On the use of guards for logics with data

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What is this talk about?

Generalizations of classical results about **regular languages** from finite-alphabet case to **infinite-alphabet case**:

- correspondence between logics, automata, and monoids
- characterizations of first-order definability
- decidability of logics

Applications of languages over infinite alphabets (**data languages**):

- Databases: XML documents with text/attributes
- Verification: programs with variables over an infinite domain

A **data language** is a set of words/trees
over a fixed infinite alphabet $\mathbb{D}$ (e.g. $\mathbb{D} = \{0, 1, 2, \ldots\}$).

To make life easier, we enforce some restrictions:

1. Only finite words! (no infinite words, no finite/infinite trees)
2. Languages are invariant under permuations of data values

   (e.g. $\boxed{1}\boxed{2}\boxed{1}\boxed{1}\boxed{3} \in L$ iff $\boxed{5}\boxed{3}\boxed{5}\boxed{5}\boxed{7} \in L$)

we focus on properties concerning equalities of data values

An example of data language

$$L = \{w \in \mathbb{D}^* \ : \ \textit{at most } 2 \textit{ distinct values in } w\}$$

$= \{\varepsilon,$  $, \ldots\}$

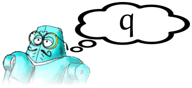
Languages over finite alphabets:

- **MSO logic**

$$\exists X.\ \texttt{first} \in X \wedge \texttt{last} \notin X \wedge \forall y.\ (y \in X \leftrightarrow y+1 \notin X)$$

- **automata**

1	0	1	1	⋯	0	1



- **finite monoids**

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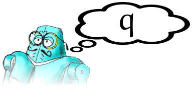
$\boxed{1}\boxed{0}\boxed{1}\boxed{1}\ \cdots\ \boxed{0}\boxed{1}$



- **finite monoids**

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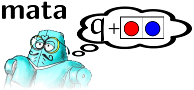
Languages over
infinite alphabets:

- **MSO logic with data tests**

$$\begin{aligned}\exists\, X.\ & \texttt{first} \in X \\ & \wedge\ \texttt{last} \notin X \\ & \wedge\ \forall\, y.\ (y \in X \leftrightarrow y+1 \notin X) \\ & \wedge\ \forall\, y,z.\ (y,z \in X \rightarrow \textcolor{red}{y \sim z})\end{aligned}$$

- **register automata**

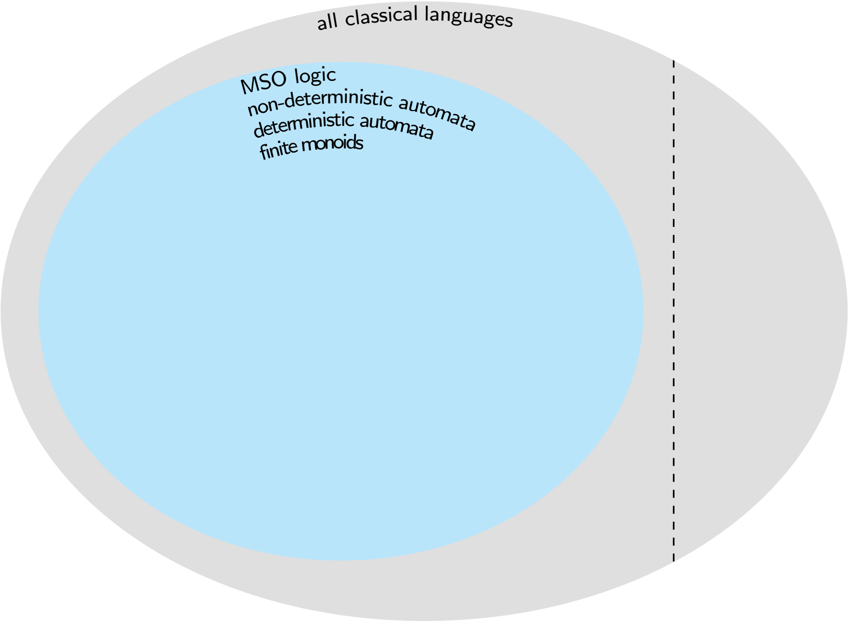
$\boxed{\textcolor{green}{\bullet}}\boxed{\textcolor{red}{\bullet}}\boxed{\textcolor{blue}{\bullet}}\boxed{\bullet}\ \cdots\ \boxed{\textcolor{magenta}{\bullet}}\boxed{\textcolor{red}{\bullet}}$



- **orbit finite data monoids**

$$s(\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}) \cdot t(\textcolor{green}{\bullet}) = s(\textcolor{blue}{\bullet}, \textcolor{green}{\bullet})$$
$$t(\textcolor{blue}{\bullet}) \cdot s(\textcolor{blue}{\bullet}, \textcolor{red}{\bullet}) = t(\textcolor{red}{\bullet})$$

Expressiveness of logics, automata, monoids over finite alphabets:



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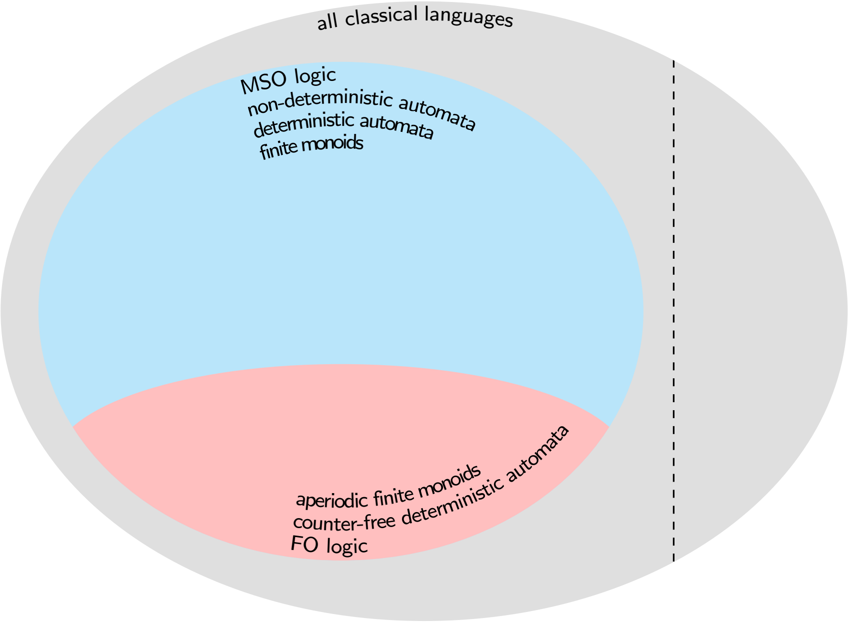

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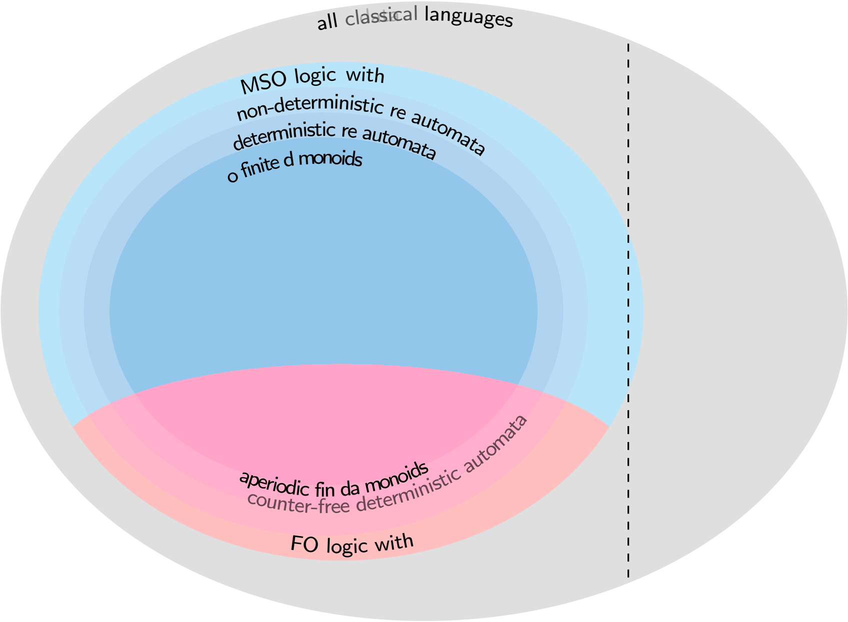

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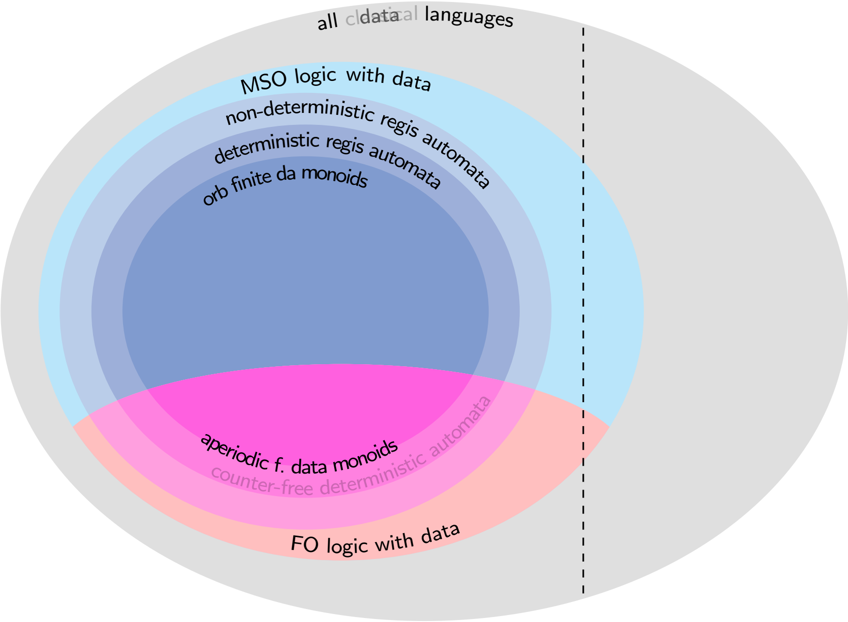

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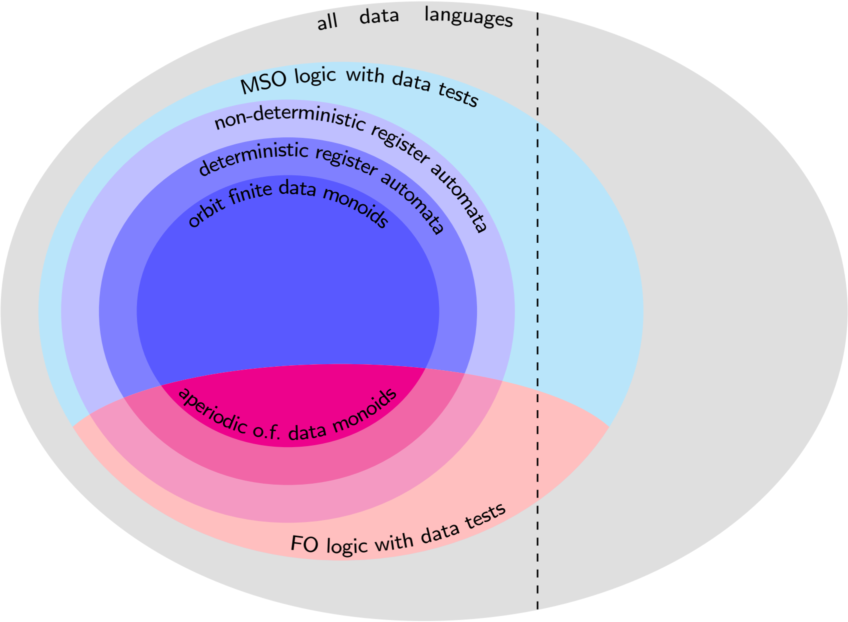

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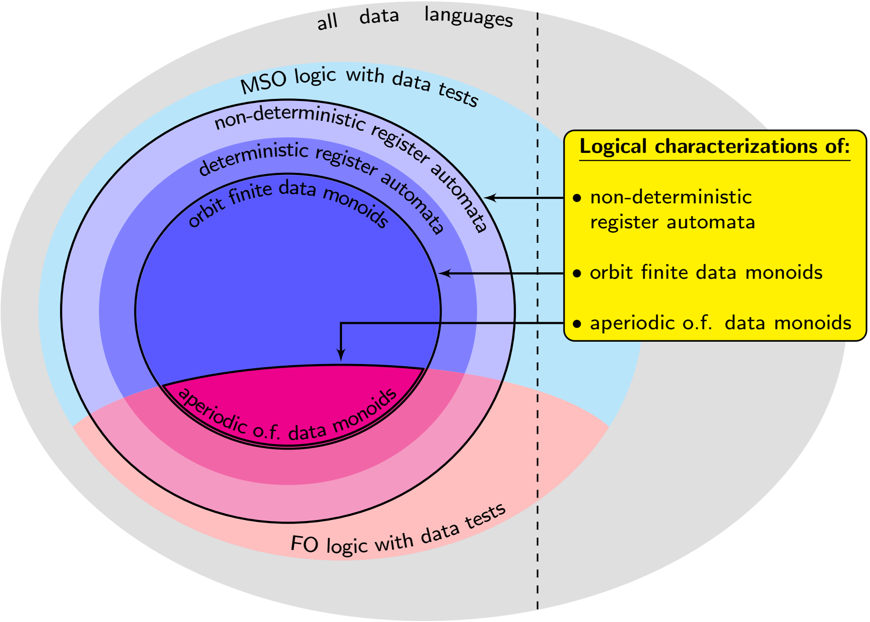
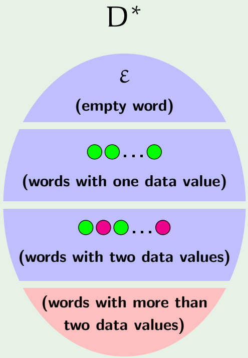


What is an orbit finite data monoid?

Consider the syntactic monoid of the language
$L = \{w \in D^* \ : \ \textit{at most 2 distinct values in } w\}$:



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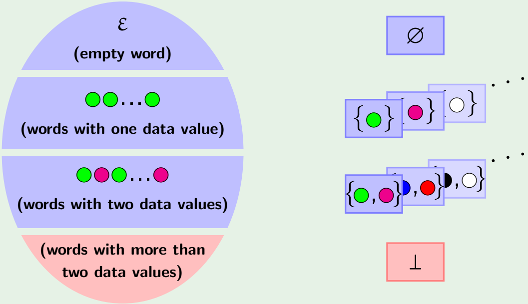

The product of $\mathcal{M}_L$ is the union of sets, up to cardinality 2.

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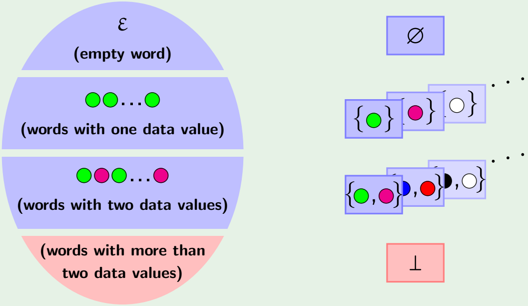

The product of $\mathcal{M}_L$ is the union of sets, up to cardinality 2.

☞ Each permutation π on D induces a permutation $\hat{\pi}$ on $\mathcal{M}_L$
e.g., if $\pi = \{● \leftrightarrow ●\}$, then $\hat{\pi}(\{●, ●\}) = \{●, ●\}$.

Examples of languages recognized by orbit finite data monoids:

- ✓ *Exactly two/three/... distinct values*
- ✓ *Any two consecutive values are different*
- ✓ *First value equals last value*
- ✓ *"Lifting" by permutations of any classical regular language*
- ✗ *First value reappears*
- ✗ *Some value appears twice*
- ✗ *All values appears at most once*

- Closure under all boolean operations!

Consider some languages recognized by orbit finite data monoids:

- words where **first value equals last value**:

  $\exists\, x, y.\, (x = \mathtt{first} \,\wedge\, y = \mathtt{last}) \,\wedge\, (x \sim y)$

- words with **at least two distinct values** (e.g. …●●…):

  $\exists\, x, y.\, (y = x + 1) \,\wedge\, (x \nsim y)$

…and some languages not recognized by orbit finite data monoids:

- words where **first value reappears**:

  $\exists\, x, y.\, (x = \mathtt{first} \,\wedge\, x < y) \,\wedge\, (x \sim y)$

- words where **all values appear at most once**:

  $\neg\exists\, x, y.\, (x < y) \,\wedge\, (x \sim y)$

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- words with **at least three distinct values** (e.g. …●●●●●…):

  $\exists\, x, y.\ \big((x \nsim x{+}1) \wedge (y \nsim y{+}1) \wedge \forall\, z.\ (x{<}z{<}y \rightarrow z \sim z{+}1)\big)$
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Definition

Rigidly guarded MSO is the fragment of MSO with data tests, defined by the following grammar:

$$\varphi \ := \ x < y \ \mid \ x \in Y \ \mid \ \neg\varphi \ \mid \ \varphi \ \wedge \ \varphi \ \mid \ \exists\, x.\ \varphi \ \mid \ \exists\, Y.\ \varphi \ \mid$$
$$\varphi_{\text{rigid}}(x,y) \ \wedge \ x \sim y$$

where $\varphi_{\text{rigid}}(x,y)$ is a rigid guard (generated by the same grammar)

($\varphi(x,y)$ is **rigid** if, in every word, x determines y and vice versa).

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- Rigidity is a semantical restriction.
  However, it can be enforced syntactically, e.g.,
  $\varphi_{\text{rigid}} \;=\; \varphi(x,y) \;\wedge\; \big(\forall\, x',y'.\, [x',y'] \subsetneq [x,y] \;\rightarrow\; \neg\varphi(x',y')\big)$

- Is $\varphi_{\text{rigid}}(x,y) \wedge x \not\sim y$ needed?
  No: $\varphi_{\text{rigid}}(x,y) \;\wedge\; \neg\big(\varphi_{\text{rigid}}(x,y) \wedge x \sim y\big)$

Main theorem (1)

Languages defined in rigidly guarded MSO

=

Languages recognized by orbit finite data monoids.

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...and as in the Schützenberger-McNaughton-Papert's theorem:

Main theorem (2)

Languages defined in rigidly guarded <u>FO</u>

=

Languages recognized by <u>aperiodic</u> orbit finite data monoids.

(A data monoid is **aperiodic** if all its sub-groups are trivial)

Proof idea (rigidly guarded MSO $\to$ orbit finite data monoid)

By induction on formulas, using closure properties of data monoids:

- negation of a formula $\Rightarrow$ easy, by definition of recognizability
- conjunction of formulas $\Rightarrow$ product of orbit finite data monoids
- existential quantification $\Rightarrow$ powerset of an orbit finite data monoid

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Given a morphism $h : (D \times \{0,1\})^* \to \mathcal{M}$ construct $h' : D^* \to 2^{\mathcal{M}}$

where $h'(w) = \big\{ h(\langle w, X \rangle) \;:\; X \subseteq \mathtt{dom}(w) \big\}$

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Given a monoid $\mathcal{M} = (M, \cdot, \hat{\ })$ construct $2^{\mathcal{M}} = (2^{M}, \odot, \hat{\hat{\ }})$

where $S \odot T = \big\{ s \cdot t \ :\ s \in S,\ t \in T \big\}$

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☞ Technical problem: this does not preserve orbit finiteness...

Proof idea (aperiodic o.f. data monoid $\rightarrow$ rigidly guarded FO)

Follow the same induction as in the Schützenberger's proof:

> “ *Given a morphism* $h : D^* \rightarrow \mathcal{M}$*,*
> *construct formulas computing* $h(w[x, y])$
> *for larger and larger infixes* $w[x, y]$ *of words.* ”

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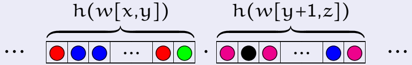
Technical problem: in order to let the induction go through, we need to simulate products of the monoid with formulas...



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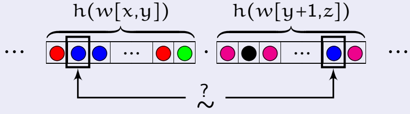

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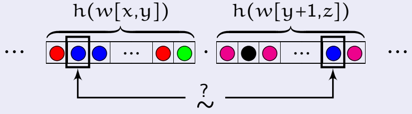

Positions with **memorable values**
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If we drop the assumption of aperiodicity,
we need <u>MSO formulas</u> to compute elements of the monoid.

☞ Unlike in the classical case, we cannot simulate runs of automata (instad, we need to further generalize Schützenberger's proof).

We also considered a relaxation of the rigidity constraints:

Definition

Semi-rigidly guarded MSO is defined by the grammar

$$\psi \;:=\; \exists\, Z_1, \dots, Z_k.\;\; \varphi(Z_1, \dots, Z_k)$$

$$\begin{aligned}\varphi(Z_1, \dots, Z_k) \;:=\;& x < y \;\mid\; x \in Y \;\mid\; x \in Z_i \;\mid \\ & \neg\varphi \;\mid\; \varphi \wedge \varphi \;\mid\; \exists\, x.\, \varphi \;\mid\; \exists\, Y.\, \varphi \;\mid \\ & \varphi_{\text{semi-rigid}}(Z_1,\dots,Z_k,x,y) \;\wedge\; x \sim y\end{aligned}$$

where $\varphi_{\text{semi-rigid}}(Z_1,\dots,Z_k,x,y)$ determines y from $Z_1, \dots, Z_k, x$.

Example

The formula below defines the language of all words where <u>some value reappears at the last even position</u>:

$$\begin{aligned}\psi \;=\;& \exists\, Z.\, \forall\, z.\, \big(z \in Z \;\leftrightarrow\; \mathtt{Even}(z)\big) \\ & \wedge\; \exists\, x, y.\, \big(x < y \;\wedge\; y = \mathtt{last}(Z)\big) \;\wedge\; x \sim y.\end{aligned}$$

Theorem (3)

Languages of data words defined in semi-rigidly guarded MSO

$$=$$

Languages recognized by non-deterministic register word automata.

Theorem (3)

Languages of data ~~words~~ trees defined in semi-rigidly guarded MSO

=

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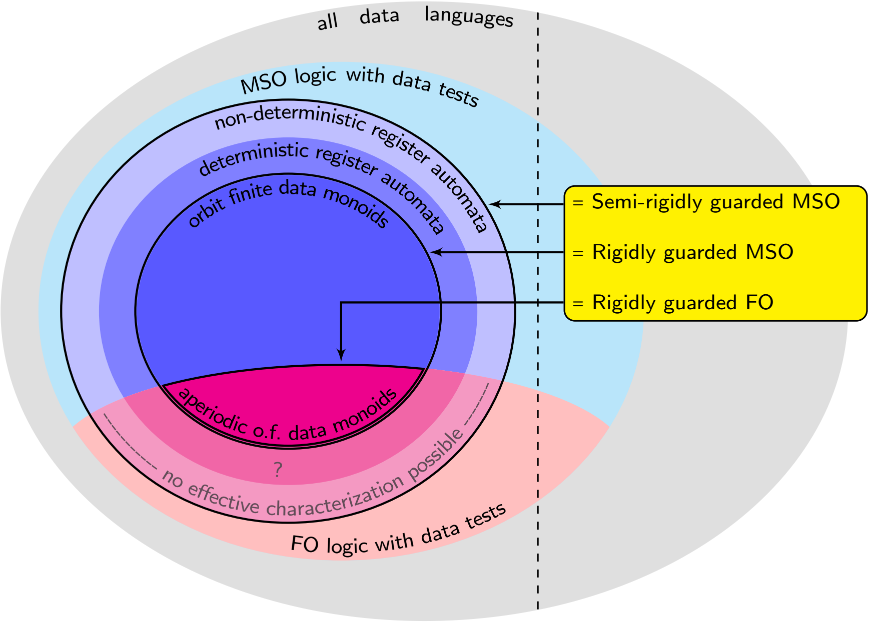
Corollary

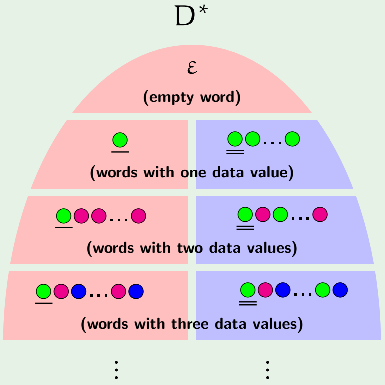
Satisfiability of semi-rigidly guarded MSO is decidable.

Back to our picture...



A data monoid with infinitely many orbits

Consider the syntactic monoid of the language
$L = \{w \in D^* \ : \ \textit{first value of } w \textit{ reappears}\}$:



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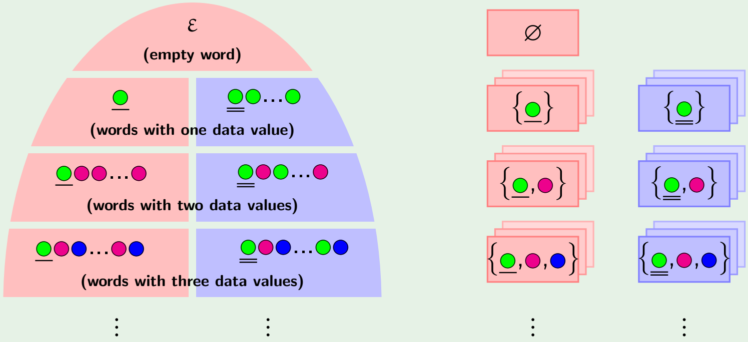

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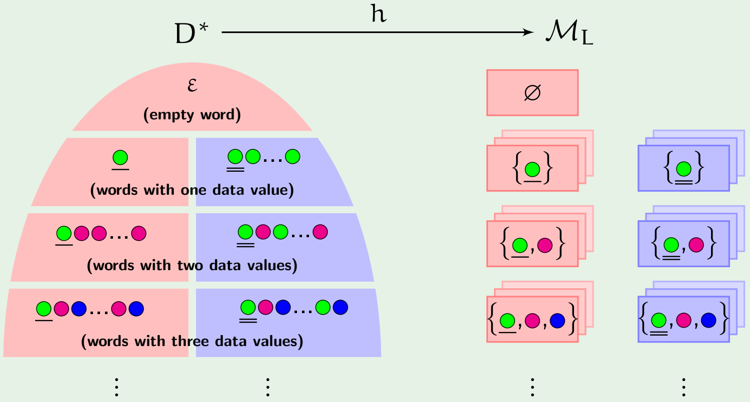

☞ L cannot be recognized with finitely many orbits
(but it is recognized by a deterministic register automaton).