

Counting planar Eulerian orientations

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WHY COUNT PLANAR MAPS?

Counting planar maps for a long time... And we know them really well!

- recursive and bijective approaches
- algebraicity of generating functions
- fine asymptotics properties
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Statistical physics and combinatorics: maps + structure?

MAPS EQUIPPED WITH A STRUCTURE

In general: maps equipped with a specialisation of the Tutte polynomial

- proper q -colouring [Tutte 73-84]
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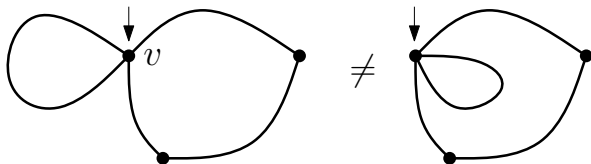
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And if the structure does not come from the Tutte polynomial?

→ Eulerian orientations

SOME DEFINITIONS

- planar maps (rooted in a corner)
- with loops and multiple edges



m : number of edges (= 4)

v is the root-vertex

d : half root-degree (= 2)

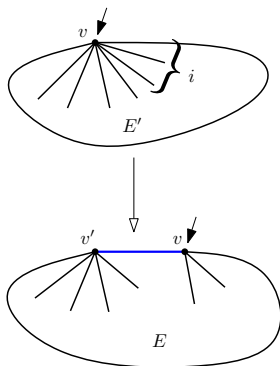
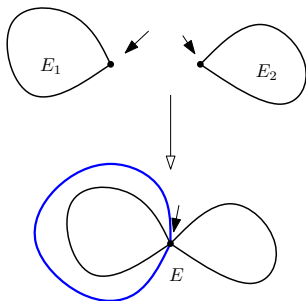
Δ : root-degree (=4)

EULERIAN MAPS

A map G is called *Eulerian* if every vertex of G has an even degree.

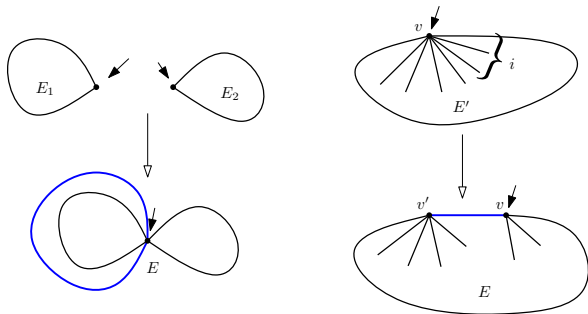
Possible operations:

- **merge** two maps and add a loop
- **split** the root-vertex at index i (i odd)



GENERATING FUNCTION FOR EULERIAN MAPS

$E(t; x) = \sum_{d \geq 0} E_d(t) x^d$: generating function (g.f.) of Eulerian maps (t counts the number of edges, x the half root-degree)



$$E(x) = 1 + txE(x)^2 + t \sum_{d \geq 0} E_d(x + x^2 + \dots + x^d)$$

$$E(x) = 1 + txE(x)^2 + tx \frac{E(x) - E(1)}{x - 1}$$

Polynomial equation with one catalytic variable, solved by Brown's quadratic method, has an algebraic equation

EULERIAN ORIENTATIONS (PEO)

A directed planar map G is a *planar Eulerian orientation (PEO)* if every vertex of G has in-degree and out-degree equal.

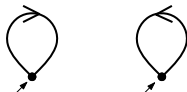
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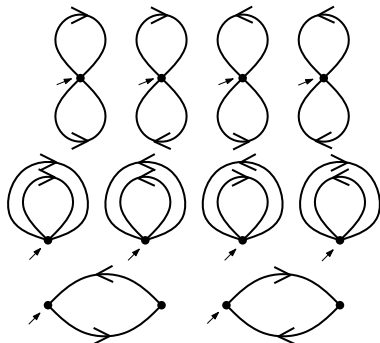
$m=0$



$m=1$

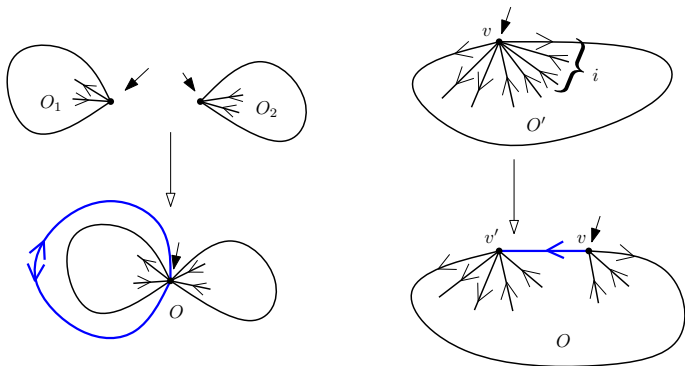


$m=2$



OPERATIONS ON PEO

\mathcal{O} is the set of PEO.

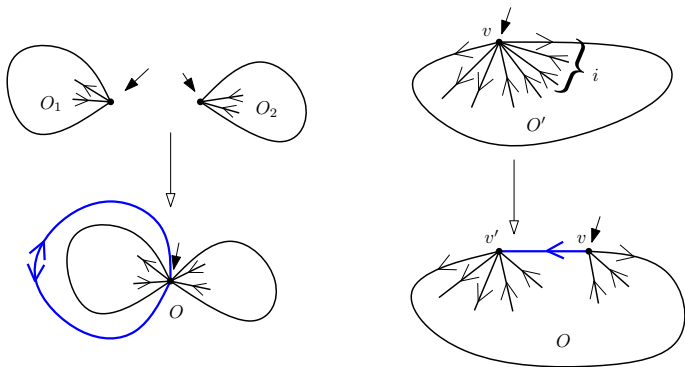


Possible operations:

- **merge** two maps and direct the new edge
- **split** the root-vertex at index i iff the resulting map is still a PEO = *legal split* (indices $i = 1$ and $i = \Delta - 1$ always give legal splits)

GENERATING FUNCTION FOR PEO?

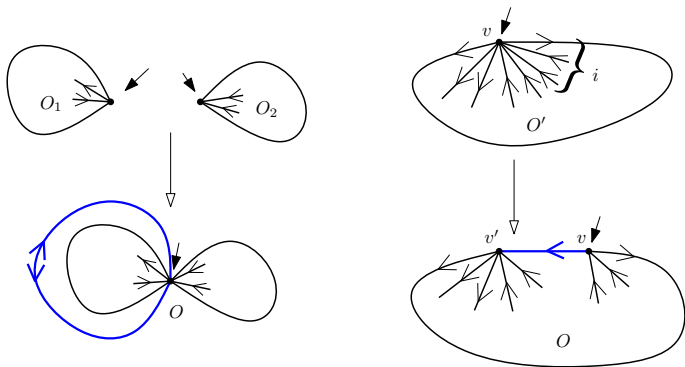
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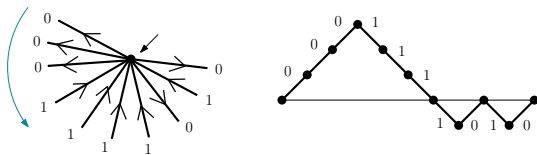
We can not keep the set of legal splits as a parameter...

FROM ENCODING...

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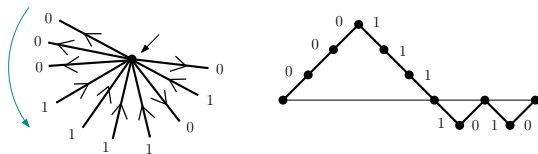
To generate a map: one only needs the orientation around the root-vertex = **bilateral Dyck path**



→ root-word of a map O : $\mathbf{w}(O) = 0001111010$

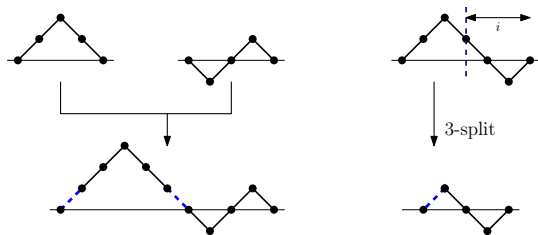
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A legal split = height 1 or -1 on the path



... TO THE FIRST TERMS

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m	$o(m)$	m	$o(m)$	m	$o(m)$
0	1	6	37.548	12	37.003.723.200
1	2	7	350.090	13	393.856.445.664
2	10	8	3.380.520	14	4.240.313.009.272
3	66	9	33.558.024	15	46.109.094.112.170
4	504	10	340.670.720		
5	4.216	11	3.522.993.656		

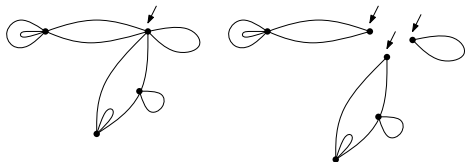
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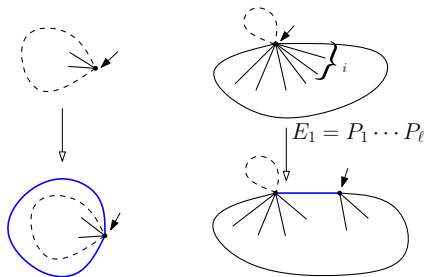
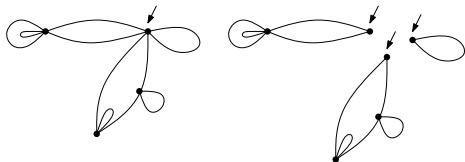
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$E(x)$ the g.f. for Eulerian maps,
 $E'(x)$ for prime Eulerian maps:

$$\begin{cases} E(x) = 1/(1 - E'(x)) \\ E'(x) = txE(x) + tE(1)x \frac{E'(x) - E'(1)}{x - 1} \end{cases}$$

APPROXIMATION OF THE GROWTH RATE

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$\{o(m)\}_{m \geq 0}$ is a *super-multiplicative sequence*, i.e.

$o(n + n') \geq o(n)o(n') \forall n, n' \in \mathbb{N}^*$ (merging two PEO with n and n' edges gives a PEO with $n + n'$ edges).

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Variant of Fekete's Lemma (1923):

$$\mu = \lim_{n \rightarrow \infty} o(n)^{1/n} = \sup_{n \geq 1} o(n)^{1/n} \in \mathbb{R}_+^* \cup \infty$$

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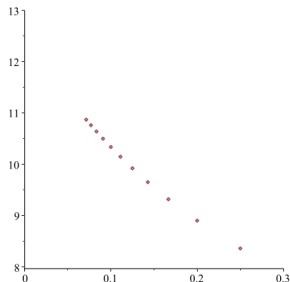
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$\frac{o(m+1)}{o(m)}$ as a function of $1/m \rightarrow$



BETTER APPROXIMATION OF THE GROWTH RATE

Two families of sets of orientations $\mathcal{O}^{(k)-}$ and $\mathcal{O}^{(k)+}$ such that
 $\mathcal{O}^{(k)-} \subset \mathcal{O}^{(k+1)-} \subset \mathcal{O} \subset \mathcal{O}^{(k+1)+} \subset \mathcal{O}^{(k)+}$

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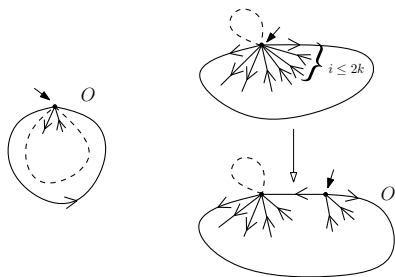
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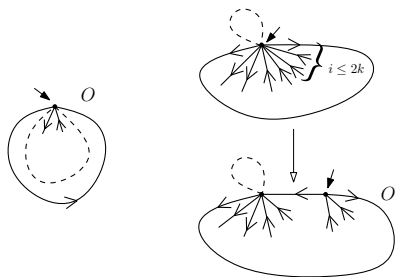


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A word w is *balanced* iff

$$|w|_0 - |w|_1 = 0.$$

It is *quasi-balanced* iff

$$|w|_0 - |w|_1 = \pm 1.$$

ALGEBRAIC SYSTEM FOR $\mathcal{O}^{(k)-} \equiv \mathcal{O}^-$

$F_w(t)$: g.f. for the set $\{O \in \mathcal{O}^- \mid \mathbf{w}(O) = w\}$

$L_w(t)$: g.f. for the set $\{O \in \mathcal{O}^- \mid \mathbf{w}(O) = uw \text{ for some } u\}$

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SOME DETAILS

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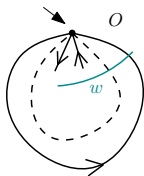
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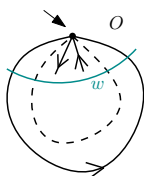
$$L'_w = tL_{w_1} + tF_{w_2} +$$

$$|w| < \Delta(O)$$



w_1 is a suffix of $\mathbf{w}(O')$

$$|w| = \Delta(O)$$



$\mathbf{w}(O') = w_2$

SOME DETAILS

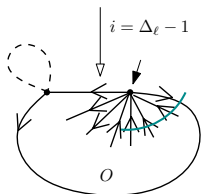
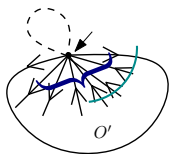
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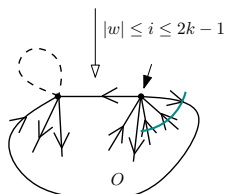
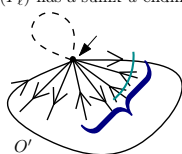
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$$tL_\varepsilon L'_w + tL_\varepsilon \sum_{\substack{u = \bar{v}w \\ u \text{ quasi-bal.} \\ |u| < 2k}} (L'_u - F'_{\bar{u}}) + tL_\varepsilon (L'_{w'} - F'_w), \text{ if } |w| \leq 2k - 2$$

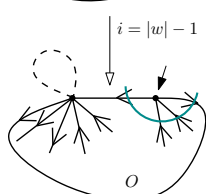
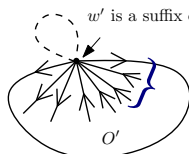
w is a suffix of $w(P_\ell)$



$w(P_\ell)$ has a suffix u ending with w



w' is a suffix of $w(P_\ell)$



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for a given value of k

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Theorem

Let \mathcal{O}_m be the number of Eulerian orientations with m edges. Then $\mathcal{O}_m^{1/m} \rightarrow \mu$
with $11.22 \leq \mu \leq 13.06$.

SOME OTHER RESULTS...

For each $k > 0$, let $C(k, n)$ be the counting sequence of the set $\mathcal{O}^{(k)-}$. Then $C(k, n) \sim \gamma n^{-3/2} \rho^{-n}$ (ρ and γ depend on k).

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The functional equations for the sets $\mathcal{O}^{(k)+}$ have one catalytic variable (counting the half root-degree) and the series relative to these sets are algebraic.

OPEN QUESTIONS

- What is the type of the generating function for PEO ?
- What if we restrict the vertices degrees of the map ?
- Find another grammar for the PEO taking the orientation into account ?

Thank you!