Decidability of the Theory of the Totally Unbounded ω-Layered Structure

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Outline

- MSO logics over tree structures
- Layered structures
- The automaton-based approach
- A solution to the decision problem
- Further work

MSO Logics over tree structures - 1

Let $\Lambda = \{1, \ldots, k\}$ be a finite set of edge labels.

We consider **tree structures extended with tuples of unary predicates**, namely, structures of the form

$$(\mathcal{T}, \bar{V}) = (S, (E_l)_{l \in \Lambda}, (V_i)_{i \in [1,m]})$$

where

- $S = \Lambda^*$ (set of vertices)
- $E_l = \{(v, vl) \,:\, v \in S\}$ (l-labeled edges)
- $V_i \subseteq S$ for all $1 \leq i \leq m$ (unary predicates)

MSO Logics over tree structures - 2

Formulas over a tree structure $\mathcal{T}$ are built up from atoms:

- $x_i = x_j$ "x_i and x_j denote the same vertex"
- $X_i \subseteq X_j$ "X_i denotes a subset of X_j"
- $E_l(x_i, x_j)$ "(x_i, x_j) denotes a l-labeled edge"
- $X_k(x_i)$ "x_i denotes a vertex in X_k"

...through connectives $\wedge$, $\vee$, $\neg$ and quantifiers $\forall$, $\exists$ over first-order and second-order variables.

Remark: we can restrict ourselves to an expressively equivalent framework devoid of *first-order variables*.

MSO Logics over tree structures - 3

Free second-order variables $X_1, \ldots, X_m$ will be interpreted by tuples of (unary) predicates $V_1, \ldots, V_m$.

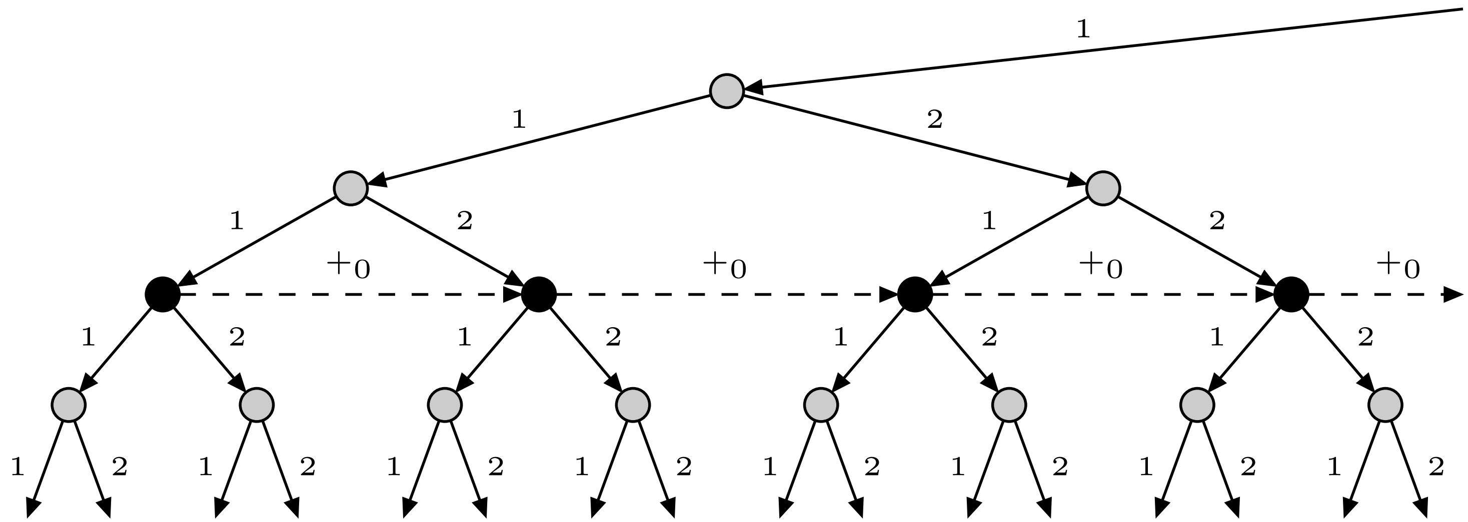
Given a formula $\varphi(\bar{X})$, we write $\mathcal{T} \models \varphi[\bar{V}]$ to say that $\varphi(\bar{X})$ holds in $\mathcal{T}$ by substituting V_i for X_i, for all $1 \leq i \leq m$.

The **decision problem** $MTh(\mathcal{T}, \bar{V})$ is the problem of deciding whether, for a given formula $\varphi(\bar{X})$,

$$\mathcal{T} \models \varphi[\bar{V}]$$

Layered structures: totally unbounded

Goal: we want to decide the MSO theory of the **totally unbounded ω-layered structure (TULS)**:

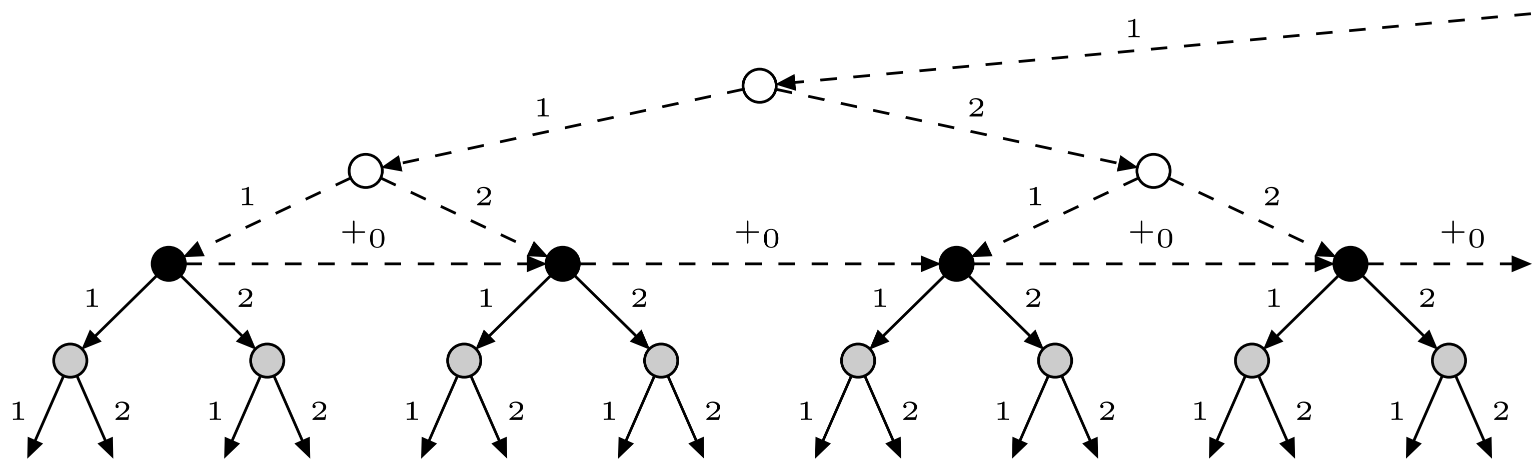


- The structure contains arbitrarily fine/coarse layers
- Arrows map elements of a given layer to elements of the immediately finer layer
- Black vertices denote the elements of a distinguished layer ("layer 0") with an (optional) successor relation $+_0$

Layered structures: downward unbounded

The TULS embeds the **downward unbounded ω-layered structure (DULS)**

(i.e., the structure with a top layer and an infinite number of finer and finer layers):

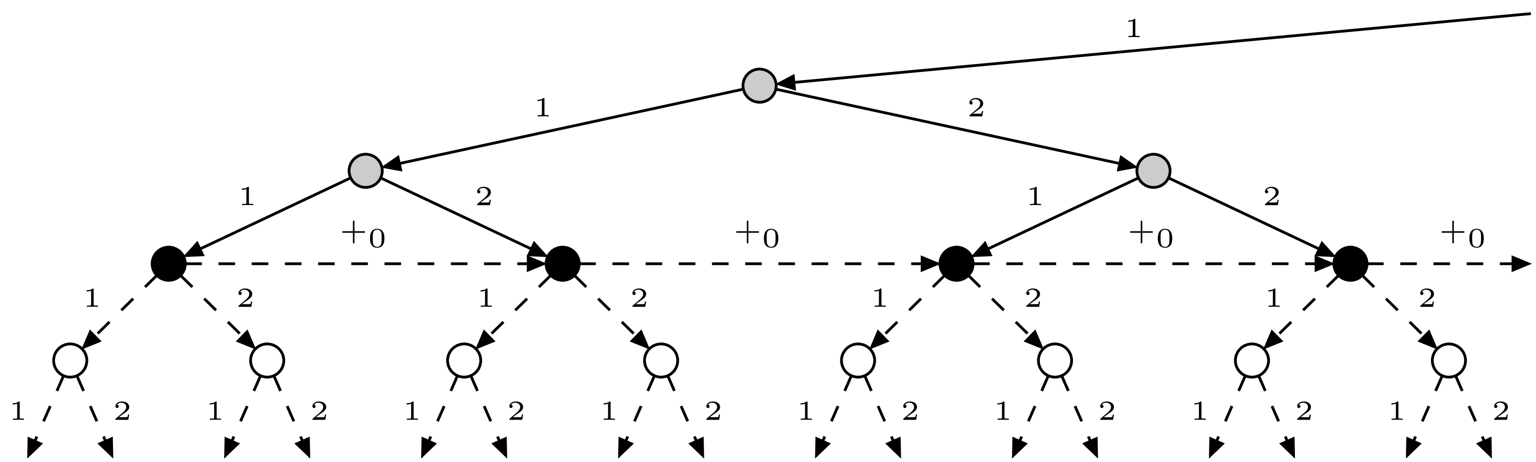


$\Rightarrow$ The DULS allows one to express properties like "*P holds true densely in an interval*".

Layered structures: upward unbounded

...and it embeds the **upward unbounded ω-layered structure (UULS)**

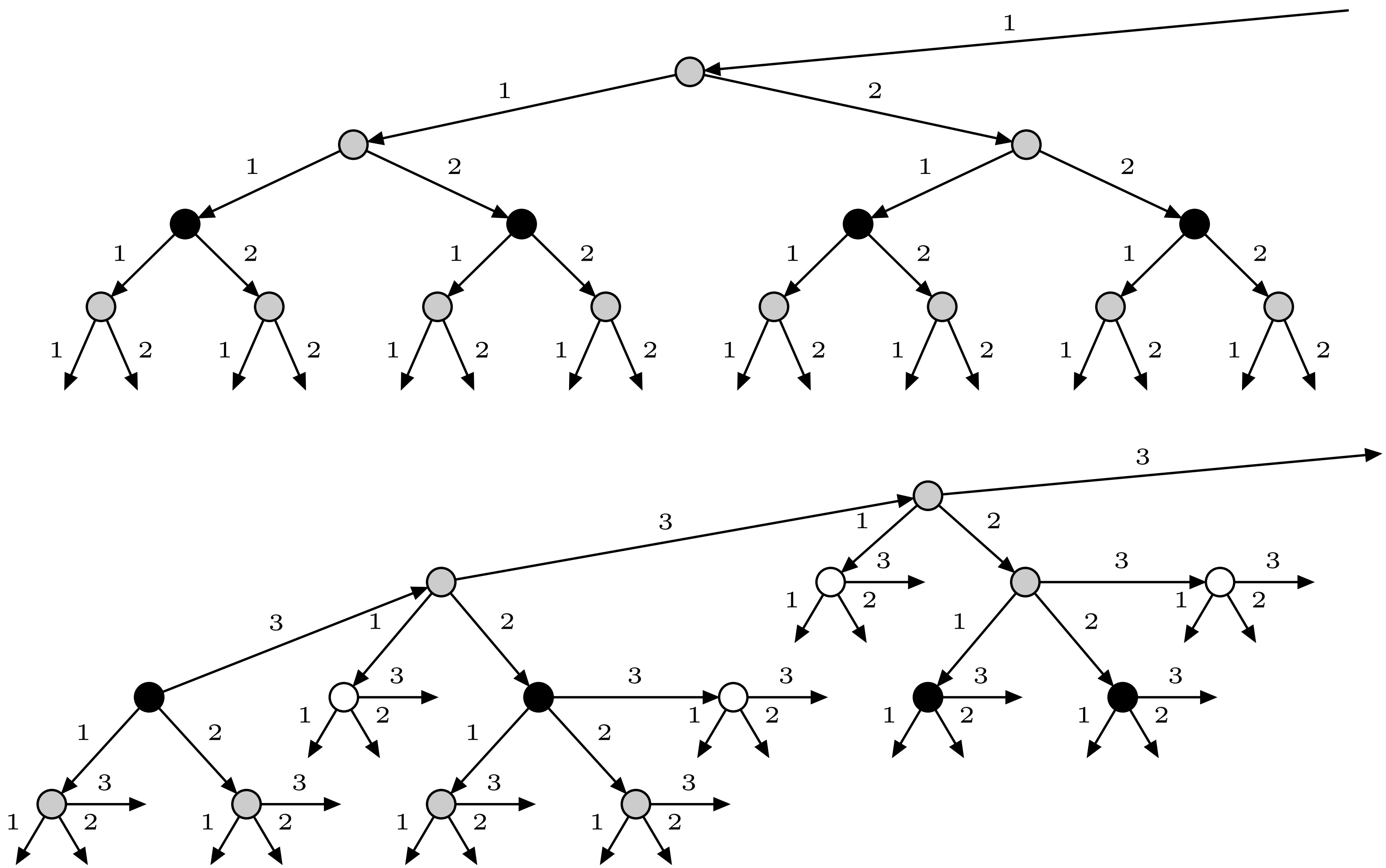
(i.e., the structure with a bottom layer and an infinite number of coarser and coarser layers):



$\Rightarrow$ The UULS allows one to express properties like "*P holds at all time points 2^i*".

Layered structures: a tree embedding

The totally unbounded ω-layered structure can, in its turn, be embedded into an infinite complete **ternary tree** $\mathcal{T}_{TULS}$:



The automaton based approach - 1

Any expanded tree structure $(\mathcal{T}, \bar{V})$ can be encoded by a (*vertex-colored*) tree $\mathcal{T}_{\bar{V}}$ (**canonical representation**).

$\Rightarrow$ **Idea:** to exploit the correspondence between logic over tree structures and Rabin tree automata in order to *reduce a decision problem to an acceptance problem*.

A **Rabin automaton** works on colored trees in a top-down fashion: it "spreads" its states inside a tree (according to the transition relation) and it verifies that suitable acceptance conditions are met.

We say that a colored tree $\mathcal{T}_{\bar{V}}$ is **accepted** by M ($\mathcal{T}_{\bar{V}} \in \mathscr{L}(M)$) if such conditions are satisfied.

The automaton based approach - 2

[Rabin's Theorem] For every formula $\varphi(\bar{X})$, there is a Rabin automaton M (and vice versa) such that

$$\mathcal{T} \models \varphi[\bar{V}] \quad \Leftrightarrow \quad \mathcal{T}_{\bar{V}} \in \mathscr{L}(M)$$

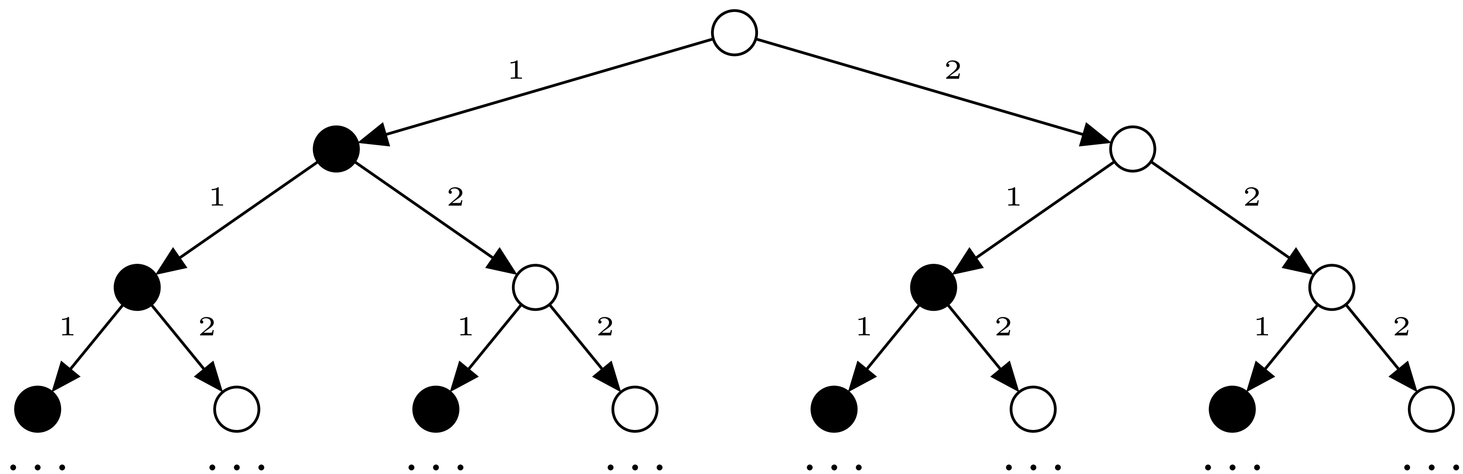
$\Rightarrow$ the decision problem $MTh(\mathcal{T}, \bar{V})$ for MSO formulas reduces to an **acceptance problem** $Acc(\mathcal{T}_{\bar{V}})$ for Rabin automata

$\Rightarrow$ we can restrict our attention to the decidability of the acceptance problem for Rabin tree automata.

Notation: Hereafter, we shall drop the subscript $\bar{V}$ from $\mathcal{T}_{\bar{V}}$.

The automaton based approach - 3

Proposition: $Acc(\mathcal{T})$ is decidable for any infinite **regular** tree $\mathcal{T}$ (i.e., a tree with *only finitely many distinct subtrees*).



However, the colored tree $\mathcal{T}_{TULS}$ that embeds the TULS is **not regular**.

$\Rightarrow$ we look for a larger class of colored trees for which the acceptance problem turns out to be decidable.

The automaton based approach - 4

Idea: Given an automaton M, we decide whether $\mathcal{T} \in \mathscr{L}(M)$ by reducing it to a simpler problem $\mathcal{T}' \in \mathscr{L}(M)$, where $\mathcal{T}'$ is a **regular tree equivalent to** $\mathcal{T}$, namely,

$$\mathcal{T} \in \mathscr{L}(M) \quad \Leftrightarrow \quad \mathcal{T}' \in \mathscr{L}(M)$$

(recall that regular trees enjoys a decidable acceptance problem)

Such a reduction works effectively for several non-regular trees.

In particular, we can reduce the acceptance problem for $\mathcal{T}_{TULS}$ to a decidable acceptance problem over an *equivalent regular tree*.

A digression into Büchi automata - 1

Given a Büchi automaton M, we can define an equivalence $\equiv_M$ over finite words s.t. $u \equiv_M u'$ iff, for every pair of states r,s,

$$r \xrightarrow{u} s \quad \Leftrightarrow \quad r \xrightarrow{u'} s$$

$$r \overset{u}{-\!\!\circ\!\!\!\longrightarrow} s \quad \Leftrightarrow \quad r \overset{u'}{-\!\!\circ\!\!\!\longrightarrow} s$$

Properties:

1. $\equiv_M$ has **finite index**
2. $\equiv_M$ is a **congruence** w.r.t. concatenation
3. $\equiv_M$-equivalent factorizations are **indistinguishable** by M, namely, if $u_i \equiv_M u'_i$ for all $i \geq 0$, then

$$u_0 u_1 u_2 \ldots \in \mathscr{L}(M) \quad \Leftrightarrow \quad u'_0 u'_1 u'_2 \ldots \in \mathscr{L}(M)$$

A digression into Büchi automata - 2

[Carton and Thomas] Given an ω-word $w = u_0u_1u_2\ldots$, if *for any congruence* $\equiv_M$ there are p, q such that $\forall\ i > p.\ u_i \equiv_M u_{i+q}$

$$w \in \mathscr{L}(M)$$

$$\Updownarrow$$

$$(u_0 \ldots u_p)(u_{p+1} \ldots u_{p+q})(u_{p+q+1} \ldots u_{p+2q}) \ldots \in \mathscr{L}(M)$$

$$\Updownarrow$$

$$(u_0 \ldots u_p) \cdot (u_{p+1} \ldots u_{p+q})^{\omega} \in \mathscr{L}(M)$$

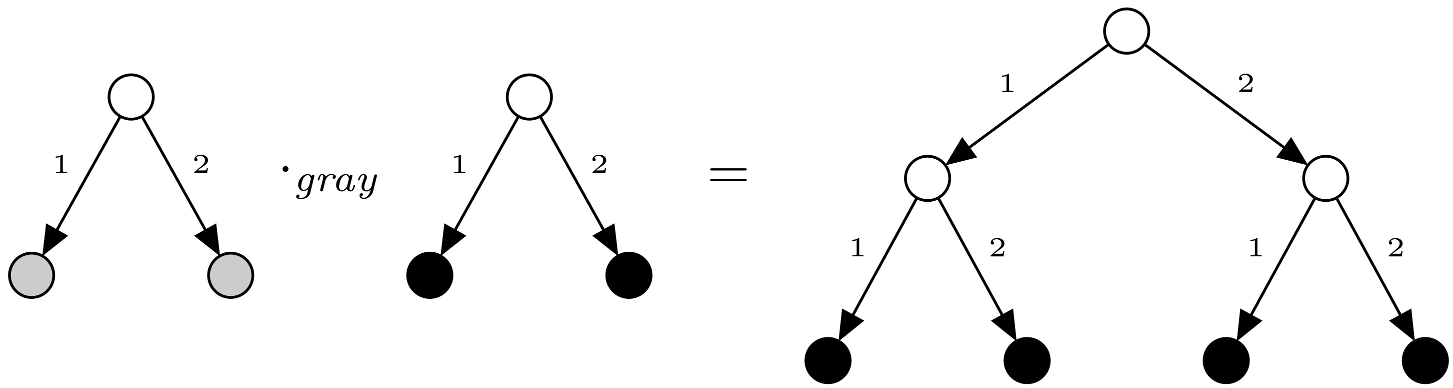
$\Rightarrow$ if such p and q are computable for any congruence $\equiv_M$, then $Acc(w)$ can be effectively reduced to a decidable acceptance problem over an **ultimately periodic word**.

Similar results hold for infinite trees...

A solution to the decision problem - 1

Basic ingredients:

- notion of **tree concatenation** $\mathcal{T}_1 \cdot_c \mathcal{T}_2$
  (defined as the substitution in $\mathcal{T}_1$ of each c-colored leaf by $\mathcal{T}_2$)



- notion of **factorization** for infinite trees
  (i.e. infinite concatenation of the form $\mathcal{T}_0 \cdot_{c_0} \mathcal{T}_1 \cdot_{c_1} \ldots$)
- notion of **congruence** $\equiv_M$ w.r.t. tree concatenations

A solution to the decision problem - 2

[Main result] Given an infinite tree $\mathcal{T}$ generated by a factorization $\mathcal{T}_0 \cdot_{c_0} \mathcal{T}_1 \cdot_{c_1} \ldots$, if *for any congruence* $\equiv_M$ *there are* p, q such that $\forall\, i > p.\ \mathcal{T}_i \equiv_M \mathcal{T}_{i+q}$, then:

$$\mathcal{T} \in \mathscr{L}(M)$$

$$\Updownarrow$$

$$\mathcal{T}_0 \cdot_{c_0} \ldots \mathcal{T}_p \cdot_{c_p} \mathcal{T}_{p+1} \cdot_{c_{p+1}} \ldots \mathcal{T}_{p+q} \cdot_{c_{p+q}} \mathcal{T}_{p+q+1} \cdot_{c_{p+q+1}} \ldots \in \mathscr{L}(M)$$

$$\Updownarrow$$

$$\mathcal{T}_0 \cdot_{c_0} \ldots \mathcal{T}_p \cdot_{c_p} \mathcal{T}_{p+1} \cdot_{c_{p+1}} \ldots \mathcal{T}_{p+q} \cdot_{c_{p+q}} \mathcal{T}_{p+1} \cdot_{c_{p+1}} \ldots \in \mathscr{L}(M)$$

Remark. The last factorization is ultimately periodic and it generates a (decidable) **regular** tree $\mathcal{T}'$.

A solution to the decision problem - 3

A tree $\mathcal{T}$ is said **residually regular** if we can provide a factorization $\mathcal{T}_0 \cdot_{c_0} \mathcal{T}_1 \cdot_{c_1} \ldots$ that is *effectively ultimately periodic w.r.t. any congruence* $\equiv_M$.

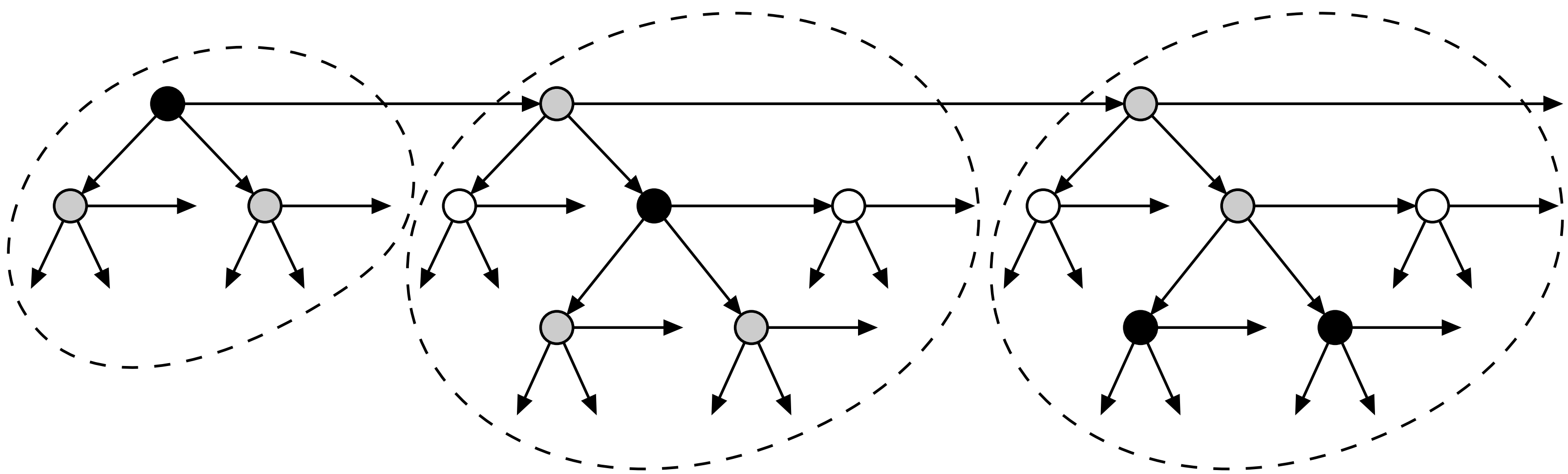
$\Rightarrow$ **we solve** $Acc(\mathcal{T})$ **as follows**:

1. we take a factorization $\mathcal{S}$ of $\mathcal{T}$ which is ultimately periodic w.r.t. any congruence $\equiv_M$
2. given automaton M, we compute an ultimately periodic factorization $\mathcal{S}'$ that is $\equiv_M$-equivalent to $\mathcal{S}$
3. we know that $\mathcal{S}'$ generates a regular tree $\mathcal{T}'$ and $\mathcal{T}' \in \mathscr{L}(M) \quad \Leftrightarrow \quad \mathcal{T} \in \mathscr{L}(M)$
4. we solve $Acc(\mathcal{T}')$ on automaton M
5. we accordingly return Yes or No to the original problem $Acc(\mathcal{T})$

A solution to the decision problem - 4

In general, residually regular trees are non-regular trees which however *exhibit a definite pattern* in their structure.

Example. The tree $\mathcal{T}_{TULS}$, which embeds the TULS, can be proved to be **residually regular**:



The sequence of factors is **ultimately periodic w.r.t. any equivalence** $\equiv_M$

$\Rightarrow$ the tree $\mathcal{T}_{TULS}$ (and hence the TULS itself) enjoys a **decidable MSO theory**.

Conclusions

Results:

- we developed an original automaton-based method to decide the TULS
- as a by-product, we obtained new uniform decidability proofs for the DULS and UULS

Further work:

- to exploit the proposed technique to decide variants of the theories of the DULS and UULS (MSO fragments extended with *equi-level*/*equi-column* predicates)
- to determine the generality of the proposed method (e.g., to compare it with the transformational approach developed by Caucal)