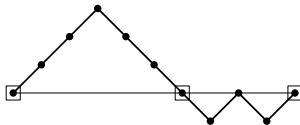
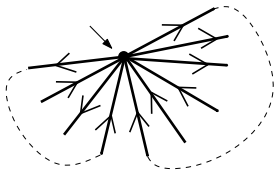
Encode with **decorated bilateral Dyck words**



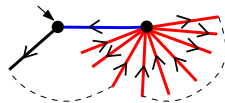
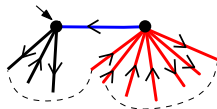
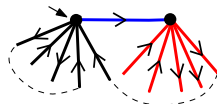
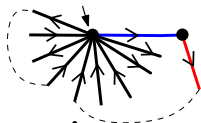
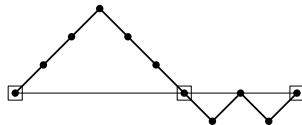
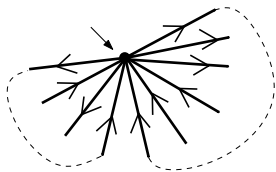
Properties:

- up step: out-going edge
- down step: in-going edge
- squared points: incidence of the root with the outer face
- 1/-1 points: legal splits

CHILDREN AFTER LOOP ADDITION



CHILDREN AFTER SPLITS



COMPUTING THE FIRST TERMS

m edges	Eulerian maps	PEO	oriented Eul. maps
0	1	1	1
1	1	2	2
2	3	10	12
3	12	66	96
4	56	504	896
5	288	4 216	9 216
6	1 584	37 548	101 376
7	9 152	350 090	1 171 456
8	54 912	3 380 520	14 057 472
9	339 456	33 558 024	173 801 472
10	2 149 888	340 670 720	2 201 485 312
11	13 891 584	3 522 993 656	28 449 964 032
growth rate	8^m	?	16^m

GENERATING FUNCTION FOR PEO ?

Useful tool to find a general formula: the *generating function*

Integer parameters (number of edges/nodes, degree of the root...)

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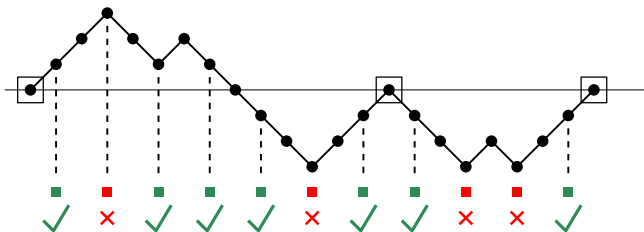
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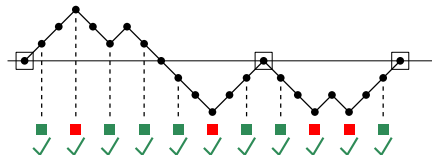
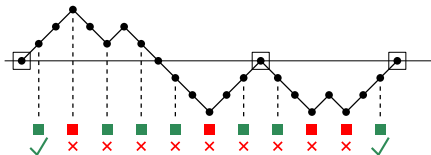


We want to allow exactly the legal splits !

→ Generate **upper/lower classes of PEO**

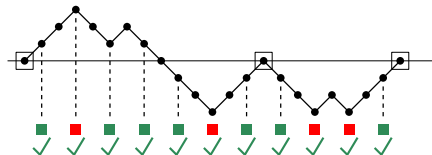
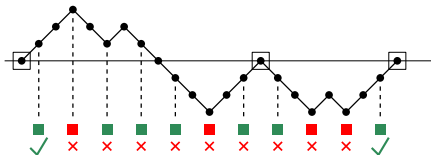
UPPER AND LOWER BOUNDS

Natural lower/upper bound: forbid/allow all odd points for splits
(except 1 and $\Delta - 1$)

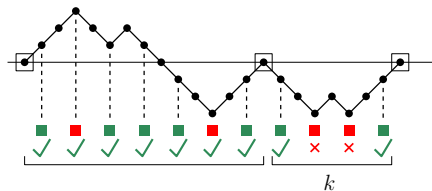
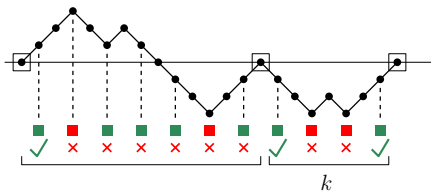


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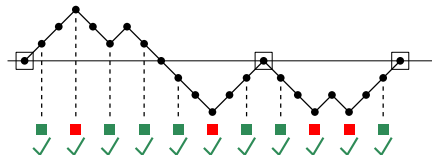
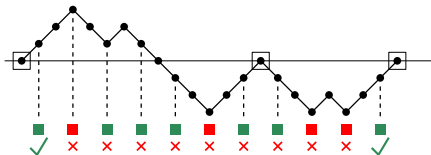


More precise lower/upper bounds: the last k odd points are exact

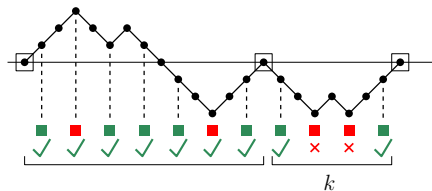
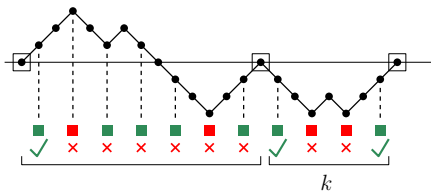


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Keep a structure of bounded length \rightarrow algebraic system of generating functions (one for each configuration)

GLOBAL RESULTS

	growth	1	2	3	4	5	6
Eulerian maps	8^m	1	3	12	56	288	1 584
inf : $k = 1$	9.68^m	2	10	66	466	3 458	26 650
inf : $k = 2$	10.16^m	2	10	66	504	4 008	32 834
inf : $k = 3$	10.51^m	2	10	66	504	4 216	36 316
inf : $k = 4$	$\sim 10.69^m$	2	10	66	504	4 216	37 548
PEO	?	2	10	66	504	4 216	37 548
sup : $k = 2$	$\sim 12.95^m$	2	10	66	504	4 234	37 998
sup : $k = 1$	13.06^m	2	10	66	506	4 266	38 418
Oriented Eulerian maps	16^m	2	12	96	896	9 216	101 376

To have the number of PEO with l edges: either use the generating tree or compute the lower bound with $k = l - 2$.

OPEN QUESTIONS

- What is the type of the generating function for PEO ?
- What if we restrict the vertices degree of the map ?
- Find another decomposition for PEO taking the orientation into account ?

Thank you for your attention !