

SOME RECURSIVELY UNSOLVABLE PROBLEMS
RELATING TO ISOLATED CUTPOINTS IN
PROBABILISTIC AUTOMATA

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1. Introduction

In the present work we examine the decidability of some questions concerning the existence of cutpoints isolated with respect to a given probabilistic automaton (PA). As is well known, the question of the existence of an isolated cutpoint, besides an intrinsic interest, has a considerable importance in connection with Rabin's theorem [1], which ensures that a cutpoint event $T(\mathcal{A}, \lambda)$ is regular if λ is an isolated cutpoint for the PA \mathcal{A} .

Rabin [2] mentions the following two open problems: for every given PA \mathcal{A} with rational transition probabilities,

- 1) decide whether every given (rational) number λ is isolated with respect to \mathcal{A} ;
- 2) decide whether \mathcal{A} has any isolated cutpoints.

These problems have been studied from the point of view of the constructive theory of PA ([3], [4]), coming to conclusion that within the frame of such theory it is undecidable whether every given PA has an isolated cutpoint. On the other hand, Metra and Smilgais [5] discovered an algorithm which decides on the existence of a cutpoint isolated with respect to a PA having only two states and rational transition probabilities.

The results of the constructive theory cannot be carried over directly to the "classical" theory: however, the first problem was solved (in a negative sense) in [6], making use of some ideas taken from [7] and [8]. Here we want to summarize and extend in part the results in [6], showing that the first problem is semidecidable if referred to a certain class of events, and then proving the unsolvability of the second problem.

2. Definitions

Let $\Sigma \equiv \{\sigma_1, \sigma_2, \dots, \sigma_m\}$ be a finite alphabet: we shall denote by $\langle \Sigma^*, \cdot, \Lambda \rangle$ the free monoid generated by Σ , in which Σ^* is the set of all strings of symbols in Σ , including the empty string Λ , and \cdot is the operation of concatenation.

Given a string x , we shall denote by $l(x)$ the length of x , that is the number of symbols in x ; $l(\Lambda) = 0$. $x \circ y$ will indicate the longest suffix common to x and y , i. e. $x \circ y = z$ iff $x = uz$, $y = vz$ and $x = u'z'$, $y = v'z'$ implies $l(z') \leq l(z)$. If $x = x_1 x_2 \dots x_m$ with $x_i \in \Sigma$, $1 \leq i \leq m$, then $\tilde{x} = x_m \dots x_2 x_1$. At last, given a set S , we shall denote by $\#(S)$ the number of elements in S . Some more specific definitions will follow.

Def. 1. A probabilistic automaton (PA) with n states is a triple

$\mathcal{A} = \langle \pi, A(\sigma_j), \eta \rangle$, where

- a) π is a stochastic vector $1 \times n$, i. e. $\pi = (\pi_1, \dots, \pi_n)$; $\sum_{k=1}^n \pi_k = 1$; $\pi_k \geq 0$;
- b) $A(\sigma_j)$ is, for every $\sigma_j \in \Sigma$, a stochastic matrix $n \times n$, that is every row in the matrix is a stochastic vector;
- c) η is a vector $n \times 1$ with components 0 or 1.

We shall indicate by η_T the transpose of η , i. e. the vector $1 \times n$.

From now on, we shall consider numbers and elements of matrices and vectors which all are rational numbers.

Def. 2. A probabilistic event (PE) generated by the PA \mathcal{A} is the function $p_\pi: \Sigma^* \rightarrow [0, 1]$ given by $p_\pi(x) = \pi A(x) \eta$, where $A(x)$ is recursively defined as follows: $A(\Lambda) = I$, the identity matrix;

$$A(x\sigma_j) = A(x)A(\sigma_j) \text{ for every } \sigma_j \in \Sigma;$$

and $[0, 1]$ denotes the interval of rational numbers r , $0 \leq r \leq 1$.

Def. 3. A rational number λ is said to be an isolated cutpoint with respect to \mathcal{A} iff there exists an $\varepsilon > 0$ such that for every $x \in \Sigma^*$

$$|p(x) - \lambda| \geq \varepsilon.$$

Def. 4. A PA is said to be quasidefinite iff for any $\varepsilon > 0$ there is a number $k(\varepsilon)$ such that for every $x \in \Sigma^*$ with $l(x) \geq k(\varepsilon)$ and any two initial distributions π and ρ , $|p_\pi(x) - p_\rho(x)| \leq \varepsilon$.

3. Unsolvability of the first problem

In this section we shall resume the results given in [6], referring to that work for details, the main difference being that here we use, instead of the longest prefix common to two strings, the longest suffix, in order to attain such results more directly. This introduces into proofs only minor changes, which substantially come down to reading strings from right to left.

Let Σ and K be two finite alphabets and consider two homomorphisms $\psi_1, \psi_2: \langle \Sigma^*, \cdot, \wedge \rangle \longrightarrow \langle K^*, \cdot, \wedge \rangle$. The well known Post's correspondence problem consists in deciding, for any two given homomorphisms, whether there exists $x \in \Sigma^*$ such that $\psi_1(x) = \psi_2(x)$. It is also well known that such a problem is recursively unsolvable.

The existence of a solution for a given instance of Post's correspondence problem obviously implies that $\#(\bigcup_{x \in \Sigma^*} \psi_1(x) \circ \psi_2(x)) = \infty$; we want to examine now the decidability of the following question: is the cardinality of the set $\bigcup_{x \in \Sigma^*} \psi_1(x) \circ \psi_2(x)$ finite or infinite for any two given homomorphisms ψ_1, ψ_2 ?

To prove the recursive unsolvability of this problem it is possible to proceed in a way similar to the one followed to prove the unsolvability of Post's correspondence problem (see for instance [10]), establishing a correspondence between the sequence of computation of every Turing machine with a given input, and the longest (for every $x \in \Sigma^*$) suffix $\psi_1(x) \circ \psi_2(x)$ common to the two homomorphisms associated with that particular Turing machine. If it were possible to determine whether

$\# \left(\bigcup_{x \in \Sigma^*} \psi_1(x) \circ \psi_2(x) \right)$ is finite or infinite, it would then be possible to decide whether every given Turing machine stops. Since the latter problem is notoriously unsolvable, it follows that our problem is unsolvable too.

Now, let $K \equiv \{0, 1, \dots, k-1\}$ be a finite set of symbols to be interpreted as digits, and let $r: K^* \rightarrow [0, 1]$ be given by $r(x_1 \dots x_m) = \sum_{j=1}^m x_j k^{j-1-m}$, where $x_j \in K, 1 \leq j \leq m$.

Any given homomorphism $\psi: \langle \Sigma^*, \cdot, \wedge \rangle \longrightarrow \langle K^*, \cdot, \wedge \rangle$ may then be interpreted as a function $\varphi \equiv r\psi: \Sigma^+ \rightarrow [0, 1]$.

Moreover, such a function is a PE: in fact the 2-state PA

$$A_{\varphi} \equiv \langle \pi \equiv (10) \rangle, \quad A(\sigma_j) \equiv \begin{pmatrix} 1-\varphi(\sigma_j) & \varphi(\sigma_j) \\ 1-\varphi(\sigma_j)-k(\sigma_j) & \varphi(\sigma_j)+k(\sigma_j) \end{pmatrix}, \quad \eta \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle, \quad (3.1)$$

where $k(\sigma_j) = k^{-1}(\varphi(\sigma_j))$, generates the probabilistic event $\varphi(x)$, as is easily verified.

Given two homomorphisms ψ_1 and ψ_2 , mapping Σ^* into $\{0, 1, \dots, k-2\}$ (note the exclusion of $k-1$), and denoted by φ_1 and φ_2 respectively the fractional k -adic representations $r\psi_1$ and $r\psi_2$, it is not difficult to prove that number 0 is isolated with respect to $\bigcup_{x \in \Sigma^*} (\varphi_1(x) - \varphi_2(x))$ iff $\# \left(\bigcup_{x \in \Sigma^*} \psi_1(x) \circ \psi_2(x) \right) < \infty$.

From the foregoing result we derive the two following theorems.

Th. 3.1. It is undecidable whether $1/2$ is isolated with respect to every given 4-state PA.

Proof. Let ψ_1 and ψ_2 be two homomorphisms and φ_1 and φ_2 their associated PEs, generated by 2-state automata according to (3.1): $1-\varphi_2$ is a PE too, as well as the convex combination $\frac{1}{2}\varphi_1 + \frac{1}{2}(1-\varphi_2) = \frac{1}{2} + \frac{1}{2}(\varphi_1 - \varphi_2) = p$, which can be generated by a 4-state PA. It is obvious that $1/2$ is isolated with respect to p iff 0 is isolated with respect to $\varphi_1 - \varphi_2$, which is undecidable, as we showed above. \square

Th. 3.2. Let λ be a rational number, $0 < \lambda < 1$: it is undecidable whether λ is isolated with respect to every given PA.

Proof. Let $0 < \lambda < 1/2$ and p be the PE defined in th. 3.1: $2\lambda p$ is a PE (which can be generated by a 5-state PA) and if it were decidable whether λ is isolated for $2\lambda p$, since $|2\lambda p - \lambda| = 2\lambda|p - 1/2|$ it would be decidable whether $1/2$ is isolated for p , contrary to the statement of th. 3.1. Moreover, if $1/2 < \lambda < 1$, $1 - 2(1-\lambda)p = q$ is a PE, and since $|q - \lambda| = 2(1-\lambda)|p - 1/2|$ a parallel argument holds. \square

As one can see, the question of the undecidability for the extreme values $\lambda=0$ and $\lambda=1$ remains open.

4. Partial decidability of the first problem for a certain class of PEs

Let $\langle \pi, A, \eta \rangle$ be a quasidefinite PA: $p_{\pi}(x) = \pi A(x)\eta$, then $p_{\pi}(zx) = \pi A(z)A(x)\eta = \pi(z)A(x)\eta$, if we write $\pi(z) = \pi A(z)$: and since $\pi(z)$ is a stochastic vector, for any $\varepsilon > 0$ there is a $k(\varepsilon)$ such that for every x with $1(x) \geq k(\varepsilon)$ and every $z \in \Sigma^*$

$$|p_{\pi(z)}(x) - p_{\pi}(x)| = |p_{\pi}(zx) - p_{\pi}(x)| \leq \varepsilon.$$

Informally, the PE generated by a quasidefinite PA is "almost insensitive" to what can be prefixed to a given string of sufficient length.

We note that the PA \mathcal{A}_φ of (3.1) is quasidefinite (see exercise 4, page 178 of [9]) and $\varphi(zx) = k^{-1(\psi(x))} \varphi(z) + \varphi(x)$, so that $|\varphi(zx) - \varphi(x)| < k^{-1(\psi(x))}$.

We can state the following theorem:

Th. 4.1. Given any PA \mathcal{A} for which a recursive nonincreasing function $k(\varepsilon)$ is known, such that for every $x, z \in \Sigma^*$ if $l(x) \geq k(\varepsilon)$ then $|p(zx) - p(x)| \leq \varepsilon$ (p being the PE generated by \mathcal{A}), the predicate " λ is isolated with respect to \mathcal{A} " is recursively enumerable for every rational λ , $0 < \lambda < 1$.

Proof. Firstly, the predicate is not recursive since the PA of th. 3.2 satisfies the conditions of the present theorem and it is undecidable whether λ is isolated with respect to it.

Then, for any fixed λ , let $\delta(n) = \min_{l(x) \leq n} |p(x) - \lambda|$: function $\delta(n)$ is obviously recursive, and suppose we compute it for increasing values of n , stopping if $\delta(n) = 0$ or $n \geq k(\delta(n)/2)$: in the former case of course λ is nonisolated, in the latter λ is isolated: in fact, if $l(y) \leq n$, then $|p(y) - \lambda| \geq \delta(n) > 0$, whereas if $l(y) > n$ we can write $y = zx$ with $l(x) = n$ and $|p(y) - \lambda| \geq ||p(y) - p(x)| - |p(x) - \lambda||$, but $|p(zx) - p(x)| \leq \delta(n)/2$, so that $|p(y) - \lambda| \geq \delta(n)/2 > 0$.

Therefore, if the computation terminates we can decide whether λ is isolated or not; on the other hand, if λ is isolated, then the computation will terminate: in fact there is an $\varepsilon > 0$ such that for every $x \in \Sigma^*$ $|p(x) - \lambda| \geq \varepsilon$, hence $\delta(n) \geq \varepsilon$, so that $n = k(\varepsilon/2)$ verifies $n \geq k(\delta/2)$ since k is nonincreasing. \square

5. Unsolvability of the second problem

We want now to prove that the problem of deciding for every given PA whether there exists an isolated outpoint is recursively unsolvable.

We followed a line of reasoning like this: given the PA

$$\mathcal{A}_t \equiv \langle \pi \equiv (1 \ 0), A(0) \equiv \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}, A(1) \equiv \begin{pmatrix} 1-t & t \\ 0 & 1 \end{pmatrix}, \eta \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle, \quad 0 \leq t \leq 1,$$

one can prove that it admits an isolated outpoint iff $t > 1/2$. Now, if t is a constructive real number, it is undecidable in general whether

$t > 1/2$, and this settles the question [3]. On the contrary, we want t to be rational and the possible unsolvability of the problem should not be due to the choice of a peculiar class of numbers, but to some features of probabilistic automata themselves.

The starting point is the following theorem:

Th. 5.1. Let $p: \{\Sigma \cup \{0\}\}^* \rightarrow [0,1]$ and $q: \{\Sigma \cup \{0\}\}^* \rightarrow [0,1]$ be two PES ($0 \notin \Sigma$). Then the event $\varphi: \{\Sigma \cup \{0, c\}\}^* \rightarrow [0,1]$, ($c \notin \Sigma$), defined as follows:

$$\left\{ \begin{array}{l} \varphi(cx_1cx_2c\dots cx_m c) = (1 \ 0) \prod_{k=1}^m \begin{pmatrix} 1-p(x_k) & p(x_k) \\ 1-q(x_k) & q(x_k) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \text{where either } x_k \in \Sigma^+ \text{ or } x_k = 0, 1 \leq k \leq m; \\ \varphi(z) = 0, z \notin S \equiv c \{ \{\Sigma^+ c\} \cup \{0c\} \}^+; \end{array} \right. \quad (5.1)$$

is a PE.

Proof. Let $p(x) = \pi_1 A_1(x) \eta_1$, $q(x) = \pi_2 A_2(x) \eta_2$ and, if η is a column vector with components 0 or 1, define $\bar{\eta}$ as the column vector such that $\eta + \bar{\eta} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Let \mathcal{A} be the PA over the alphabet $\Sigma \cup \{0, c\}$ defined as follows:

$$\pi = (\bar{\eta}_{1T} \ \bar{\eta}_{2T}) \cdot \frac{1}{\bar{\eta}_{1T} \bar{\eta}_1 + \bar{\eta}_{2T} \bar{\eta}_2};$$

$$A(\sigma) = \begin{pmatrix} A_1(\sigma) & 0 \\ 0 & A_2(\sigma) \end{pmatrix} \quad \text{for every } \sigma \in \Sigma \cup \{0\};$$

$$A(c) = \begin{pmatrix} \bar{\eta}_1 \pi_1 & \eta_1 \pi_2 \\ \bar{\eta}_2 \pi_1 & \eta_2 \pi_2 \end{pmatrix} = \begin{pmatrix} \bar{\eta}_1 & \eta_1 \\ \bar{\eta}_2 & \eta_2 \end{pmatrix} \cdot \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix};$$

$$\eta = \begin{pmatrix} \eta_1 - \bar{\eta}_1 \\ \eta_2 + \bar{\eta}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix};$$

and ψ be the PE defined by \mathcal{A} .

Let then $\chi: \{\Sigma \cup \{0, c\}\}^* \rightarrow \{0,1\}$ be the characteristic ^{function} of the (regular) event $S \equiv c \{ \{\Sigma^+ c\} \cup \{0c\} \}^+ : \chi$ is a PE and, if $\varphi(x) = \chi(x) \cdot \psi(x)$, φ is a PE and satisfies (5.1). \square

If we call $f_t(x)$ the PE generated by PA \mathcal{A}_t , defined at the beginning of this section, it is well known that $f_{1/2}(x) = 0.\tilde{x}$, where $0.\tilde{x}$ is the binary fraction whose part after the point is given by the symbols of x in inverted order, and $f_t(x)$ is a continuous function in t , so that

$$\lim_{t \rightarrow 1/2} f_t(x) = f_{1/2}(x).$$

Let us now suitably specify the PES p and q in (5.1): let $\varphi_1: \{\Sigma \cup \{0\}\}^* \rightarrow [0,1]$, $\varphi_2: \{\Sigma \cup \{0\}\}^* \rightarrow [0,1]$ be two PES and let

$g = \frac{\varphi_1^2}{4} + \frac{\varphi_2^2}{4} + \frac{1 - \varphi_1 \varphi_2}{2} = \frac{1}{2} + \frac{(\varphi_1 - \varphi_2)^2}{4}$: g is a PE since it is a convex combination of PEs, and if we say χ_0 the characteristic function of the regular event $\{0\}$, $(1 - \chi_0)g$ is a PE too. Let

$$p(x) = (1 - \chi_0(x))g(x), \quad (5.2)$$

so that $p(x) = g(x)$ if $x \neq 0$, while $p(0) = 0$; and

$$q(x) = (1 \ 0)A(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad (5.3)$$

$A(0) = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$, $A(\sigma) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ for every $\sigma \in \Sigma$: by this choice $q(x) = 1$ for every $x \in \Sigma^+$ and $q(0) = 1/2$. Then the following theorem holds:

Th. 5.2. The PE φ defined in (5.1), with p and q as specified in (5.2) and (5.3), has an isolated cutpoint iff 0 is isolated with respect to $\varphi_1 - \varphi_2$.

Proof. a) Suppose 0 isolated for $\varphi_1 - \varphi_2$, that is there exists $\delta > 0$ such that $|\varphi_1(x) - \varphi_2(x)| \geq 2\sqrt{\delta}$ for every $x \in \Sigma^+$: then for every $x \in \Sigma^+$

$$g(x) = \frac{1}{2} + \frac{(\varphi_1(x) - \varphi_2(x))^2}{4} \geq \frac{1}{2} + \delta.$$

Let z be a word in $S \equiv c \left\{ \left\{ \Sigma^+ c \right\} \cup \left\{ 0c \right\} \right\}^+$: then we can write $z = xyc$ with $x \in S \cup \{c\}$ and either $y \in \Sigma^+$, or $y = 0$. In the former case

$$\varphi(xyc) = (1 - \varphi(x) \quad \varphi(x)) \begin{pmatrix} 1 - p(y) & p(y) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = p(y) + \varphi(x)(1 - p(y)) \geq p(y) = g(y) \geq \frac{1}{2} + \delta;$$

in the latter $\varphi(x0c) = (1 - \varphi(x) \quad \varphi(x)) \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \varphi(x) \leq 1/2$.

At last, $\varphi(z) = 0$ if $z \notin S$.

We can conclude that the value $1/2 + \delta/2$, for instance, is isolated.

b) Suppose 0 is not isolated for $\varphi_1 - \varphi_2$: in that case there exists a sequence $\{x_n\}$ of words in Σ^+ such that $\lim_{n \rightarrow \infty} p(x_n) = 1/2$. But, because of (5.1), (5.2) and (5.3), if

$$z = c(x_n c)^{i_1} (0c)^{k_1} (x_n c)^{i_2} (0c)^{k_2} \dots (x_n c)^{i_m} (0c)^{k_m}$$

with $i_j, k_j \geq 0$, $1 \leq j \leq m$, then $\varphi(z) = f_{p(x_n)}(y)$, where

$$y = 1^{i_1} 0^{k_1} 1^{i_2} 0^{k_2} \dots 1^{i_m} 0^{k_m} \text{ and}$$

$$\lim_{n \rightarrow \infty} \varphi(z) = \lim_{n \rightarrow \infty} f_{p(x_n)}(y) = f_{1/2}(y) = 0. \tilde{y}.$$

Since the set of rational numbers whose representation is of type $0.\tilde{y}$ is dense in $[0, 1]$, we conclude that there cannot exist an isolated cutpoint with respect to φ . \square

Th. 5.3. The problem of deciding whether every given PA with rational probability transitions has an isolated cutpoint is recursively unsol-

vable.

Proof. Suppose the problem is solvable for the above-mentioned PA \mathcal{Q} : it would then be possible to decide whether 0 is isolated with respect to $\varphi_1 - \varphi_2$, contrary to what we proved in §3. \square

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