First-order theories

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Definition

Fix a class $\mathcal{C}$ of structures (e.g. graphs) and a logic $\mathcal{L}$ (e.g. FO).

The **$\mathcal{L}$-theory of $\mathcal{C}$** is the set of all formulas in $\mathcal{L}$ that can be *satisfied by some structure* in $\mathscr{C}$.

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Examples

- first-order theory of the class of all graphs
- monadic theory of the class of all linear orders
- monadic theory of $\mathbb{N}$
- monadic theory of the grid

Undecidability of first-order theory

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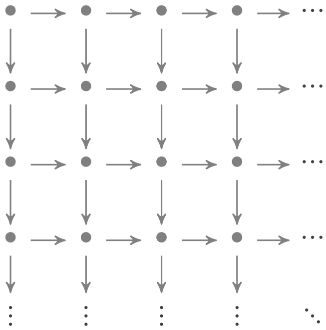

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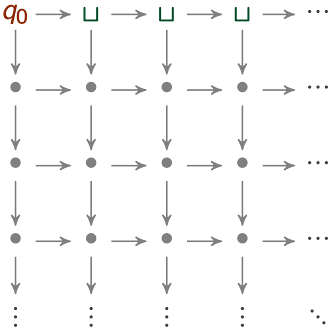

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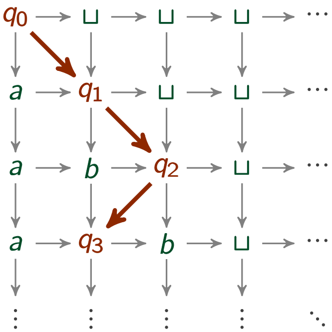

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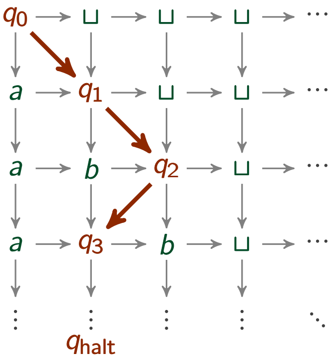

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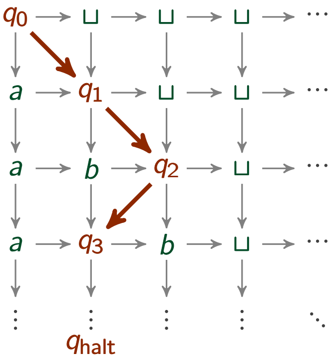

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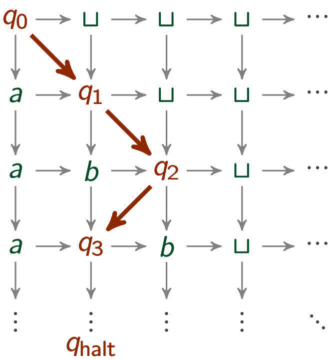

☞ MSO can even guess the labelling!

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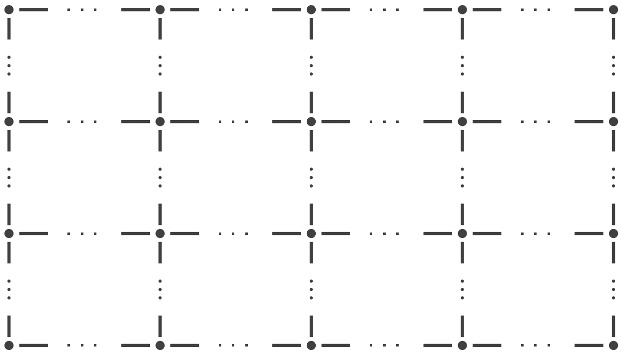

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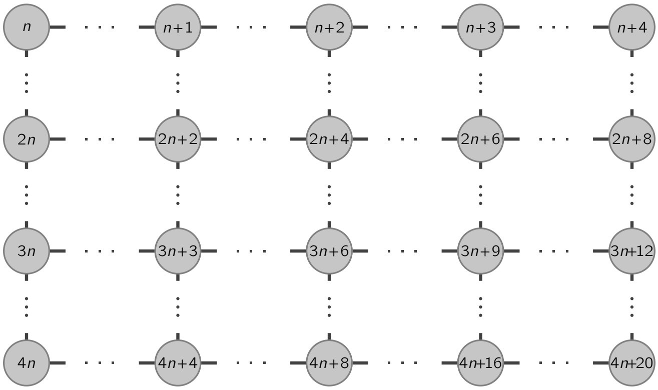
The MSO theory of $(\mathbb{N}, +)$ is undecidable.



Definition

Presburger arithmetic is the first-order theory of $(\mathbb{N}, +)$

Examples of Presburger formulas

$$\begin{aligned}
\psi_0 &= \exists x.\ \forall y.\ (x + y = y) \\
\varphi_{\leq}(x, y) &= \exists z.\ (y = x + z) \\
\varphi_{2\times}(x, y) &= (x + x = y) \\
\psi_\omega &= \forall x.\ \exists y.\ (x \leq y \ \wedge\ \neg x = y)
\end{aligned}$$

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   e.g. $\big[+(3, 1, 4)\big] \;=\; [3] \otimes [1] \otimes [4] \;=\; \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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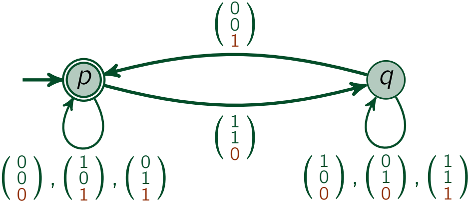

Decidability of Presburger arithmetic (Presburger '29)

One can decide if a Presburger sentence ψ holds over $(\mathbb{N}, +)$.

We inductively translate every Presburger formula $\varphi(x_1, \ldots, x_m)$ into a finite automaton $\mathcal{A}_\varphi$ over $\Sigma_m = \mathbb{B}^m$ such that

$$\mathscr{L}(\mathcal{A}_\varphi) \;=\; \left\{ [x_1] \otimes \cdots \otimes [x_m] \in \Sigma_m^\star \;\middle|\; (\mathbb{N}, +) \models \varphi(x_1, \ldots, x_m) \right\}$$

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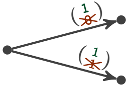
- For existential quantification $\exists y.\ \varphi(\bar{x}, y)$, project $\mathcal{A}_\varphi$ from Σ_{m+1} to Σ_m



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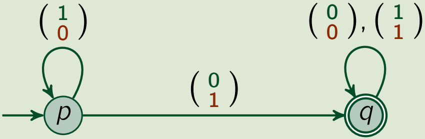

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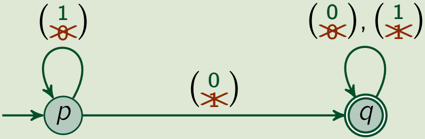

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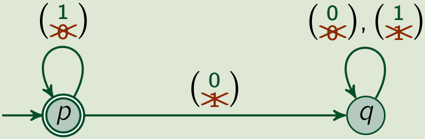

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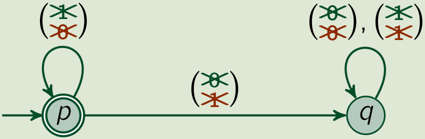

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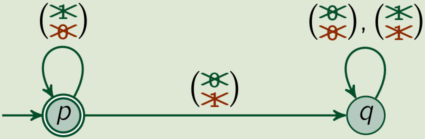

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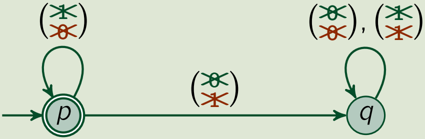

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Languages of encodings should be closed under padding with 0's
After complement, keep only final states that are stable under 0.

The previous result can be generalized to many other structures:

Definition

An **automatic structure** is a structure that is isomorphic to

$$\big(L,\ R_1,\ \ldots,\ R_n \big)$$

where

- L is a regular language of words over Σ
  (each word identifies a precise element of the structure)
- each relation R_i has arity k_i and is represented
  by a regular language L_i over $\big(\Sigma \uplus \{\#\}\big)^{k_i}$
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Examples of automatic structures

- $(\mathbb{N}, +, |_p)$, with $x \,|_p\, y$ iff $x = p^n$ divides y
- Binary tree with successor, ancestor, and equi-level predicates
- Unlabelled grid $(\mathbb{N} \times \mathbb{N}, \rightarrow, \downarrow)$

Theorem (Büchi '60, Hodgson '76, Khoussainov & Nerode '94)

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On the other hand...

Theorem

Some automatic structures have an **undecidable reachability problem**.

The **transition graph** of a Turing machine is automatic!

- configurations are encoded by words $a_1 \ldots a_{i-1}\, q\, a_i\, a_{i+1} \ldots a_n$
- transitions are of the following forms

$$a_1 \ldots a_{i-1}\, q\, a_i\, a_{i+1} \ldots a_n \quad a_1 \ldots a_{i-1}\, q\, a_i\, a_{i+1} \ldots a_n \quad a_1 \ldots a_{i-1}\, a_i\, q\, a_{i+1} \ldots a_n$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$a_1 \ldots a_{i-1}\, q'\, a'_i\, a_{i+1} \ldots a_n \quad a_1 \ldots a_{i-1}\, a'_i\, q'\, a_{i+1} \ldots a_n \quad a_1 \ldots a_{i-1}\, q'\, a'_i\, a_{i+1} \ldots a_n$$

Next