

# SKETCHY TWEETS: TEN MINUTE CONJECTURES IN GRAPH THEORY

ANTHONY BONATO AND RICHARD J. NOWAKOWSKI

Comments by Richard Hamming in his address *You and Your Research* [16] resonated with us. On the one hand, Hamming says:

“*What are the most important problems in your field?*”

which suggests working on hard problems. Yet on the other, he exhorts us to:

“*Plant the little acorns from which mighty oak trees grow.*”

Following this advice, we should look over the big questions, then doodle and sketch out some approaches. If you are an expert, then this is easy to do, but most people do not want to wait the requisite 10,000 hours before looking at interesting problems. Graph Theory, our area of expertise, has many hard-to-state and hard-to-solve questions. However, like Number Theory, it has many easy-to-state but difficult conjectures. Some hark back to the recreational roots of the area yet still keep their mystery. These “acorns” can be planted on the backs of envelopes, on a blackboard, and over a coffee.

Our goal is to collect some of these conjectures—arguably some of the most intriguing—in one place. We present 10 conjectures in Graph Theory, and you can read about each one in at most 10 minutes. As we live in the era of Twitter, all the conjectures we state are 140 characters or less (so “minute” here has a double meaning). We might even call these *sketchy tweets*, as we present examples for each conjecture that you can doodle on as you read.

Hamming also references *ambiguity*: good researchers can work both on proving and disproving the same statement, so we approach the conjectures with an open mind. He also mentions

that a good approach is to reframe the problem, and change the point of view. One example, from Vizing’s conjecture, is the three page paper [2] which, with a new way of thinking, reduced most of the published work of 20 years to a corollary of its main result! Given the size of modern Graph Theory with its many smaller subfields (such as structural graph theory, random graphs, topological graph theory, graph algorithms, spectral graph theory, graph minors, or graph homomorphisms, to name a few) it would be impossible to list all, or even the bulk of these conjectures in the field. We are content instead to focus on a few family jewels, which have an intrinsic beauty and have provided some challenges for Graph Theorists for at least two decades. There is something for everyone here, from the undergraduate student taking their first course in Graph Theory, to the seasoned researcher in the field. For additional reading on problems and conjectures in Graph Theory and other fields, see the Open Problem Garden maintained by IRMACS at Simon Fraser University [24].

We consider only finite and undirected graphs, with no multiple edges or loops (unless otherwise stated). We assume the reader has some basic familiarity with graphs and their terminology (including notions such as cycles  $C_n$ , paths  $P_n$ , complete graphs  $K_n$ , the complete bipartite graphs  $K_{m,n}$ , degrees, and connected graphs). All the background we need can be found in any text in Graph Theory, such as those of Diestel [9] and West [41]. For a graph  $G$ , we write  $V(G)$  for its *vertices*, and  $E(G)$  for its *edges*. If two vertices are joined by an edge, then we say they are *adjacent*. The cardinality  $|V(G)|$  is the *order* of  $G$ .

## THE CONJECTURES

Some conjectures we present (such as Meyniel’s) are lesser known and deserve more exposure, while others (such as Hadwiger’s, for example) are more well known. We provide no justification for our bias toward one problem over another, so we apologize upfront if your favorite conjecture is missing.

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For each conjecture, we present it using minimal technical jargon. To shorten the number references on partial results, we cite surveys whenever possible; we always cite the original authors of the problem. To show no preference among the problems, we present the conjectures in alphabetical order.

**Double the fun.** There is an old puzzle, found in many books that feature “pencil-and-paper” problems, of attempting to trace a diagram without lifting the pen off the paper nor retracing any line of the figure. Euler in his famous 1736 solution to the Königsberg bridge problem, essentially solved this problem. The problem can be restated as covering the diagram (represented by a graph) with cycles, where every edge is in exactly one cycle. Such graphs are now called *Eulerian*, and are exactly those connected graphs with every vertex of even degree. Our next conjecture may be thought of as a generalization of this kind of problem to graphs with some vertices of odd degree; however, we replace “exactly one cycle” by “exactly two cycles”.

A *bridge* is an edge whose deletion disconnects the graph. A graph with no bridges is *bridgeless*.<sup>1</sup> For example, each edge of a tree is a bridge.

**Cycle Double Cover Conjecture:** Every bridgeless graph contains a set of cycles so that every edge is contained in exactly two cycles.

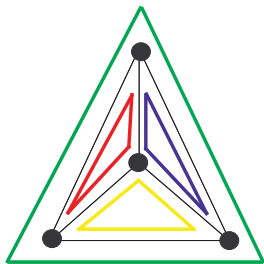


FIGURE 1. A CDC in  $K_4$ , with the cycles in different colors. What would be a CDC of  $K_n$  for general  $n$ ?

<sup>1</sup>The famous graph of the Königsberg Bridge problem is about bridges, but it is itself bridgeless!

Such a set of cycles, as in the conjecture, is called a *CDC*. See Figure 1 for an example. The conjecture was formulated independently by Szekeres in 1973 [34] and Seymour in 1979 [30]. See the survey [19] and book [44] for additional background and references on the conjecture.

The conjecture has connections to *embeddings of graphs on surfaces*; that is, drawings of graphs on different surfaces so that no two edges cross. The simplest case being *planar graphs* which have an embedding in the plane (see Hadwiger’s conjecture for more on planar graphs). If each face in the embedding corresponds to a cycle in the graph, then the faces form a CDC as in Figure 1, as is true for all connected, bridgeless, planar graphs. That there is an embedding where each face corresponds to a cycle is the *Strong Embedding conjecture*.

In the other direction, much is known about a minimum counter-example, if it exists: every vertex has degree 3, it is not *3-edge-colorable* (that is, there is no coloring of the edges with three colors so that each edge is incident with an edge of the same color), it is *cyclically-4-connected* (that is every partition of the vertices into two parts with a cycle in each part has at least four edges that go between the parts), and the smallest cycle has length at least 10. Such graphs, without the smallest cycle restriction, are called *snarks*. A different conjecture says that there are not any snarks with the smallest cycle length at least 10. See Figure 2 for an example of a snark.

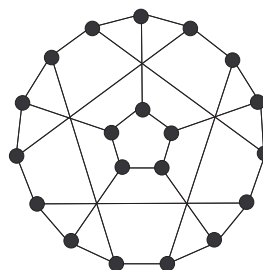


FIGURE 2. The flower snark  $J_5$ .

A related conjecture is the *Small Cycle Double Cover conjecture*: every bridgeless graph on  $n$  vertices contains a set of at most  $n$  cycles so that every edge is contained in exactly two cycles.

**Party, but know your limits.** Frank Ramsey, who died at the early age of 26, wrote a paper [26] in mathematical logic that has gone on to have applications in many fields, including Graph Theory. To motivate Ramsey numbers, consider a party with 6 people, some of whom are friends, and some are strangers. An observation is that there are always 3 people who are all mutually friends, or all mutually strangers. The reader should convince themselves this never happens in smaller parties. All of this together can be neatly summarized by saying that  $R(3) = 6$ .

Ramsey numbers generalize this setting from 3 to  $n$  mutual friends or strangers. For a positive integer  $n$ , define the  $n$ th Ramsey number, written  $R(n)$ , to be the minimum integer  $r$  such that any coloring of the edges of  $K_r$  with red or blue (red equals friends and blue equal strangers) results in a complete subgraph of order  $n$  which has edges all one color. How large a party do we need to realize such patterns?

One immediate question is whether the Ramsey numbers even exist. Calculating them directly is hard; while  $R(4) = 18$ , the value of  $R(5)$  is unknown (although it is between 43 and 49; see [25] for a dynamic survey of the known small Ramsey numbers). So we must be content with lower and upper bounds. An inductive argument gives that  $R(n) \leq \binom{2n-2}{n-1}$ . In an early application of the probabilistic method, Erdős [11] proved the lower bound:

$$(1 + o(1)) \frac{1}{e\sqrt{2}} n 2^{n/2} \leq R(n),$$

which has not been substantially improved to this day (Spencer [32] improved the constant  $\frac{1}{\sqrt{2}}$  to  $\sqrt{2}$ ). The best known upper bound for  $R(n)$  was given by Thomason [36]:

$$R(n) \leq n^{-1/2+c/\sqrt{\log n}} \binom{2n-2}{n-1}.$$

Erdős 1947 posed the following asymptotic conjecture, and it remains one of the major topics in Ramsey numbers.

**Erdős' Ramsey Number Conjecture:**  $\lim_{n \rightarrow \infty} R(n)^{1/n}$  exists.

Solve this conjecture and you will be awarded \$100. There are, however, much easier ways to

make this money! From the bounds stated above, if the limit exists, then it is between  $\sqrt{2}$  and 4 (finding the value of the limit is worth \$250). Hence, we can think of the conjecture as way to better understand which bound for  $R(n)$  stated above is more accurate. For more on Erdős and his questions on Ramsey numbers, see [8].

**Saving grace.** Graph labelling problems, where vertices or edges are labelled by numbers subject to some constraints, are often tough in general. A graph with  $n$  vertices is *graceful* if there is a numbering  $1, 2, \dots, n$  of the vertices such that  $n - 1$  distinct differences appear on the edges. Graceful graphs have received ample attention in the literature. See Figure 3 for an example.

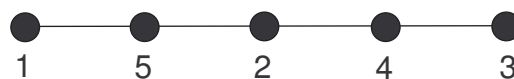


FIGURE 3. The path  $P_5$  is graceful. Show that all the paths  $P_n$  are graceful.

Ringel 1964 [27] introduced the conjecture, which is now sometimes called the *Ringel-Kotzig conjecture* (since, if the conjecture were true, it would imply conjectures of both authors on certain decompositions of complete graphs).

**Graceful Tree Conjecture:** Every tree is graceful.

Over 200 papers have been written on proving special cases of this conjecture, and a bewildering number of variants on graceful labellings have been proposed and studied. See the dynamic survey of Gallian [13] for further background and references on graceful (and other) labellings. Kotzig labelled the collective work on proving the conjecture a “disease”. We note that graceful labellings of graphs were introduced by Rosa as  $\beta$ -labellings, and renamed “graceful” by Golumb. A few of the classes of trees where we know the conjecture holds include: caterpillars (that is, a tree where the removal of its end-vertices leaves a path), trees with at most four end-vertices, trees with diameter at most 5, and trees with at most 35 vertices.

Much of the research on the conjecture tries to settle it in the affirmative. One class of trees where the conjecture remains open are lobsters (a tree where the removal of the end-vertices leaves a caterpillar).

**No minors allowed.** Coloring has both fascinated and perplexed Graph Theorists since the early days of the field to the present. The *chromatic number* of  $G$ , written  $\chi(G)$ , is the minimum integer  $n$  with the property that  $V(G)$  may be partitioned into  $n$  many independent sets; that is, the minimum  $n$  so that  $G$  is  $n$ -colorable. The most famous theorem proved so far in Graph Theory is the Four Color theorem [1], which states that every planar graph is 4-colorable. All the proofs of this fact that are known are computer-assisted.

A graph is a *minor* of  $G$  if it results by repeatedly performing one of the following operations: i) deleting a vertex, ii) deleting an edge, or iii) contracting an edge (that is, shrinking an edge to a vertex and preserving adjacencies and non-adjacencies with vertices outside the edge). A beautiful result of Kuratowski [23] states that a graph is planar if and only if it contains no  $K_5$  or  $K_{3,3}$  minor. The reader can show that the Petersen graph (see Figure 8) has a  $K_5$  minor, and so is not planar.

Hadwiger's conjecture, dating back to 1943 [15] relates graph coloring to minors.

**Hadwiger's Conjecture:** For  $m \geq 2$ , a graph with no  $K_m$  minor is  $(m - 1)$ -colorable.

Hadwiger's conjecture is open for all  $m \geq 7$ . The most startling case for small  $m$  is  $m = 5$ , which was shown by Wagner [40] to reduce to the Four Color theorem. Hence, Hadwiger's conjecture may be viewed as a broad generalization of that theorem. The case  $m = 6$  was settled by Robertson, Seymour, and Thomas [28] by showing that a minimal counter-example to the conjecture is planar after the removal of one vertex (and so also reduces also to the Four Color Theorem).

The cases  $m = 2$ , and 3 are elementary (for example, a graph with no  $K_2$  minor has no edges, and with no  $K_3$  minor is a forest). Dirac [10] and Hadwiger [15] proved the case  $m = 4$ , by showing

that graphs with no  $K_4$  minor have a vertex of degree at most 2, and so can be 3-colored using a greedy algorithm. Although  $m = 7$  is open, in 2005, Kawarabayashi and Toft [20] proved that any 7-chromatic graph has  $K_7$  or  $K_{4,4}$  as a minor.

**X marks the spot.** As with Hadwiger's conjecture, our next conjecture also deals with coloring, but adds graphs products to the mix. All the references in this section can be found in three surveys on the conjecture: [29, 35, 45].

A graph product makes new graphs from old. We consider one of the most well known products: the *categorical product*. For graphs  $G$  and  $H$ , define  $G \times H$  to have vertices  $V(G) \times V(H)$ , with  $(a, b)$  adjacent to  $(c, d)$  if  $a$  is joined to  $c$  in  $G$ , and  $b$  is joined to  $d$  in  $H$ . See Figure 4 which justifies the notation for this product.

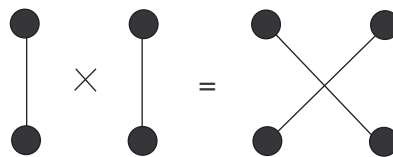


FIGURE 4. The graph  $K_2 \times K_2$ .

Hedetniemi's conjecture gives a simple formula for the chromatic number of the categorical product, and was posed by him in 1966 [17] while he was a graduate student.

**Hedetniemi's Conjecture:** For graphs  $G$  and  $H$ ,

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

We note that the conjecture was stated independently by Burr, Erdős and Lovász in 1976. Most experts think the conjecture is true. For starters,  $G \times H$  may be visualized as replacing each vertex  $v$  of  $G$  by a copy of the vertices of  $H$ . Label these vertices as  $(v, h)$ . Then add the edges  $(v, h)(w, j)$  just if  $v$  is adjacent to  $w$  and  $h$  is adjacent to  $j$ . See Figure 5. Now take a proper coloring of  $G$ . For each vertex  $v$  of  $G$ , color all vertices  $(v, h)$  with the same color as  $v$ . Since  $(v, h)$  and  $(v, h')$  are not adjacent, then this is also a proper coloring of  $G \times H$ . Hence,  $\chi(G \times H) \leq \chi(G)$ . The same construction, but considering vertices

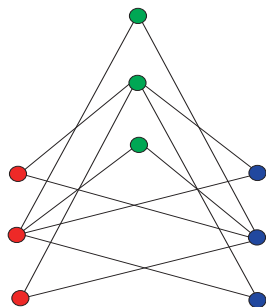


FIGURE 5. The graph  $P_3 \times K_3$  with the independent sets different colors.

of  $H$ , gives  $\chi(G \times H) \leq \chi(H)$ . The reader should try this coloring with the graph in Figure 5.

Since this conjecture has received a lot of attention over the past 45 years, if there was a counter-example surely it would have been found by now! The conjecture has a restatement which is often used. For a positive integer  $n$ , define

$H(n)$ : If  $\chi(G \times H) = n$ , then either  $\chi(G) = n$  or  $\chi(H) = n$ .

Hedetniemi's conjecture is equivalent to  $H(n)$  being true for all  $n \geq 1$ , and so allows for an incremental approach. Indeed, it is not too difficult to show that  $H(1)$  and  $H(2)$  are true. El-Zahar and Sauer proved in 1985 that  $H(3)$  is true, but nothing is known about  $H(n)$  for  $n > 3$ .

Burr, Erdős, and Lovász in their 1976 paper showed that if  $G$  is a graph which has every vertex in a  $K_n$  and  $H$  is a connected graph with  $\chi(G \times H) = n$ , then  $\min\{\chi(G), \chi(H)\} = n$ . This is not too surprising since the presence of  $K_n$  is a trivial reason why the chromatic number is at least  $n$ . Proofs of the conjecture then, or the search for a counter-example, have to take into account graphs that have large chromatic number and small complete subgraphs.

The strangest result arising out of the work on the conjecture has to do with a special case. Define the function

$$g(n) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = n\}.$$

It is known that  $g(1) = 1$ ,  $g(2) = 2$ ,  $g(3) = 3$ , and  $g(4) = 4$ . However, several authors discovered the striking fact that either  $g$  is unbounded or  $g(n) \leq 9$ , for all  $n$ .

We mention in passing (and without explanation of the jargon!) that Hedetniemi's conjecture is equivalent to the meet-irreducibility of the complete graphs in the lattice of cores. For this reason, there is ample interest in the conjecture not only among experts in graph coloring, but those working on graph homomorphisms.

**The long arm of the law.** Many of us played games like Cops and Robbers (or other pursuit games) as children, and our next conjecture considers such a game played on graphs. In the graph game of *Cops and Robbers* there are two players, a set of cops and a robber, who move from vertex-to-vertex along edges in the graph or can pass. The game is played with alternate moves of the players, with the cops going first. The cops win if eventually they capture or land on the vertex with the robber; the robber wins if he can indefinitely evade capture. The game is perfect information, in the sense that both players can see and remember each others moves. Placing a cop on each vertex gives an easy win for the cops. The minimum number of cops needed to win the game is the *cop number* of a graph. The readers should verify that the cop number of the snark  $J_5$  in Figure 2 is 3.

As the cop number of a disconnected graph is the sum of the cop number of its components, a more interesting situation arises if we consider only connected graphs. For functions  $f$  and  $g$  on positive integers taking positive real numbered values, we set  $f = O(g)$  if there is a constant  $d$ , such that for large enough  $n$ ,  $f(n) \leq dg(n)$ .

**Meyniel's Conjecture:** If  $G$  is a connected graph, then

$$c(G) = O(\sqrt{|V(G)|}).$$

So Meyniel's conjecture tells us that about  $\sqrt{n}$  many cops are sufficient to capture the robber in a connected graph of order  $n$  (and there are examples of graphs needing this many cops). Interestingly, Aigner and Fromme 1984 proved that the cop number of a planar graph is at most three.

Meyniel's conjecture may be one of the more lesser known major unsolved conjectures in Graph Theory, but it has received a fair bit of recent

attention. For further background on the conjecture, see Chapter 3 of the book [3]. Meyniel's conjecture was communicated by Frankl [12], who could only prove that

$$c(G) = O\left(n \frac{\log \log n}{\log n}\right).$$

The best known bound for general graphs we have is the following, recently discovered by three independent sets of researchers:

$$c(G) = O\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}\right).$$

Even to prove that  $c(G) = O(n^{1-\varepsilon})$ , for some  $\varepsilon > 0$  is open! The conjecture was settled for bipartite graphs with diameter three, and Andrae proved it is true in graph classes formed by omitting a fixed graph as a minor (in fact, the cop number is bounded by a constant in such graphs).

**House of cards.** The Reconstruction conjecture appears to be notoriously difficult, and suggests how much more there is to learn about graphs. The *deck* of a graph  $G$  is the multiset consisting of all subgraphs of  $G$  formed by deleting a vertex. Each such point-deleted subgraph is a *card*. See Figure 6 for an example.

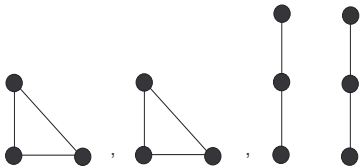


FIGURE 6. Which graph has this deck?

The conjecture was posed independently by both Kelly 1957 [21] and Ulam 1960 [38].

**Reconstruction Conjecture:** If two graphs with at least three vertices have the same deck, then they are isomorphic.

It is easy to see that a path of length two and its complement have the same deck, hence, the modest requirement in the conjecture on the order. Given a deck, we immediately know the order of  $G$ , and some thought derives the number

of edges and degrees of all the vertices. For references to results on the conjecture, see the survey [4]. Kelly 1957 proved that regular graphs are reconstructible from their deck. McKay 1997 shows that the conjecture is true for all graphs with at most 11 vertices. The conjecture also holds for trees, for disconnected graphs, and outerplanar graphs. Bollobás proved that with probability tending to 1 as  $n$  tends to infinity, there exist three cards which determine the graph. Surprisingly, the conjecture remains open for planar graphs.

**Go with the flow.** A fundamental application of Graph Theory is to networks in the real world. We may view the edges as a series of pipes transporting some liquid (or electric current, or information, and so on) between nodes. Usually edges have a maximum capacity for carrying materials, and what enters into a node must equal what must come out. Further, these *flows*, as they are called, usually move in one direction, so some orientation must be assigned to the edges of the network. Flows have deep connections to the Four Color Theorem, and Tutte's conjecture on flows extends these connections beyond the context of planar graphs.

To be more precise an *integer flow* on a graph is a pair consisting of an orientation of the graph, and an assignment of integer weights to the edges such that for each vertex, the total weight on exiting edges equals the total weight on entering edges. It is a  $k$ -*flow* if all weights have absolute value less than  $k$ , and it is *nowhere-zero* if weight 0 is never used. See Figure 7.

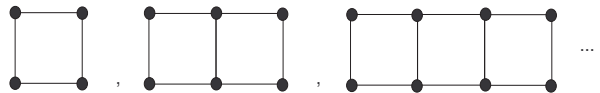


FIGURE 7. Find a nowhere-zero 4-flow for this sequence of graphs. Do CDCs help?

Nowhere-zero  $k$ -flows were introduced by Tutte [37] as a generalization of face coloring problems in planar graphs (where we color the faces so no adjacent faces receive the same color). The famous Four Color Theorem is equivalent to saying that every planar bridgeless graph has a nowhere-zero 4-flow. Unfortunately, this result cannot be

extended to arbitrary bridgeless graphs since the Petersen graph has no nowhere-zero 4-flow. See Figure 8.

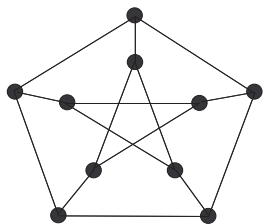


FIGURE 8. The Petersen graph.

Tutte 1954 therefore, considered 5-flows instead, and conjectured the following in [37].

**Tutte’s 5-Flow Conjecture:** Every bridgeless graph has a nowhere-zero 5-flow.

The conjecture holds for planar graphs, and follows from the duality of flows and coloring, and the Five Color theorem (every planar graph is 5-colorable). We may therefore view the 5-flow conjecture as a generalization of the Five Color theorem to graphs which are not planar.

Jaeger [18] proved that every bridgeless graph has a nowhere-zero 8-flow. Seymour [31] improved upon this result by showing that bridgeless graphs have nowhere-zero 6-flows. Celmins [6] proved that a smallest counter-example to the conjecture must be a cyclically 5-edge-connected snark with girth at least 7 (see also the Cycle Double Cover conjecture). The conjecture can be reduced to the 3-regular case, and Steinberg [33] proved the conjecture for graphs which can be drawn in the projective plane without edge crossings.

**You are so square.** We state another conjecture about products, this time related to domination. No, not the kinds with whips and chains! For graphs  $G$  and  $H$ , define the *Cartesian product* of  $G$  and  $H$ , written  $G \square H$ , to have vertices  $V(G) \times V(H)$ , with  $(a, b)$  adjacent to  $(c, d)$  if  $a = c$  and  $b$  is joined to  $d$  in  $H$ , or  $b = d$  and  $a$  is joined to  $c$  in  $G$ . See Figure 9 which justifies the notation for this product.

In a graph  $G$ , a set  $S$  of vertices is a *dominating set* if every vertex not in  $S$  has a neighbor in  $S$ . The *domination number* of  $G$ , written  $\gamma(G)$ ,

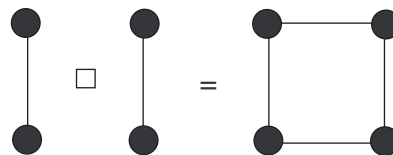


FIGURE 9. The graph  $K_2 \square K_2$ .

is the minimum order of a dominating set. For example, see Figure 10.

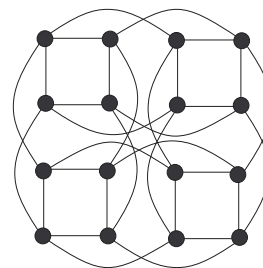


FIGURE 10. Find a dominating set of order 4 in  $C_4 \square C_4$ .

The following was proposed by Vizing 1968 [39].

**Vizing’s Conjecture:** For graphs  $G$  and  $H$ ,

$$\gamma(G \square H) \geq \gamma(G)\gamma(H).$$

This is a conjecture everyone thinks is true. All the references below appear in [5].

For graphs which have *efficient dominating sets* (where most vertices are adjacent to at most one vertex of the dominating set, such as in complete graphs) it is easy to show that the conjecture is true. What about graphs with *inefficient dominating sets* such as  $C_4$ ? In this case,  $\gamma(C_4 \square C_4) = \gamma(C_4)\gamma(C_4) = 4$ . Check this in Figure 10.

An important theorem from 1979 was not appreciated (or apparently not even known) for the next 16 years. During some collaboration of Rall and the second author of this article, the **Math Review MR0544028** was discovered. From that brief description, in a couple of days Hartnell and Rall were able to piece together the whole of the original paper (which was in Russian)<sup>2</sup>.

<sup>2</sup>No translation was involved.

The main theorem of [2] reduced much of the work from 1968 through to 1996, to a corollary of their main theorem!

From [2] a graph  $G$  is called *decomposable* if  $\gamma(G) = k$  and the vertex set of  $G$  is contained in the union of  $k$  complete graphs. Hartnell and Rall 1995 and Brešar and Rall 2009 used decompositions with subgraphs other than complete graphs to extend the family of graphs for which Vizing's conjecture is known to be true.

As with several of the conjectures considered so far, many experts think that if the conjecture were false then a minimal counter-example would have been found. But since a proof has not been found, where would one look for a counter-example? For example, a minimal counter-example to Vizing's conjecture must have domination number larger than 3; adding an edge between two non-adjacent vertices decreases the domination number; and every vertex belongs to a minimum dominating set.

**Don't get cross.** Our next and final conjecture has its origins in a 1940's labor camp in Budapest. The famous Mathematician Turán was imprisoned there, watching trucks to move bricks along rails from kilns to storage areas. Every once in awhile, the trucks would cross each other's path and the bricks would come crashing down. No doubt as a kind of liberation from the monotony, Turán began thinking about minimizing the crossings of the trucks, assuming the general situation that there were  $m$  kilns and  $n$  trucks.

We may formalize Turán's problem in the following way. The *crossing number* of  $G$ , written  $cr(G)$ , is the minimum number of pairwise crossings of edges in a drawing of  $G$  in the plane. Some readers may recall the *Three Utilities Problem*, which reduces to showing that  $K_{3,3}$  has crossing number 1.

Crossing number problems tend to be hard to calculate exactly, in no small part from the fact that determining the crossing number of a graph is an NP-complete problem. The following conjecture is named after Zarankiewicz who published a flawed proof of it [43], but it is also called *Turán's Brick Factory conjecture* (see [14] for a survey of the history of the conjecture).

### Zarankiewicz's Conjecture:

$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

The best exact result on the conjecture was proved by Kleitman 1970 [22] who confirmed it for  $n \leq 6$ . Kleitman also proved that the smallest counter-example must occur for  $m$  and  $n$  odd. Woodall [42] discovered the crossing numbers  $cr(K_{7,7}) = 81$  and  $cr(K_{7,9}) = 144$ . Hence, the smallest unsolved cases are for  $K_{7,11}$  and  $K_{9,9}$ . As  $\lfloor \frac{4}{2} \rfloor \lfloor \frac{3}{2} \rfloor \lfloor \frac{5}{2} \rfloor \lfloor \frac{4}{2} \rfloor = 8$ , the drawing in Figure 11 satisfies the conjectured bound.

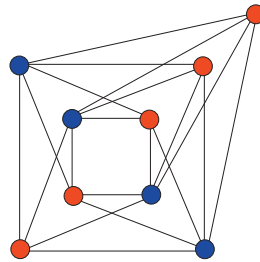


FIGURE 11. A drawing of  $K_{4,5}$  with 8 crossings, which is the minimum number possible.

A recent result in [7] states that if for a fixed  $m$ , the conjecture holds for all values  $n$  smaller than some constant depending on  $m$ , then the conjecture holds for all  $n$ . Hence, for each  $m$  there is an algorithm which verifies the conjecture for all  $n$  or gives a counter-example.

### EPILOGUE

Now it is your turn to finish this survey and solve (or partially solve) one or more of these conjectures in Graph Theory. And when you are done with those, we have a few others that might keep you busy such as Barnette's conjecture, the Berge-Fulkerson conjecture, the Erdős-Sós conjecture, the Middle Levels conjecture, or Sheehan's conjecture.

We close with our version of another Hamming quote [16], which nicely summarizes where we would like to leave off.

“Go forth, then, and doodle great work!”



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DEPARTMENT OF MATHEMATICS, RYERSON UNIVERSITY,  
TORONTO, ON, CANADA, M5B 2K3  
*E-mail address:* [abonato@ryerson.ca](mailto:abonato@ryerson.ca)

DEPARTMENT OF MATHEMATICS AND STATISTICS, DAL-  
HOUSIE UNIVERSITY, HALIFAX, NS, CANADA, B3H 3J5  
*E-mail address:* [rjn@mathstat.dal.ca](mailto:rjn@mathstat.dal.ca)