

A *tournament* is an orientation of a complete graph, i.e., an oriented graph where for each pair of vertices u and v there is either the arc uv or the arc vu .

Exercise 1

Prove that every tournament contains a Hamilton path (an oriented path passing through each vertex exactly once). What is the structure of tournaments having only one Hamilton path?

Hint : Try by induction on the number of vertices.

Exercise 2

Prove that for each n , there exists a tournament on n vertices having at least $\frac{n!}{2^{n-1}}$ Hamilton paths.

Hint : Calculate the expected number of Hamilton paths (defined as the sum of the indicators for all possible Hamilton paths) in a random tournament.

Exercise 3

Let $d \geq 1$ be a real number and let G be a graph with n vertices and $nd/2$ edges (i.e. a graph with bounded average degree). Prove that $\alpha(G) \geq n/2d$ (i.e. it has a large independent set).

Hint : Consider a random subset of vertices S (attention, the graph is fixed, we can only choose randomly for each vertex whether we take it into S or not). Instead of proving that S itself is a stable set (probability of which is very small), prove that the expected number of edges in S (compared to its number of vertices) is small enough so that we can remove some vertices from S (at most one per edge) in a way that what remains is a stable set.

Exercise 4

A tournament satisfies *Property P_k* if for every set S of k vertices, there exists a vertex dominating all the vertices of S (i.e., for some vertex $u \in S$, we have $uv \in E(G)$ for every vertex $v \in S \setminus u$).

Show that if $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there exists a tournament on n vertices satisfying Property P_k .

Deduce that there exists a tournament on (at least) $\lceil 4k^2 2^k \rceil$ vertices which satisfies Property P_k .