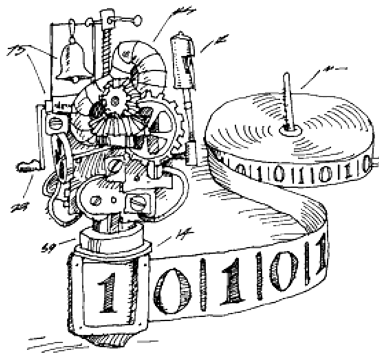


# The monadic theory of one successor

Gabriele Puppis

LaBRI / CNRS



While FO talks of elements of  $\mathbb{N}$ , MSO talks of **subsets of  $\mathbb{N}$** .

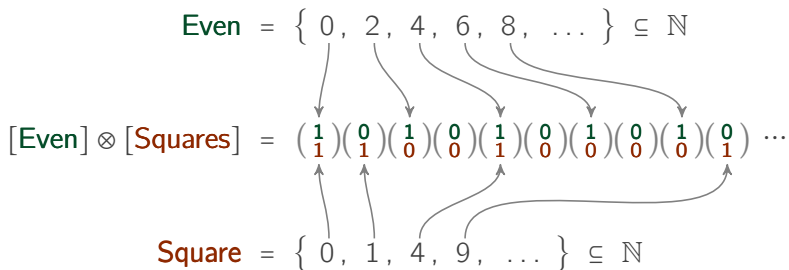
These subsets can be encoded by **infinite words**:

$$\text{Even} = \{ 0, 2, 4, 6, 8, \dots \} \subseteq \mathbb{N}$$

$$\text{Square} = \{ 0, 1, 4, 9, \dots \} \subseteq \mathbb{N}$$

While FO talks of elements of  $\mathbb{N}$ , MSO talks of **subsets of  $\mathbb{N}$** .

These subsets can be encoded by **infinite words**:



While FO talks of elements of  $\mathbb{N}$ , MSO talks of **subsets** of  $\mathbb{N}$ .

These subsets can be encoded by **infinite words**:

$$\begin{array}{l} \text{Even} = \{ 0, 2, 4, 6, 8, \dots \} \subseteq \mathbb{N} \\ \text{[Even]} \otimes \text{[Squares]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dots \\ \text{Square} = \{ 0, 1, 4, 9, \dots \} \subseteq \mathbb{N} \end{array}$$

👉 Accordingly, a language  $L \subseteq \mathbb{B}^\omega$  encodes a **set of subsets** of  $\mathbb{N}$ .

## Definition

A **Büchi automaton** is a tuple  $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ , where

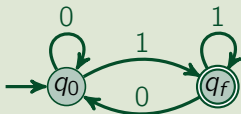
- $Q$  is a finite set of control states
- $\Sigma$  is a finite alphabet for transition labels
- $\Delta \subseteq Q \times \Sigma \times Q$  is a finite set of transition rules
- $I \subseteq Q$  is a set of initial states
- $F \subseteq Q$  is a set of final states

$\mathcal{A}$  **accepts** a word  $w \in \Sigma^\omega$  if it admits a run  $\rho$  on  $w$  such that

$$\text{inf}(\rho) \cap F \neq \emptyset$$

where  $\text{inf}(\rho) = \{q \in Q \mid \forall i. \exists j \geq i. \rho(j) = q\}$

## Example



$$\mathcal{L}(\mathcal{A}) = (0^* 1)^\omega$$

## Decidability of S1S (Büchi '60)

One can decide if a given sentence  $\psi$  of **MSO[+1]** holds over  $\mathbb{N}$ .

- 1 Replace first-order variables  $x$  with set variables  $X$  satisfying

$$\varphi_{\text{singleton}}(X) = (X \neq \emptyset) \wedge \forall Y. (Y \subseteq X) \rightarrow (Y = \emptyset \vee Y = X)$$

## Decidability of S1S (Büchi '60)

One can decide if a given sentence  $\psi$  of **MSO[+1]** holds over  $\mathbb{N}$ .

- 1 Replace first-order variables  $x$  with set variables  $X$  satisfying

$$\varphi_{\text{singleton}}(X) = (X \neq \emptyset) \wedge \forall Y. (Y \subseteq X) \rightarrow (Y = \emptyset \vee Y = X)$$

- 2 By induction on all subformulas  $\varphi(X_1, \dots, X_m)$  of  $\psi$ ,  
construct Büchi automata  $\mathcal{A}_\varphi$  over  $\Sigma_m = \mathbb{B}^m$  such that

$$\mathcal{L}(\mathcal{A}_\varphi) = \{ [X_1] \otimes \dots \otimes [X_m] \in \Sigma_m^\omega \mid (\mathbb{N}, +1) \models \varphi(X_1, \dots, X_m) \}$$

## Decidability of S1S (Büchi '60)

One can decide if a given sentence  $\psi$  of **MSO[+1]** holds over  $\mathbb{N}$ .

- 1 Replace first-order variables  $x$  with set variables  $X$  satisfying

$$\varphi_{\text{singleton}}(X) = (X \neq \emptyset) \wedge \forall Y. (Y \subseteq X) \rightarrow (Y = \emptyset \vee Y = X)$$

- 2 By induction on all subformulas  $\varphi(X_1, \dots, X_m)$  of  $\psi$ , construct Büchi automata  $\mathcal{A}_\varphi$  over  $\Sigma_m = \mathbb{B}^m$  such that

$$\mathcal{L}(\mathcal{A}_\varphi) = \{ [X_1] \otimes \dots \otimes [X_m] \in \Sigma_m^\omega \mid (\mathbb{N}, +1) \models \varphi(X_1, \dots, X_m) \}$$

$$\varphi(X, Y) = (Y = X + 1)$$





## Decidability of S1S (Büchi '60)

One can decide if a given sentence  $\psi$  of **MSO[+1]** holds over  $\mathbb{N}$ .

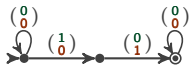
- 1 Replace first-order variables  $x$  with set variables  $X$  satisfying

$$\varphi_{\text{singleton}}(X) = (X \neq \emptyset) \wedge \forall Y. (Y \subseteq X) \rightarrow (Y = \emptyset \vee Y = X)$$

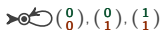
- 2 By induction on all subformulas  $\varphi(X_1, \dots, X_m)$  of  $\psi$ , construct Büchi automata  $\mathcal{A}_\varphi$  over  $\Sigma_m = \mathbb{B}^m$  such that

$$\mathcal{L}(\mathcal{A}_\varphi) = \{ [X_1] \otimes \dots \otimes [X_m] \in \Sigma_m^\omega \mid (\mathbb{N}, +1) \models \varphi(X_1, \dots, X_m) \}$$

$$\varphi(X, Y) = (Y = X + 1)$$



$$\varphi(X, Y) = (X \subseteq Y)$$



## Decidability of S1S (Büchi '60)

One can decide if a given sentence  $\psi$  of **MSO[+1]** holds over  $\mathbb{N}$ .

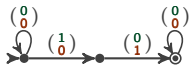
- 1 Replace first-order variables  $x$  with set variables  $X$  satisfying

$$\varphi_{\text{singleton}}(X) = (X \neq \emptyset) \wedge \forall Y. (Y \subseteq X) \rightarrow (Y = \emptyset \vee Y = X)$$

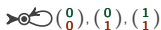
- 2 By induction on all subformulas  $\varphi(X_1, \dots, X_m)$  of  $\psi$ , construct Büchi automata  $\mathcal{A}_\varphi$  over  $\Sigma_m = \mathbb{B}^m$  such that

$$\mathcal{L}(\mathcal{A}_\varphi) = \{ [X_1] \otimes \dots \otimes [X_m] \in \Sigma_m^\omega \mid (\mathbb{N}, +1) \models \varphi(X_1, \dots, X_m) \}$$

$$\varphi(X, Y) = (Y = X + 1)$$



$$\varphi(X, Y) = (X \subseteq Y)$$



$$\varphi(\bar{X}) = \varphi_1(\bar{X}) \vee \varphi_2(\bar{X})$$

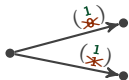
union

$$\varphi(\bar{X}) = \neg \varphi_1(\bar{X})$$

complementation

$$\varphi(\bar{X}) = \exists Y. \varphi_1(\bar{X}, Y)$$

projection



## Decidability of S1S (Büchi '60)

One can decide if a given sentence  $\psi$  of **MSO[+1]** holds over  $\mathbb{N}$ .

- 1 Replace first-order variables  $x$  with set variables  $X$  satisfying

$$\varphi_{\text{singleton}}(X) = (X \neq \emptyset) \wedge \forall Y. (Y \subseteq X) \rightarrow (Y = \emptyset \vee Y = X)$$

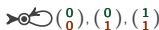
- 2 By induction on all subformulas  $\varphi(X_1, \dots, X_m)$  of  $\psi$ , construct Büchi automata  $\mathcal{A}_\varphi$  over  $\Sigma_m = \mathbb{B}^m$  such that

$$\mathcal{L}(\mathcal{A}_\varphi) = \{ [X_1] \otimes \dots \otimes [X_m] \in \Sigma_m^\omega \mid (\mathbb{N}, +1) \models \varphi(X_1, \dots, X_m) \}$$

$$\varphi(X, Y) = (Y = X + 1)$$



$$\varphi(X, Y) = (X \subseteq Y)$$



$$\varphi(\bar{X}) = \varphi_1(\bar{X}) \vee \varphi_2(\bar{X})$$

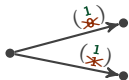
union

$$\varphi(\bar{X}) = \neg \varphi_1(\bar{X})$$

complementation

$$\varphi(\bar{X}) = \exists Y. \varphi_1(\bar{X}, Y)$$

projection



A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathcal{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$\mathcal{L}_{p,q} = -\infty$	if there is no path from $p$ to $q$
$\mathcal{L}_{p,q} = 1$	if there is a path from $p$ to $q$ that visits a final state
$\mathcal{L}_{p,q} = 0$	otherwise

A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathcal{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$\mathcal{L}_{p,q} = -\infty$	if there is no path from $p$ to $q$
$\mathcal{L}_{p,q} = 1$	if there is a path from $p$ to $q$ that visits a final state
$\mathcal{L}_{p,q} = 0$	otherwise

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathcal{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\begin{array}{ll} \mathcal{L}_{p,q} = -\infty & \text{if there is no path from } p \text{ to } q \\ \mathcal{L}_{p,q} = 1 & \text{if there is a path from } p \text{ to } q \\ & \text{that visits a final state} \\ \mathcal{L}_{p,q} = 0 & \text{otherwise} \end{array}$$

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel_f \quad \parallel_f \quad \parallel_f \quad \quad \quad \downarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad \quad \quad w' \in \mathcal{L}(\mathcal{A})$

A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$\mathcal{L}_{p,q} = -\infty$  if there is no path from  $p$  to  $q$   
 $\mathcal{L}_{p,q} = 1$  if there is a path from  $p$  to  $q$  that visits a final state  
 $\mathcal{L}_{p,q} = 0$  otherwise

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel_f \quad \parallel_f \quad \parallel_f \quad \quad \quad \updownarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad \quad \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_2) = f(w_2) = \dots$



A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\mathcal{L}_{p,q} = -\infty \quad \text{if there is no path from } p \text{ to } q$$

$$\mathcal{L}_{p,q} = 1 \quad \text{if there is a path from } p \text{ to } q \text{ that visits a final state}$$

$$\mathcal{L}_{p,q} = 0 \quad \text{otherwise}$$

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel_f \quad \parallel_f \quad \parallel_f \quad \quad \quad \updownarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_1) = f(w_2) = \dots$





A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\mathcal{L}_{p,q} = -\infty \quad \text{if there is no path from } p \text{ to } q$$

$$\mathcal{L}_{p,q} = 1 \quad \text{if there is a path from } p \text{ to } q \text{ that visits a final state}$$

$$\mathcal{L}_{p,q} = 0 \quad \text{otherwise}$$

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel f \quad \parallel f \quad \parallel f \quad \quad \quad \downarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_1) = f(w_2) = \dots$



A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\begin{array}{ll} \mathcal{L}_{p,q} = -\infty & \text{if there is no path from } p \text{ to } q \\ \mathcal{L}_{p,q} = 1 & \text{if there is a path from } p \text{ to } q \\ & \text{that visits a final state} \\ \mathcal{L}_{p,q} = 0 & \text{otherwise} \end{array}$$

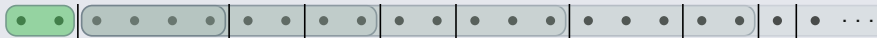
①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel_f \quad \parallel_f \quad \parallel_f \quad \quad \quad \updownarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad \quad \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_2) = f(w_2) = \dots$



A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\mathcal{L}_{p,q} = -\infty \quad \text{if there is no path from } p \text{ to } q$$

$$\mathcal{L}_{p,q} = 1 \quad \text{if there is a path from } p \text{ to } q \text{ that visits a final state}$$

$$\mathcal{L}_{p,q} = 0 \quad \text{otherwise}$$

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel_f \quad \parallel_f \quad \parallel_f \quad \quad \quad \updownarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_2) = f(w_2) = \dots$



A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\mathcal{L}_{p,q} = -\infty \quad \text{if there is no path from } p \text{ to } q$$

$$\mathcal{L}_{p,q} = 1 \quad \text{if there is a path from } p \text{ to } q \text{ that visits a final state}$$

$$\mathcal{L}_{p,q} = 0 \quad \text{otherwise}$$

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel_f \quad \parallel_f \quad \parallel_f \quad \quad \quad \updownarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_2) = f(w_2) = \dots$



A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\mathcal{L}_{p,q} = -\infty \quad \text{if there is no path from } p \text{ to } q$$

$$\mathcal{L}_{p,q} = 1 \quad \text{if there is a path from } p \text{ to } q \text{ that visits a final state}$$

$$\mathcal{L}_{p,q} = 0 \quad \text{otherwise}$$

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel_f \quad \parallel_f \quad \parallel_f \quad \quad \quad \updownarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad \quad \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_2) = f(w_2) = \dots$



A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$\mathcal{L}_{p,q} = -\infty$  if there is no path from  $p$  to  $q$   
 $\mathcal{L}_{p,q} = 1$  if there is a path from  $p$  to  $q$  that visits a final state  
 $\mathcal{L}_{p,q} = 0$  otherwise

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel_f \quad \parallel_f \quad \parallel_f \quad \quad \quad \updownarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad \quad \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_2) = f(w_2) = \dots$



A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\mathcal{L}_{p,q} = -\infty \quad \text{if there is no path from } p \text{ to } q$$

$$\mathcal{L}_{p,q} = 1 \quad \text{if there is a path from } p \text{ to } q \text{ that visits a final state}$$

$$\mathcal{L}_{p,q} = 0 \quad \text{otherwise}$$

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel_f \quad \parallel_f \quad \parallel_f \quad \quad \quad \updownarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_2) = f(w_2) = \dots$



A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{\mathbb{Q}^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\begin{array}{ll} \mathcal{L}_{p,q} = -\infty & \text{if there is no path from } p \text{ to } q \\ \mathcal{L}_{p,q} = 1 & \text{if there is a path from } p \text{ to } q \\ & \text{that visits a final state} \\ \mathcal{L}_{p,q} = 0 & \text{otherwise} \end{array}$$

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

②  $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots \Rightarrow w \in \mathcal{L}(\mathcal{A})$   
 $\quad \quad \quad \parallel f \quad \parallel f \quad \parallel f \quad \quad \quad \updownarrow$   
 $w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots \quad \quad \quad w' \in \mathcal{L}(\mathcal{A})$

③  $\mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

Proof:  $\supseteq$  straightforward,  $\subseteq$  follows from Ramsey's Theorem

Every word can be factorized into  $w_1, w_2, w_3, \dots$  with  $f(w_2) = f(w_2) = \dots$





A Büchi automaton  $\mathcal{A}$  induces a function  $f : \Sigma^* \rightarrow \{-\infty, 0, 1\}^{Q^2}$  such that

$$f(w) = \rightarrow_p \left[ \begin{array}{c} \downarrow_q \\ \mathcal{L}_{p,q} \end{array} \right]$$

$$\begin{array}{ll} \mathcal{L}_{p,q} = -\infty & \text{if there is no path from } p \text{ to } q \\ \mathcal{L}_{p,q} = 1 & \text{if there is a path from } p \text{ to } q \\ & \text{that visits a final state} \\ \mathcal{L}_{p,q} = 0 & \text{otherwise} \end{array}$$

①  $f$  is **compositional**:  $f(w_1 \cdot w_2) = f(w_1) \times f(w_2)$

$$\begin{array}{ccc} \textcircled{2} & w = w_1 \cdot w_2 \cdot w_3 \cdot \dots & \Rightarrow & w \in \mathcal{L}(\mathcal{A}) \\ & \parallel_f \quad \parallel_f \quad \parallel_f & & \downarrow \\ & w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots & & w' \in \mathcal{L}(\mathcal{A}) \end{array}$$


$$\textcircled{3} \quad \mathcal{L}(\mathcal{A}) = \bigcup_{\mathcal{L}(\mathcal{A}) \ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$$

**Complementation:**  $\mathcal{L}(\mathcal{A})^c = \bigcup_{\mathcal{L}(\mathcal{A}) \not\ni f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$

## A converse translation: from automata to logic

One can translate every Büchi automaton  $\mathcal{A}$  into a formula  $\psi_{\mathcal{A}} = \exists \bar{X}. \varphi(\bar{X})$ , where  $\varphi$  is a first-order formula, such that

$$\mathcal{L}(\mathcal{A}) = \{w \in \Sigma^\omega \mid w \models \psi_{\mathcal{A}}\}$$


 Encode an accepting run of  $\mathcal{A}$  into monadic variables  $\bar{X}$ :

$w \in \mathcal{L}(\mathcal{A})$  iff  $\exists \rho$  *accepting run of  $\mathcal{A}$  on  $w$*   
iff  $\exists (X_t)_{t \in \Delta}$ . *exactly one transition on each position*  
 $\wedge$  *transitions respect symbols of  $w$*   
 $\wedge$  *consecutive transitions agree on states*  
 $\wedge$  *first transition departs from initial state*  
 $\wedge$  *some final state is visited infinitely often*

## A converse translation: from automata to logic

One can translate every Büchi automaton  $\mathcal{A}$  into a formula  $\psi_{\mathcal{A}} = \exists \bar{X}. \varphi(\bar{X})$ , where  $\varphi$  is a first-order formula, such that

$$\mathcal{L}(\mathcal{A}) = \{w \in \Sigma^\omega \mid w \models \psi_{\mathcal{A}}\}$$

 Encode an accepting run of  $\mathcal{A}$  into monadic variables  $\bar{X}$ :

$w \in \mathcal{L}(\mathcal{A})$  iff  $\exists \rho$  *accepting run of  $\mathcal{A}$  on  $w$*   
iff  $\exists (X_t)_{t \in \Delta}$ . *exactly one transition on each position*  
 $\wedge$  *transitions respect symbols of  $w$*   
 $\wedge$  *consecutive transitions agree on states*  
 $\wedge$  *first transition departs from initial state*  
 $\wedge$  *some final state is visited infinitely often*

## Corollary (collapse of quantifier hierarchy)

$$\text{MSO}[\langle \rangle] = \text{Büchi automata} = \exists \text{MSO}[+1]$$

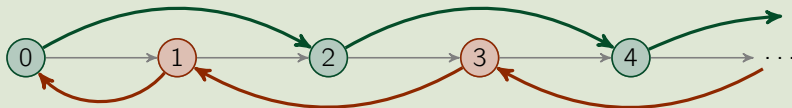
## Application example 1 (interpretation of the integers)



One can **logically define**  $\mathbb{Z}$  inside  $\mathbb{N}$ :

$$\begin{aligned}\varphi_{\leq \mathbb{Z}}(x, y) &= (\text{Even}(x) \wedge \text{Even}(y) \wedge x \leq_{\mathbb{N}} y) \\ &\vee (\text{Odd}(x) \wedge \text{Odd}(y) \wedge y \leq_{\mathbb{N}} x) \\ &\vee (\text{Odd}(x) \wedge \text{Even}(y))\end{aligned}$$

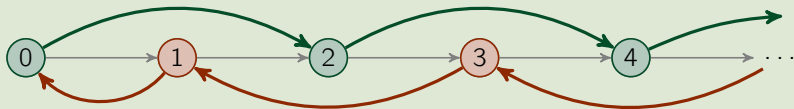
## Application example 1 (interpretation of the integers)



One can **logically define**  $\mathbb{Z}$  inside  $\mathbb{N}$ :

$$\begin{aligned}\varphi_{\leq \mathbb{Z}}(x, y) &= (\text{Even}(x) \wedge \text{Even}(y) \wedge x \leq_{\mathbb{N}} y) \\ &\vee (\text{Odd}(x) \wedge \text{Odd}(y) \wedge y \leq_{\mathbb{N}} x) \\ &\vee (\text{Odd}(x) \wedge \text{Even}(y))\end{aligned}$$

## Application example 1 (interpretation of the integers)



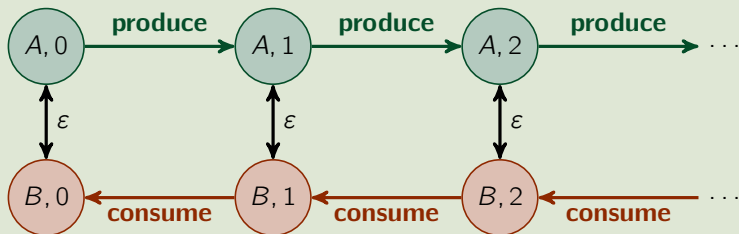
One can logically define  $\mathbb{Z}$  inside  $\mathbb{N}$ :

$$\begin{aligned}\varphi_{\leq \mathbb{Z}}(x, y) &= (\text{Even}(x) \wedge \text{Even}(y) \wedge x \leq_{\mathbb{N}} y) \\ &\vee (\text{Odd}(x) \wedge \text{Odd}(y) \wedge y \leq_{\mathbb{N}} x) \\ &\vee (\text{Odd}(x) \wedge \text{Even}(y))\end{aligned}$$

### Corollary

The  $\text{MSO}[\leq]$  theory of  $\mathbb{Z}$  is decidable.

## Application example 2 (interpretation of a counter system)



Any property of the above system expressed by an MSO formula

$$\psi = \dots \forall X \dots \left( y \xleftarrow{\text{consume}} z \right) \dots \left( y \updownarrow_{\epsilon} z \right) \dots$$

can be translated into an **equi-satisfiable** formula over  $(\mathbb{N}, +1)$

$$\hat{\psi} = \dots \forall X_1, X_2 \dots \left( y_2 + 1 = z_2 \right) \dots \left( y_1 = z_2 \vee y_2 = z_1 \right) \dots$$

and then checked for validity.

### Application example 3 (expanded theories)

Recall inductive invariant:  $\mathcal{L}(\mathcal{A}_\varphi) = \{[\bar{X}] \in \Sigma_m^\omega \mid \mathbb{N} \models \varphi(\bar{X})\}$

and recall  $f$ -equivalence on factors of infinite words:

$$\begin{array}{ccc} W = W_1 \cdot W_2 \cdot W_3 \cdot \dots & \Rightarrow & W \in \mathcal{L}(\mathcal{A}) \\ \parallel_f \quad \parallel_f \quad \parallel_f & & \downarrow \\ W' = W'_1 \cdot W'_2 \cdot W'_3 \cdot \dots & & W' \in \mathcal{L}(\mathcal{A}) \end{array}$$

---



### Application example 3 (expanded theories)

Recall inductive invariant:  $\mathcal{L}(\mathcal{A}_\varphi) = \{[\bar{X}] \in \Sigma_m^\omega \mid \mathbb{N} \models \varphi(\bar{X})\}$

and recall  $f$ -equivalence on factors of infinite words:

$$\begin{array}{ccc} w = w_1 \cdot w_2 \cdot w_3 \cdot \dots & \Rightarrow & w \in \mathcal{L}(\mathcal{A}) \\ \parallel_f \quad \parallel_f \quad \parallel_f & & \downarrow \\ w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots & & w' \in \mathcal{L}(\mathcal{A}) \end{array}$$

We have

$$(\mathbb{N}, +1, \text{Squares}) \models \varphi$$

$$\text{iff } w = 1 \ 1001000010000001 \ \dots \in \mathcal{L}(\mathcal{A}_\varphi)$$

### Application example 3 (expanded theories)

Recall inductive invariant:  $\mathcal{L}(\mathcal{A}_\varphi) = \{[\bar{X}] \in \Sigma_m^\omega \mid \mathbb{N} \models \varphi(\bar{X})\}$

and recall  $f$ -equivalence on factors of infinite words:

$$\begin{array}{ccc} w = w_1 \cdot w_2 \cdot w_3 \cdot \dots & \Rightarrow & w \in \mathcal{L}(\mathcal{A}) \\ \parallel_f \quad \parallel_f \quad \parallel_f & & \downarrow \\ w' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots & & w' \in \mathcal{L}(\mathcal{A}) \end{array}$$

We have

$$(\mathbb{N}, +1, \text{Squares}) \models \varphi$$

$$\text{iff } w = 1 \underbrace{100}_{w_0} \underbrace{10000}_{w_1} \underbrace{1000000}_{w_2} \underbrace{10000000}_{w_3} \dots \in \mathcal{L}(\mathcal{A}_\varphi) \quad \text{where } w_n = (00)^n$$

## Application example 3 (expanded theories)

Recall inductive invariant:  $\mathcal{L}(\mathcal{A}_\varphi) = \{[\bar{X}] \in \Sigma_m^\omega \mid \mathbb{N} \models \varphi(\bar{X})\}$

and recall  $f$ -equivalence on factors of infinite words:

$$\begin{array}{rcl}
 W = W_1 \cdot W_2 \cdot W_3 \cdot \dots & \Rightarrow & W \in \mathcal{L}(\mathcal{A}) \\
 \parallel_f \quad \parallel_f \quad \parallel_f & & \downarrow \\
 W' = W'_1 \cdot W'_2 \cdot W'_3 \cdot \dots & & W' \in \mathcal{L}(\mathcal{A})
 \end{array}$$

We have

$$(\mathbb{N}, +1, \text{Squares}) \models \varphi$$

$$\text{iff } w = 1 \underbrace{100}_{w_0} \underbrace{10000}_{w_1} \underbrace{1000000}_{w_2} \underbrace{10000000}_{w_3} \dots \in \mathcal{L}(\mathcal{A}_\varphi)$$

where  $w_n = (00)^n$

since  $f(w_n) = f(w_{n+\ell})$  for some  $\ell > 0$  and all  $n > n_0$

## Application example 3 (expanded theories)

Recall inductive invariant:  $\mathcal{L}(\mathcal{A}_\varphi) = \{[\bar{X}] \in \Sigma_m^\omega \mid \mathbb{N} \models \varphi(\bar{X})\}$

and recall  $f$ -equivalence on factors of infinite words:

$$\begin{array}{rcl}
 W = w_1 \cdot w_2 \cdot w_3 \cdot \dots & \Rightarrow & W \in \mathcal{L}(\mathcal{A}) \\
 \parallel_f \quad \parallel_f \quad \parallel_f & & \downarrow \\
 W' = w'_1 \cdot w'_2 \cdot w'_3 \cdot \dots & & W' \in \mathcal{L}(\mathcal{A})
 \end{array}$$

We have

$$(\mathbb{N}, +1, \text{Squares}) \models \varphi$$

$$\text{iff } w = 1 \underbrace{100}_{w_0} \underbrace{10000}_{w_1} \underbrace{1000000}_{w_2} \underbrace{10000000}_{w_3} 1 \dots \in \mathcal{L}(\mathcal{A}_\varphi)$$

where  $w_n = (00)^n$

$$\text{iff } w' = (1w_0 \dots 1w_{n_0}) \cdot (1w_{n_0+1} \dots 1w_{n_0+\ell})^\omega \in \mathcal{L}(\mathcal{A}_\varphi)$$

since  $f(w_n) = f(w_{n+\ell})$  for some  $\ell > 0$  and all  $n > n_0$

## Contraction method (Elgot & Rabin '66)

Let  $P$  be a subset of  $\mathbb{N}$  that is (effectively) **profinately periodic**  
i.e.  $[P] = w_0 w_1 w_2 \dots$  and for every *semigroup morphism*  $f$   
the series  $f(w_0) f(w_1) f(w_2) \dots$  is (effectively) periodic.

Then one can decide whether  $(\mathbb{N}, +, 1, P) \models \psi$ .

## Examples of effectively profinitely periodic subsets

- Squares =  $\{n^2 \mid n \in \mathbb{N}\}$
- Powers =  $\{2^n \mid n \in \mathbb{N}\}$
- Factorials =  $\{n! \mid n \in \mathbb{N}\}$
- Fibonacci =  $\{0, 1, 2, 3, 5, 8, 11, \dots\}$
- *basically all recursive series defined with  $+$  and  $\cdot$*

## Contraction method (Elgot & Rabin '66)

Let  $P$  be a subset of  $\mathbb{N}$  that is (effectively) **profinely periodic**

i.e.  $[P] = w_0 w_1 w_2 \dots$  and for every *semigroup morphism*  $f$   
the series  $f(w_0) f(w_1) f(w_2) \dots$  is (effectively) periodic.

Then one can decide whether  $(\mathbb{N}, +1, P) \models \psi$ .

## Examples of effectively profinitely periodic subsets

- Squares =  $\{n^2 \mid n \in \mathbb{N}\}$
- Powers =  $\{2^n \mid n \in \mathbb{N}\}$
- Factorials =  $\{n! \mid n \in \mathbb{N}\}$
- Fibonacci =  $\{0, 1, 2, 3, 5, 8, 11, \dots\}$
- *basically all recursive series defined with  $+$  and  $\cdot$*



## Composition method (Shelah '75)

$m$  = number of free variables

$k$  = number of nested quantifiers

### Definition

Let  $g_m^k$  map words  $w$  over  $\Sigma_m = \{0, 1\}^m$  to **logical types**:

$$g_m^k(w) = \{ \varphi(X_1, \dots, X_m) \text{ with } k \text{ nested quantifiers such that } w \models \varphi \}$$

Example:  $g_2^0\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \{ X_1 \subseteq X_2, \dots \}$

## Composition method (Shelah '75)

$m$  = number of free variables

$k$  = number of nested quantifiers

### Definition

Let  $g_m^k$  map words  $w$  over  $\Sigma_m = \{0, 1\}^m$  to **logical types**:

$$g_m^k(w) = \{ \varphi(X_1, \dots, X_m) \text{ with } k \text{ nested quantifiers such that } w \models \varphi \}$$

Example:  $g_2^0\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \{ X_1 \subseteq X_2, \dots \}$

 Up to logical equivalence (e.g.  $\neg\wedge = \vee\neg$ ,  $\neg\forall = \exists\neg$ ,  $\exists\vee = \vee\exists$ )

$$g_m^0(w) = \left\{ \begin{array}{l} X_i \subseteq X_j \mid \forall n. w(n)[i] = 1 \rightarrow w(n)[j] = 1 \\ \cup \\ \neg X_i \subseteq X_j \mid \exists n. w(n)[i] = 1 \wedge w(n)[j] = 0 \end{array} \right\}$$

$$g_m^{k+1}(w) = \left\{ \begin{array}{l} \exists Y. \varphi(X_1, \dots, X_m, Y) \mid \\ Y \subseteq \text{dom}(w) \\ \varphi \in g_{m+1}^k(w \otimes Y) \end{array} \right\} \\ \cup \left\{ \begin{array}{l} \neg \exists Y. \varphi(X_1, \dots, X_m, Y) \mid \\ Y \subseteq \text{dom}(w) \\ \varphi \notin g_{m+1}^k(w \otimes Y) \end{array} \right\}$$



## Composition method (Shelah '75)

Like  $f$ , the type function  $g_m^k$  is **compositional**, namely

$$g_m^k(w_1 \cdot w_2) = g_m^k(w_1) \odot g_m^k(w_2)$$

## Composition method (Shelah '75)

Like  $f$ , the type function  $g_m^k$  is **compositional**, namely

$$g_m^k(w_1 \cdot w_2) = g_m^k(w_1) \odot g_m^k(w_2)$$



Given a formula  $\varphi \in g_m^k(w_1 \cdot w_2)$ , i.e. such that  $w_1 \cdot w_2 \models \varphi$ , separate syntactically the quantifications over  $w_1$  and over  $w_2$ :

## Composition method (Shelah '75)

Like  $f$ , the type function  $g_m^k$  is **compositional**, namely

$$g_m^k(w_1 \cdot w_2) = g_m^k(w_1) \odot g_m^k(w_2)$$



Given a formula  $\varphi \in g_m^k(w_1 \cdot w_2)$ , i.e. such that  $w_1 \cdot w_2 \models \varphi$ , separate syntactically the quantifications over  $w_1$  and over  $w_2$ :

### Example

$$\begin{aligned} \exists x. \exists y. x < y & \quad \mapsto \quad (\exists x_1. \exists y_1. x_1 < y_1) \vee (\exists x_2. \exists y_2. x_2 < y_2) \\ & \quad \vee (\exists x_1. \exists y_2. \text{true}) \quad \vee (\exists x_2. \exists y_1. \text{false}) \end{aligned}$$

$$\exists X. \exists Y. X \subseteq Y \quad \mapsto \quad (\exists X_1. \exists Y_1. X_1 \subseteq Y_1) \wedge (\exists X_2. \exists Y_2. X_2 \subseteq Y_2)$$

Transform  $\varphi$  into  $\bigvee_{i=1 \dots n} (\varphi_{i,1} \wedge \varphi_{i,2})$  in such a way that

$$w_1 \cdot w_2 \models \varphi \quad \text{iff} \quad w_1 \models \varphi_{i,1} \text{ and } w_2 \models \varphi_{i,2} \text{ for some } i = 1 \dots n$$

Accordingly, construct  $g_m^k(w_1 \cdot w_2)$  on the basis of  $g_m^k(w_1)$  and  $g_m^k(w_2)$ .

In a similar way, one can “compute” **types of  $\omega$ -products**:

$$g_m^k(w_1 \cdot w_2 \cdot w_3 \cdot \dots) = g_m^k(w_1) \odot g_m^k(w_2) \odot g_m^k(w_3) \odot \dots$$

In a similar way, one can “compute” **types of  $\omega$ -products**:

$$g_m^k(w_1 \cdot w_2 \cdot w_3 \cdot \dots) = g_m^k(w_1) \odot g_m^k(w_2) \odot g_m^k(w_3) \odot \dots$$

### Corollary

One can decide whether a formula  $\psi$  of  $\text{MSO}[\Sigma, <]$  holds over  $\mathbb{N}$ .

- 1 Start by computing types of singleton words  $a$ , for all  $a \in \Sigma$
- 2 Saturate by  $\odot$ : this gives all types of finite words!
- 3 Choose any two types  $\tau_1, \tau_2$  and compute  $\tau_1 \odot \tau_2^{(\omega)}$
- 4 Check if  $\psi \in \tau_1 \odot \tau_2^{(\omega)}$

In a similar way, one can “compute” **types of  $\omega$ -products**:

$$g_m^k(w_1 \cdot w_2 \cdot w_3 \cdot \dots) = g_m^k(w_1) \odot g_m^k(w_2) \odot g_m^k(w_3) \odot \dots$$

### Corollary

One can decide whether a formula  $\psi$  of  $\text{MSO}[\Sigma, <]$  holds over  $\mathbb{N}$ .

- 1 Start by computing types of singleton words  $a$ , for all  $a \in \Sigma$
- 2 Saturate by  $\odot$ : this gives all types of finite words!
- 3 Choose any two types  $\tau_1, \tau_2$  and compute  $\tau_1 \odot \tau_2^{(\omega)}$
- 4 Check if  $\psi \in \tau_1 \odot \tau_2^{(\omega)}$

 Ramsey's Theorem implies completeness of the above procedure!

## From Ramsey's Theorem to **Factorization Forests** (Simon '90)

Fix a semigroup morphism  $f : \Sigma^* \rightarrow (S, \odot)$  (e.g. logical types)  
and recall that every infinite word has a Ramseyan factorization, i.e.



with **idempotent** factors:  $f(w_2) = f(w_3) = f(w_2) \odot f(w_3) = \dots$

## From Ramsey's Theorem to **Factorization Forests** (Simon '90)

Fix a semigroup morphism  $f : \Sigma^* \rightarrow (S, \odot)$  (e.g. logical types)  
and recall that every ~~infinite~~ **finite** word has a Ramseyan factorization, i.e.



with **idempotent** factors:  $f(w_2) = f(w_3) = f(w_2) \odot f(w_3) = \dots$



## From Ramsey's Theorem to **Factorization Forests** (Simon '90)

Fix a semigroup morphism  $f : \Sigma^* \rightarrow (S, \odot)$  (e.g. logical types)  
and recall that every ~~infinite~~ word has a Ramseyan factorization, i.e.

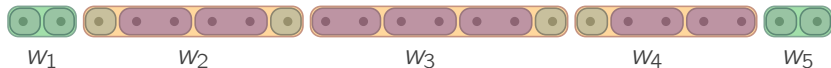


with **idempotent** factors:  $f(w_2) = f(w_3) = f(w_2) \odot f(w_3) = \dots$

- ⊕ the first and last factors are “**short**” w.r.t.  $w$   
(i.e. they do not contain proper factors with similar  $f$ -images)

## From Ramsey's Theorem to **Factorization Forests** (Simon '90)

Fix a semigroup morphism  $f : \Sigma^* \rightarrow (S, \odot)$  (e.g. logical types)  
and recall that every ~~infinite~~ word has a Ramseyan factorization, i.e.



with **idempotent** factors:  $f(w_2) = f(w_3) = f(w_2) \odot f(w_3) = \dots$

- ⊕ the first and last factors are “**short**” w.r.t.  $w$   
(i.e. they do not contain proper factors with similar  $f$ -images)

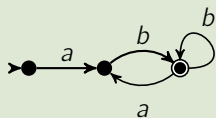


We can recursively factorize factors until we get single characters

- only a **few nested factorizations** (linear in  $|S|$ , independent of  $|w|$ )
- each factor is divided into a **few non-idempotent sub-factors**

## An example of factorization forest

Consider the automaton recognizing  $(ab^+)^+$  and the induced function  $f$ :



$$f(a) = \begin{bmatrix} -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & 1 & -\infty \end{bmatrix}$$

$$f(b) = \begin{bmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \\ -\infty & -\infty & 1 \end{bmatrix}$$

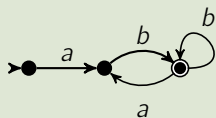
$$f(ab) = \begin{bmatrix} -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \end{bmatrix}$$

$$f(abab) = \begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & -\infty \end{bmatrix}$$

$w = a \quad b \quad b \dots b \quad a \quad b \quad b \dots b \quad \dots \quad a \quad b \quad b \dots b$

## An example of factorization forest

Consider the automaton recognizing  $(ab^+)^+$  and the induced function  $f$ :



$$f(a) = \begin{bmatrix} -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & 1 & -\infty \end{bmatrix}$$

$$f(b) = \begin{bmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \\ -\infty & -\infty & 1 \end{bmatrix}$$

$$f(ab) = \begin{bmatrix} -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \end{bmatrix}$$

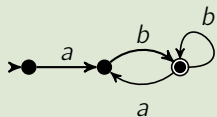
$$f(aba) = \begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & -\infty \end{bmatrix}$$

$$w = \begin{array}{cccccccccccc} \mathbf{a} & \mathbf{b} & \mathbf{b} & \dots & \mathbf{b} & \mathbf{a} & \mathbf{b} & \mathbf{b} & \dots & \mathbf{b} & \dots & \mathbf{a} & \mathbf{b} & \mathbf{b} & \dots & \mathbf{b} \\ | & | & | & & | & | & | & | & & | & & | & | & | & | & | \\ f(a) & f(b) & f(b) & & f(b) & f(a) & f(b) & f(b) & & f(b) & & f(a) & f(b) & f(b) & & f(b) \end{array}$$



## An example of factorization forest

Consider the automaton recognizing  $(ab^+)^+$  and the induced function  $f$ :

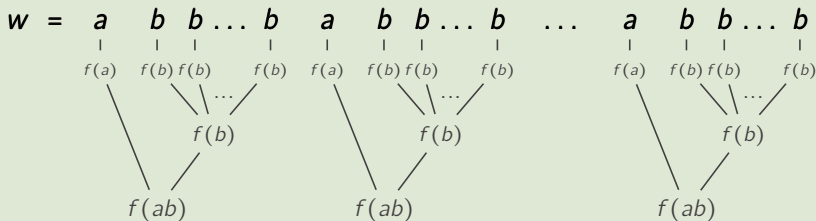


$$f(a) = \begin{bmatrix} -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & 1 & -\infty \end{bmatrix}$$

$$f(b) = \begin{bmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \\ -\infty & -\infty & 1 \end{bmatrix}$$

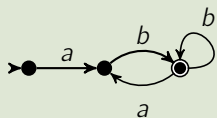
$$f(ab) = \begin{bmatrix} -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \end{bmatrix}$$

$$f(aba) = \begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & -\infty \end{bmatrix}$$



## An example of factorization forest

Consider the automaton recognizing  $(ab^+)^+$  and the induced function  $f$ :

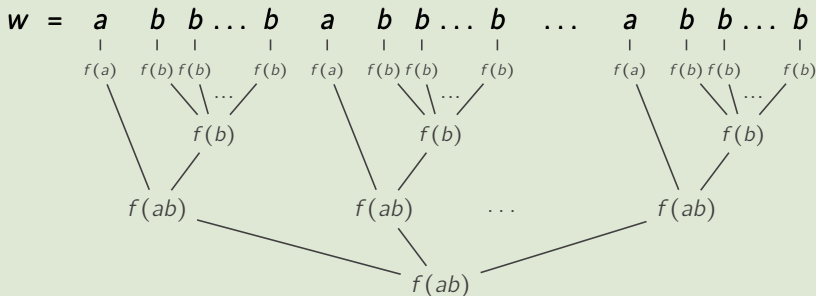


$$f(a) = \begin{bmatrix} -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & 1 & -\infty \end{bmatrix}$$

$$f(b) = \begin{bmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \\ -\infty & -\infty & 1 \end{bmatrix}$$

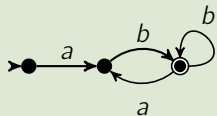
$$f(ab) = \begin{bmatrix} -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \end{bmatrix}$$

$$f(ab a) = \begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & -\infty \end{bmatrix}$$



## An example of factorization forest

Consider the automaton recognizing  $(ab^+)^+$  and the induced function  $f$ :

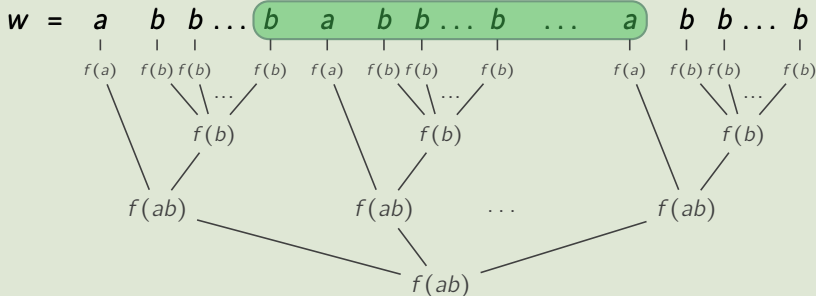


$$f(a) = \begin{bmatrix} -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & 1 & -\infty \end{bmatrix}$$

$$f(b) = \begin{bmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \\ -\infty & -\infty & 1 \end{bmatrix}$$

$$f(ab) = \begin{bmatrix} -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \end{bmatrix}$$

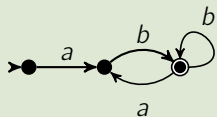
$$f(ab a) = \begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & -\infty \end{bmatrix}$$





## An example of factorization forest

Consider the automaton recognizing  $(ab^+)^+$  and the induced function  $f$ :

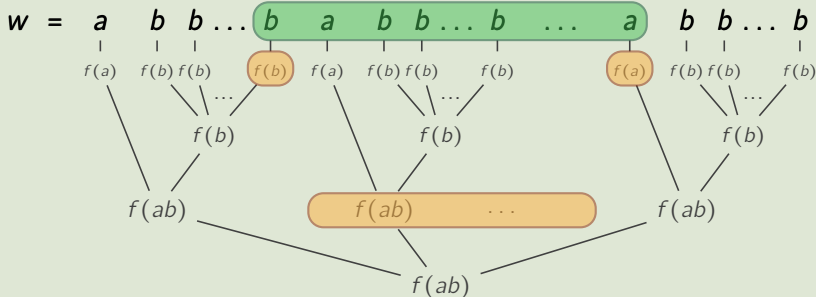


$$f(a) = \begin{bmatrix} -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & 1 & -\infty \end{bmatrix}$$

$$f(b) = \begin{bmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \\ -\infty & -\infty & 1 \end{bmatrix}$$

$$f(ab) = \begin{bmatrix} -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \end{bmatrix}$$

$$f(aba) = \begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & -\infty \end{bmatrix}$$





## An application: infix matching

For a fixed morphism  $f$  (or automaton, or formula), one can receive a word  $w$ , construct its factorization forest, and then use it as an **index structure** to evaluate in **constant time** the  $f$ -image of any given infix of  $w$ .

Other applications:

- constant-delay enumeration of answers to a query
- number of nested Kleene  $\star$  needed in a regular expression
- determinization of Büchi automata ( $\rightarrow$  parity automata)
- convert semigroups to formulas

▶ Next