

A combinatorial approach of representations of symmetric groups : application to Kerov's polynomials

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Plan

- 1 Introduction
 - Free cumulants and Kerov's polynomials
 - Combinatorial formula for characters

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- 2 Combinatorics of Kerov's polynomials
 - Decomposition of maps in forests
 - Intervals and cumulants

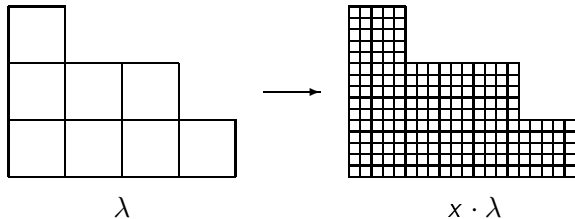
Free cumulants

Young diagram $\lambda \rightarrow$ Transition measure
 \rightarrow Free cumulants $(R_i(\lambda))_{i \geq 2}$

Proposition (Biane)

Homogeneous $R_i(x \cdot \lambda) = x^i R_i(\lambda)$

Asymptotics $\chi^{x \cdot \lambda}(1 \dots k) \sim_{x \rightarrow \infty} R_{k+1}(\lambda) |x \cdot \lambda|^{-k/2}$



Kerov's polynomials

If $\mu \in S(k) \subset S(n)$ and $\lambda \vdash n$, let

$$\Sigma_{\mu}^{\lambda} = n(n-1) \dots (n-k+1) \frac{\chi^{\lambda}(\mu)}{\chi^{\lambda}(Id_n)}$$

where χ^{λ} is the character of the irreducible representation indexed by λ .

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Theorem (Existence of Kerov's polynomials, Kerov, Biane)

Let $k \geq 1$, there exists a **universal** polynomial K_k such that :

$$\Sigma_{(1\dots k)}^{\lambda} = K_k(R_2(\lambda), \dots, R_{k+1}(\lambda))$$

It does not depend on the diagram λ !

Description of the coefficients

Asymptotics property of free cumulants implies :

Proposition

$$K_k = R_{k+1} + \text{lower degree terms}$$

Moreover :

- It has integer coefficients.

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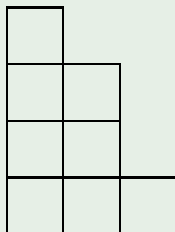
- It has integer coefficients.
- We will sketch the proof of their positivity thanks to a combinatorial description using permutations in $S(k)$.

Irreducible representations of symmetric groups

- They are indexed by partitions $\lambda \vdash n$, or equivalently by Young diagrams.

Example

- $\lambda_1 = 3; \lambda_2 = \lambda_3 = 2;$
 $\lambda_4 = 1; \lambda_5 = \dots = 0,$

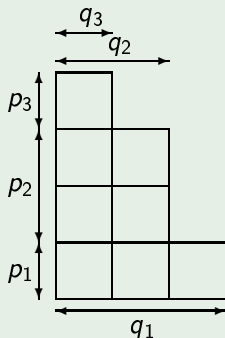


Irreducible representations of symmetric groups

- They are indexed by partitions $\lambda \vdash n$, or equivalently by Young diagrams.
- Other notation : $\lambda = \mathbf{p} \times \mathbf{q}$.

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- $\lambda_1 = 3; \lambda_2 = \lambda_3 = 2;$
 $\lambda_4 = 1; \lambda_5 = \dots = 0,$
- $\lambda = (1, 2, 1) \times (3, 2, 1)$



Map of a pair of permutations

pair of permutations \mapsto bicolored edge-labeled map

Example

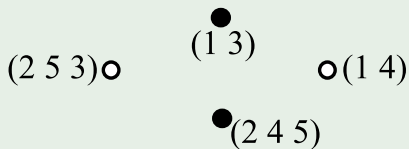
$$\tau = (14)(325), \bar{\tau} = (13)(254)$$

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white vertices \leftrightarrow cycles of τ

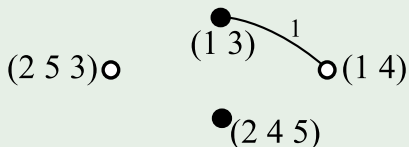
black vertices \leftrightarrow cycles of $\bar{\tau}$

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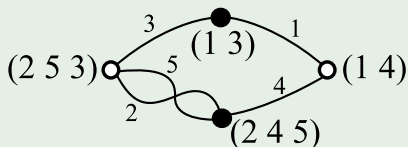
The edge labeled 1 links the two vertices corresponding to cycles containing 1.

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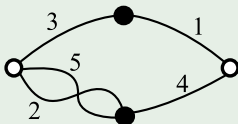
Same thing for the integers between 2 and k . The cyclic order at each vertex is given by the corresponding cycle.

Map of a pair of permutations

pair of permutations \mapsto bicolored edge-labeled map

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$$\tau = (14)(325), \bar{\tau} = (13)(254)$$



We can recover the pair of permutations from the map.

Power series associated to a bicolored map

A colouring of the white vertices of M is :

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Let define the power series in indeterminates \mathbf{p} and \mathbf{q} :

$$N(M) = \sum_{\varphi \text{ colouring of the white vertices}} \left(\prod_{w \in V_w(M)} p_{\varphi(w)} \prod_{b \in V_b(M)} q_{\psi(b)} \right)$$

Combinatorial formulas for character values and cumulants

Theorem (Stanley, Féray, Śniady)

With these notations, the character values is given by :

$$\Sigma_{\mu}^{\mathbf{p} \times \mathbf{q}} = \sum_{\substack{\tau, \bar{\tau} \in S(k) \\ \tau \cdot \bar{\tau} = \mu}} (-1)^{|C(\bar{\tau})|} N(M^{\tau, \bar{\tau}})(\mathbf{p}, \mathbf{q})$$

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From asymptotic property of cumulants, we have :

$$R_{k+1}(\mathbf{p} \times \mathbf{q}) = \sum_{\substack{\tau, \bar{\tau} \in S(k) \\ \tau \cdot \bar{\tau} = (1 \dots k) \\ |C(\tau)| + |C(\bar{\tau})| = k+1}} (-1)^{|C(\bar{\tau})|} N(M^{\tau, \bar{\tau}})(\mathbf{p}, \mathbf{q})$$

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The factorisation appearing in the second equation are in bijection with $NC(k)$ (**non-crossing partitions** of $[k]$). They are exactly the pair of permutations whose map is a **planar tree**.

Idea

Recall that, as power series in \mathbf{p} and \mathbf{q} :

$$\Sigma_k = K_k(R_2, \dots, R_{k+1})$$

Replace R_i by their combinatoric expression and expand, we obtain something of the kind:

$$\Sigma_k = \sum \pm \text{series associated to forests}$$

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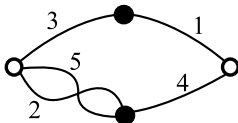
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We will write each summand under the form :

$$N^{\tau, \bar{\tau}} = \sum \pm \text{series associated to forests}$$

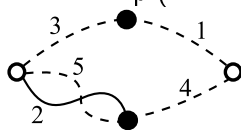
Elementary transformation

Description on our favorite example



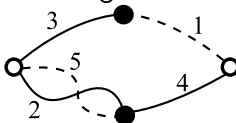
Elementary transformation

We choose a loop (here dotted)



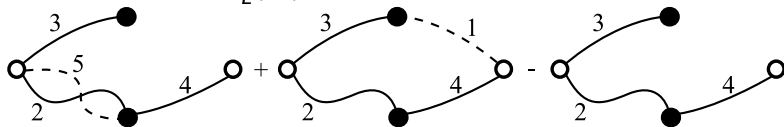
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Call erasable one edge over two of this loop



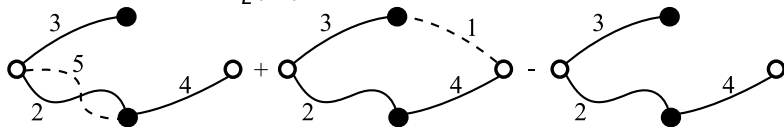
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Let $T_{\vec{L}}(M)$ be the formal expression :



Elementary transformation

Let $T_{\bar{L}}(M)$ be the formal expression :

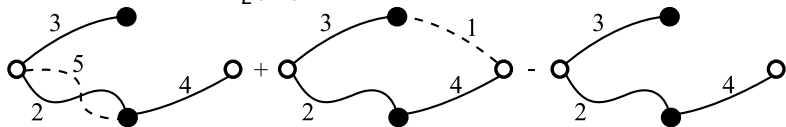


Proposition

$$N(T_{\bar{L}}(M)) = N(M)$$

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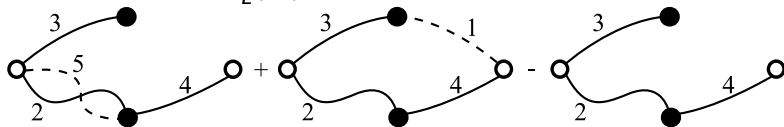
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Some choices of loops and erasable edges \longrightarrow Some way to write $N(M)$ as $\sum N(\text{forest})$

Elementary transformation

Let $T_{\bar{L}}(M)$ be the formal expression :



Proposition

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Different choices of
loops and erasable edges \longrightarrow

Maybe different ways to write
 $N(M)$ as $\sum N(\text{forest})$

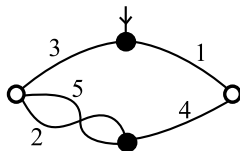
Restriction of choices

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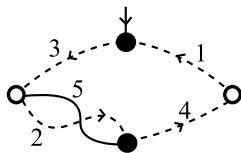
- 1 Add an external half-edge of black extremity to connected components which do not have one (after the edge of smallest label) and draw it on top on the map.



Restriction of choices

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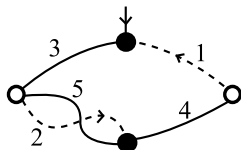
- ① Add an external half-edge to connected components which do not have one.
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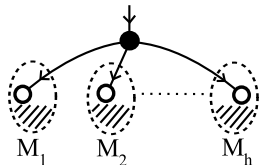
- 1 Add an external half-edge to connected components which do not have one.
- 2 In any connected component, choose an admissible oriented loop.
- 3 Select the edges which are oriented **from** their **white** extremity **to** their **black** extremity in \vec{L} .



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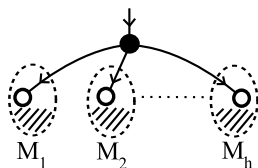
- ① Add an external half-edge to connected components which do not have one.
- ② In any connected component, choose an admissible oriented loop : *if there is no loop going through \star , take an admissible oriented loop of one of the M_i .*
- ③ Select the edges which are oriented from white to black in \vec{L} .



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If we iterate transformations with such choices of erasable edges, we obtain an algebraic sum of forests, whose associated polynomial is equal to $N(M)$.

Invariance of the result

There is still some choices to do, but :

Proposition

*If we follow the rules above, we always obtain the **same** sum of forests which we will denote $D(M)$.*

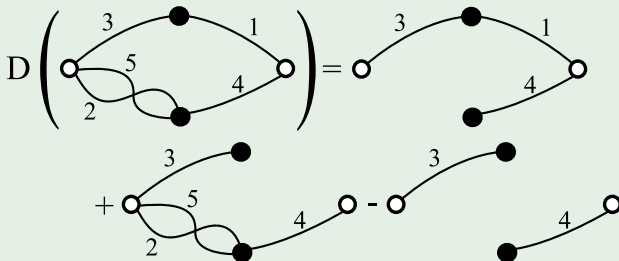
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Example



Properties of our decomposition

As we iterate N -invariant transformations :

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Thanks to our choice of loops, one has :

Proposition

The sign of the coefficient of M' in $(-1)^{\# \text{ c.c. of } M} D(M)$ is $(-1)^{\# \text{ c.c. of } M'}$

Back to Kerov's polynomials

Recall :

$$\Sigma_k = \sum_{\substack{\tau, \bar{\tau} \in \mathcal{S}(k) \\ \tau \cdot \bar{\tau} = (1 \dots k)}} (-1)^{|C(\bar{\tau})|} N(M^{\tau, \bar{\tau}})$$

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Replace each term $N(M)$ by $N(D(M))$, we have something like :

$$\Sigma_k = \sum \pm N(\text{forests})$$

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To understand Kerov's polynomial we have to put terms together and make appear free cumulants.

Order on the symmetric group

Definition

$$|\sigma| \stackrel{\text{def}}{=} \min \left\{ h \text{ s.t. } \exists \text{ transpositions } \tau_1, \dots, \tau_h \right. \\ \left. \text{with } \sigma = \tau_1 \cdot \tau_2 \cdot \dots \cdot \tau_h \right\}$$

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$$\sigma \leq \sigma' \stackrel{\text{def}}{\Leftrightarrow} |\sigma'| = |\sigma| + |\sigma'^{-1}\sigma|$$

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Proposition

If $\sigma \leq \sigma'$ and $\sigma^{-1}\sigma' = c_1 \cdot \dots \cdot c_t$ (decomposition in cycles of disjoint supports),

$$[\sigma; \sigma'] \simeq [id_k; \sigma^{-1}\sigma'] \simeq \prod [e; c_i] \simeq \prod NC(|c_i| + 1)$$

Intervals and cumulants

Let ϕ be an isomorphism :

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If $\tau \in [\sigma; \sigma']$, denote :

$$N_\phi(\tau) = N(M^{\phi(\tau)})$$

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$$\sum_{\tau \in \mathcal{S}(k)} N_\phi(\tau) = \prod R_{|c_i|+2}$$

If we choose well ϕ , $N_\phi(\tau)$ appears in $N(D(M))$. So intervals are a good tool to make appear products of free cumulants in Σ_k .

Main theorem

With an appropriate family of isomorphisms ϕ , we prove :

Theorem

If $\mu \in S(k)$, let

$$\Sigma'_\mu \stackrel{\text{def}}{=} \sum_{\substack{\tau, \bar{\tau} \in S(k) \\ \tau \cdot \bar{\tau} = \mu \\ \langle \tau, \bar{\tau} \rangle \text{ trans.}}} (-1)^{|C(\bar{\tau})| + |C(\mu)| - 1} N(M^{\tau, \bar{\tau}})$$

then there exists a polynomial with **non-negative** integer coefficients such that :

$$\Sigma'_\mu = K'_\mu(R_2, \dots, R_k)$$

The case $|C(\mu)| = 1$ is the result we claimed for classical Kerov's polynomial.

Computation of coefficients

Proposition

The coefficient of monomial $\prod_{i=1}^t R_{j_i+1}$ in K'_μ is the coefficient of the disjoint union of t trees with one black and respectively j_1, \dots, j_t white vertices in

$$\sum_{\substack{\tau, \bar{\tau} \in S(k) \\ \tau \bar{\tau} = \sigma, \langle \tau, \bar{\tau} \rangle \text{ trans.} \\ |C(\bar{\tau})| = t}} D(M^{\tau, \bar{\tau}}).$$

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Corollary

The coefficient of the linear monomial R_d in K_k is the number of cycles $\sigma \in S(k)$ such that $\sigma^{-1}(12 \dots k)$ has $d - 1$ cycles.

Computation of coefficients

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The coefficient of the linear monomial R_d in K_k is the number of cycles $\sigma \in S(k)$ such that $\sigma^{-1}(12 \dots k)$ has $d - 1$ cycles.

Proof.

If $|C(\bar{\tau})| = 1$, the map $M = M^{\tau, \bar{\tau}}$ has one black vertex, so $D(M)$ is a tree with one black vertex and as many white vertices as M . \square

End

Thank you!