

# Every planar graph without cycles of lengths 4 to 12 is acyclically 3-choosable

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## Abstract

An acyclic coloring of a graph  $G$  is a coloring of its vertices such that : (i) no two adjacent vertices in  $G$  receive the same color and (ii) no bicolored cycles exist in  $G$ . A list assignment of  $G$  is a function  $L$  that assigns to each vertex  $v \in V(G)$  a list  $L(v)$  of available colors. Let  $G$  be a graph and  $L$  be a list assignment of  $G$ . The graph  $G$  is acyclically  $L$ -list colorable if there exists an acyclic coloring  $\phi$  of  $G$  such that  $\phi(v) \in L(v)$  for all  $v \in V(G)$ . If  $G$  is acyclically  $L$ -list colorable for any list assignment  $L$  with  $|L(v)| \geq k$  for all  $v \in V(G)$ , then  $G$  is acyclically  $k$ -choosable. In this paper, we prove that every planar graph without cycles of lengths 4 to 12 is acyclically 3-choosable.

## 1 Introduction

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph such that two adjacent vertices do not use the same color. A  $k$ -*coloring* of  $G$  is a proper coloring of  $G$  using  $k$  colors; a graph admitting a  $k$ -coloring is said to be  $k$ -*colorable*. An *acyclic coloring* of a graph  $G$  is a proper coloring of  $G$  such that  $G$  contains no bicolored cycles; in other words, the graph induced by every two color classes is a forest. A list assignment of  $G$  is a function  $L$  that assigns to each vertex  $v \in V(G)$  a list  $L(v)$  of available colors. Let  $G$  be a graph and  $L$  be a list assignment of  $G$ . The graph  $G$  is *acyclically  $L$ -list colorable* if there is an acyclic coloring  $\phi$  of  $G$  such that  $\phi(v) \in L(v)$  for all  $v \in V(G)$ . If  $G$  is acyclically  $L$ -list colorable for any list assignment  $L$  with  $|L(v)| \geq k$  for all  $v \in V(G)$ , then  $G$  is *acyclically  $k$ -choosable*. The *acyclic choice number* of  $G$ ,  $\chi_a^l(G)$ , is the smallest integer  $k$  such that  $G$  is acyclically  $k$ -choosable. Borodin *et al.* [4] first investigated the acyclic choosability of planar graphs proving that:

**Theorem 1** [4] *Every planar graph is acyclically 7-choosable.*

and put forward to the following challenging conjecture:

**Conjecture 1** [4] *Every planar graph is acyclically 5-choosable.*

This conjecture if true strengthens Borodin's Theorem [1] on the acyclic 5-colorability of planar graphs and Thomassen's Theorem [12] on the 5-choosability of planar graphs.

In 1976, Steinberg conjectured that every planar graph without cycles of lengths 4 and 5 is 3-colorable (see Problem 2.9 [8]). This problem remains open. In 1990, Erdős suggested the following relaxation of Steinberg's Conjecture: what is the smallest integer  $i$  such that every planar graph without cycles of lengths 4 to  $i$  is 3-colorable? The best known result is  $i = 7$  [5]. This question is also studied in the choosability case: what is the smallest integer  $i$  such that every planar graph without cycles of lengths 4 to  $i$  is 3-choosable? In [13], Voigt proved that Steinberg's Conjecture can not be extended to list coloring; hence,  $i \geq 6$ . Nevertheless, in 1996, Borodin [2] proved that every planar graph without cycles of lengths 4 to 9 is 3-colorable; in fact, 3-choosable. So,  $i \leq 9$ .

In this paper, we study the question of Erdős in the acyclic choosability case:

**Problem 1** *What is the smallest integer  $i$  such that every planar graph without cycles of lengths 4 to  $i$  is acyclically 3-choosable?*

Note that it is proved that every planar graph without cycles of lengths 4 to 6 is acyclically 4-choosable [11]. Also, the relationship between the maximum average degree of  $G$  (or the girth of  $G$ ) and its acyclic choice number was studied (see for example [10, 9, 6]).

Our main result is the following:

**Theorem 2** *Every planar graph without cycles of lengths 4 to 12 is acyclically 3-choosable.*

Hence, in Problem 1,  $6 \leq i \leq 12$ .

Section 2 is dedicated to the proof of Theorem 2. First we list some notations we will use:

**Notations** Let  $G$  be a planar graph. We use  $V(G)$ ,  $E(G)$  and  $F(G)$  to denote the set of vertices, edges and faces of  $G$  respectively. Let  $d(v)$  denote the degree of a vertex  $v$  in  $G$  and  $r(f)$  the length of a face  $f$  in  $G$ . A vertex of degree  $k$  (resp. at least  $k$ , at most  $k$ ) is called a  $k$ -vertex (resp.  $\geq k$ -vertex,  $\leq k$ -vertex). We use the same notations for faces : a  $k$ -face (resp.  $\geq k$ -face,  $\leq k$ -face) is a face of length  $k$  (resp. at least  $k$ , at most  $k$ ). A  $k$ -face having the boundary vertices  $x_1, x_2, \dots, x_k$  in the cyclic order is denoted by  $[x_1x_2\dots x_k]$ . For a vertex  $v \in V(G)$ , let  $n_i(v)$  denote the number of  $i$ -vertices adjacent to  $v$  for  $i \geq 1$ , and  $m_3(v)$  the number of 3-faces incident to  $v$ . A 3-vertex is called a  $3^*$ -vertex if it is incident to a 3-face and adjacent to a 2-vertex (for example in Figure 1, the vertex  $t$  is a  $3^*$ -vertex). A 3-face  $[rst]$  with  $d(r) = d(s) = d(t) = 3$  and with a  $3^*$ -vertex on its boundary is called a  $3^*$ -face. Two 3-faces  $[rst]$  and  $[uvw]$  are called linked if there exists an edge  $tv$  which connects these two 3-faces such that  $d(t) = d(v) = 3$  (see Figure 2). A vertex  $v$  is linked to a 3-face  $[rst]$  if there exists an edge between  $v$  and one vertex of the boundary of  $[rst]$ , say  $t$ , such that  $d(t) = 3$  (for example in Figure 1, the vertex  $v$  is linked to the 3-face  $[rst]$ ). Let  $n^*(v)$  be the number of  $3^*$ -face linked to  $v$ .

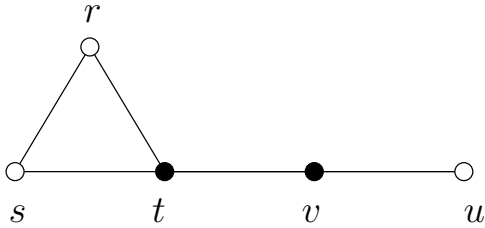


Figure 1: The vertex  $t$  is a  $3^*$ -vertex and the vertex  $v$  is linked to the 3-face  $[rst]$

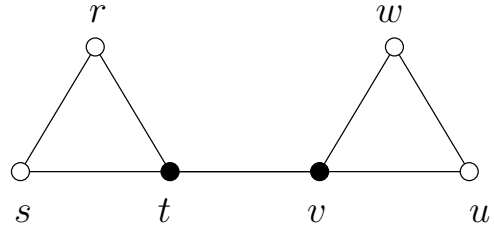


Figure 2: The two 3-faces  $[rst]$  and  $[uvw]$  are linked

## 2 Proof of Theorem 2

To prove Theorem 2 we proceed by contradiction. Suppose that  $H$  is a counterexample with the minimum order to Theorem 2 which is embedded in the plane. Let  $L$  be a list assignment with  $|L(v)| = 3$  for all  $v \in V(H)$  such that  $H$  does not admit an acyclic  $L$ -coloring.

Without loss of generality we can suppose that  $H$  is connected. We will first investigate the structural properties of  $H$  (Section 2.1), then using Euler's formula and the discharging technique we will derive a contradiction (Section 2.2).

## 2.1 Structural properties of $H$

**Lemma 1** *The minimal counterexample  $H$  to Theorem 2 has the following properties:*

(C1)  $H$  contains no 1-vertices.

(C2) A 3-face has no 2-vertex on its boundary.

(C3) A 2-vertex is not adjacent to a 2-vertex.

(C4) A 3-face has at most one  $3^*$ -vertex on its boundary.

(C5) A 3-face  $[rst]$  with  $d(r) = d(s) = d(t) = 3$  is linked to at most one  $3^*$ -face.

(C6) Two  $3^*$ -faces cannot be linked.

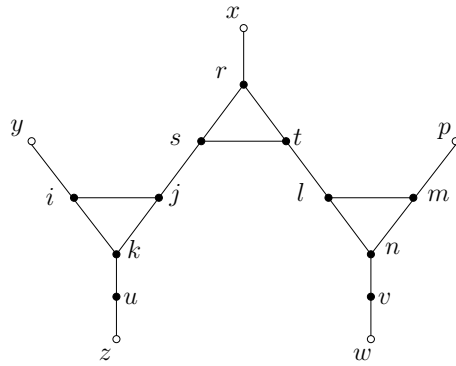


Figure 3:  $[rst]$  is linked to two  $3^*$ -faces  $[ijk]$  and  $[lmn]$

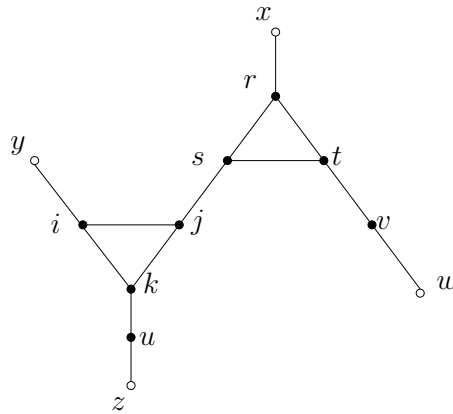


Figure 4: The two  $3^*$ -faces  $[rst]$  and  $[ijk]$  are linked

### Proof

(C1) Suppose  $H$  contains a 1-vertex  $u$  adjacent to a vertex  $v$ . By minimality of  $H$ , the graph  $H' = H \setminus \{u\}$  is acyclically 3-choosable. Consequently, there exists an acyclic  $L$ -coloring  $c$  of  $H'$ . To extend this coloring to  $H$  we just color  $u$  with  $c(u) \in L(u) \setminus \{c(v)\}$ . The obtained coloring is acyclic, a contradiction.

- (C2) Suppose  $H$  contains a 2-vertex  $u$  incident to a 3-face  $[uvw]$ . By minimality of  $H$ , the graph  $H' = H \setminus \{u\}$  is acyclically 3-choosable. Consequently, there exists an acyclic  $L$ -coloring  $c$  of  $H'$ . We show that we can extend this coloring to  $H$  by coloring  $u$  with  $c(u) \in L(u) \setminus \{c(v), c(w)\}$ .
- (C3) Suppose  $H$  contains a 2-vertex  $u$  adjacent to a 2-vertex  $v$ . Let  $t$  and  $w$  be the other neighbors of  $u$  and  $v$  respectively. By minimality of  $H$ , the graph  $H' = H \setminus \{u\}$  is acyclically 3-choosable. Consequently, there exists an acyclic  $L$ -coloring  $c$  of  $H'$ . We show that we can extend this coloring to  $H$ . Assume first that  $c(t) \neq c(v)$ . Then we just color  $u$  with  $c(u) \in L(u) \setminus \{c(t), c(v)\}$ . Now, if  $c(t) = c(v)$ , we color  $u$  with  $c(u) \in L(u) \setminus \{c(v), c(w)\}$ . In the two cases, the obtained coloring is acyclic, a contradiction.
- (C4) Suppose  $H$  contains a 3-face  $[rst]$  with two  $3^*$ -vertices  $s$  and  $t$ . Suppose that  $t$  (resp.  $s$ ) is adjacent to a 2-vertex  $v$  (resp.  $x$ ) with  $v \neq r, s$  by (C2) (resp.  $x \neq r, t$ ). Let  $u$  (resp.  $y$ ) be the other neighbor of  $v$  (resp.  $x$ ) with  $u \neq r, s$  (resp.  $y \neq r, t$ ). By the minimality of  $H$ ,  $H' = H \setminus \{v\}$  is acyclically 3-choosable. Consequently, there exists an acyclic  $L$ -coloring  $c$  of  $H'$ . We show now that we can extend  $c$  to  $H$ . If  $c(u) \neq c(t)$ , we color then  $v$  with a color different from  $c(u)$  and  $c(t)$  and the coloring obtained is acyclic. Otherwise,  $c(u) = c(t)$ . If we cannot color  $v$ , this implies without loss of generality  $L(v) = \{1, 2, 3\}$ ,  $c(u) = c(t) = c(x) = 1$ ,  $c(r) = 2$  and  $c(s) = c(y) = 3$ . Observe that necessarily  $L(t) = \{1, 2, 3\}$  (otherwise we can recolor  $t$  with  $\alpha \in L(t) \setminus \{1, 2, 3\}$  and color  $v$  properly *i.e*  $v$  receives a color distinct of those of these neighbors). For a same reason  $L(s) = \{1, 2, 3\}$  and  $L(x) = \{1, 2, 3\}$ . Now, we recolor  $t$  with the color 3,  $s$  with the color 1 and  $x$  with the color 2, then we can color  $v$  with the color 2. It is easy to see that the coloring obtained is acyclic.
- (C5) Suppose  $H$  contains a 3-face  $[rst]$  incident to three 3-vertices such that two of them are linked to two  $3^*$ -faces  $[ijk]$  and  $[lmn]$ . Suppose  $[ijk]$  and  $[lmn]$  are linked to  $[rst]$  respectively by the edges  $sj$  and  $tl$ . Call  $y$  the third neighbor of  $i$ ,  $x$  the third neighbor of  $r$ , and  $p$  the third neighbor of  $m$ . Suppose that the 2-vertex  $u$  (resp.  $v$ ) is adjacent to  $k$  and  $z$  (resp.  $n$  and  $w$ ). See Figure 3. By the minimality of  $H$ ,  $H' = H \setminus \{v\}$  is acyclically 3-choosable. Consequently, there exists an acyclic  $L$ -coloring  $c$  of  $H'$ . We show now that we can extend  $c$  to  $H$ . If  $c(w) \neq c(n)$ , we color then  $v$  with a color different from  $c(w)$  and  $c(n)$  and the coloring obtained is acyclic. Otherwise,  $c(w) = c(n)$ . If we cannot color  $v$ , this implies without loss of generality  $L(v) = \{1, 2, 3\} = L(l) = L(m)$ ,  $c(w) = c(n) = c(t) = c(p) = 1$ , and by permuting the colors of  $l$  and  $m$ , we are sure that  $L(r) = \{1, 2, 3\} = L(s)$  and  $c(x) = c(j) = 1$ , then by permuting the colors of  $r$  and  $s$ , we are sure that  $L(i) = \{1, 2, 3\} = L(k)$ ,  $c(y) = c(u) = 1$ , and  $c(z) \in \{2, 3\}$ . Let  $\alpha = \{2, 3\} \setminus \{c(z)\}$ . We recolor  $k, s, l, v$  with  $\alpha$  and  $m, r, i$  with  $c(z)$ . The coloring obtained is acyclic.
- (C6) Suppose  $H$  contains two  $3^*$ -faces  $[rst]$  and  $[ijk]$  which are linked by the edge  $sj$ . Call  $y$  the third neighbor of  $i$ ,  $x$  the third neighbor of  $r$ . Suppose that the 2-vertex  $u$  (resp.  $v$ ) is adjacent to  $k$  and  $z$  (resp.  $t$  and  $w$ ). See Figure 4. By the minimality of  $H$ ,  $H' = H \setminus \{v\}$  is acyclically 3-choosable. Consequently, there exists an acyclic  $L$ -coloring  $c$  of  $H'$ . We show now that we can extend  $c$  to  $H$ . If  $c(w) \neq c(t)$ , we color then  $v$  with a color different from  $c(w)$  and  $c(t)$  and the coloring obtained is acyclic. Otherwise,  $c(w) = c(t)$ . If we cannot color  $v$ , this implies without loss of generality  $L(v) = \{1, 2, 3\} = L(r) = L(s)$ ,  $c(w) = c(t) = c(x) = c(j) = 1$ , and by permuting the colors of  $r$  and  $s$ , we are sure that  $L(i) = \{1, 2, 3\} = L(k)$ ,  $c(y) = c(u) = 1$ , and  $c(z) \in \{2, 3\}$ . Let  $\alpha = \{2, 3\} \setminus \{c(z)\}$ . We recolor  $k, s, v$  with  $\alpha$  and  $r, i$  with  $c(z)$ . The coloring obtained is acyclic.

□

**Lemma 2** *Let  $H$  be a connected plane graph with  $n$  vertices,  $m$  edges and  $r$  faces. Then, we have the following:*

$$\sum_{v \in V(H)} (11d(v) - 26) + \sum_{f \in F(H)} (2r(f) - 26) = -52 \quad (1)$$

**Proof**

Euler's formula  $n - m + f = 2$  can be rewritten as  $(22m - 26n) + (4m - 26f) = -52$ . The relation  $\sum_{v \in V(H)} d(v) = \sum_{f \in F(H)} r(f) = 2m$  completes the proof.  $\square$

**2.2 Discharging procedure**

Let  $H$  be a counterexample to Theorem 2 with the minimum order. Then,  $H$  satisfies Lemma 1.

We define the weight function  $\omega : V(H) \cup F(H) \rightarrow \mathbb{R}$  by  $\omega(x) = 11d(x) - 26$  if  $x \in V(H)$  and  $\omega(x) = 2r(x) - 26$  if  $x \in F(H)$ . It follows from Equation (1) that the total sum of weights is equal to  $-52$ . In what follows, we will define discharging rules (R1) and (R2) and redistribute weights accordingly. Once the discharging is finished, a new weight function  $\omega^*$  is produced. However, the total sum of weights is kept fixed when the discharging is achieved. Nevertheless, we will show that  $\omega^*(x) \geq 0$  for all  $x \in V(H) \cup F(H)$ . This leads to the following obvious contradiction:

$$0 \leq \sum_{x \in V(H) \cup F(H)} \omega^*(x) = \sum_{x \in V(H) \cup F(H)} \omega(x) = -52 < 0$$

and hence demonstrates that no such counterexample can exist.

The discharging rules are defined as follows:

- (R1.1) Every  $\geq 3$ -vertex  $v$  gives 2 to each adjacent 2-vertex.
- (R1.2) Every  $\geq 4$ -vertex  $v$  gives 9 to each incident 3-face and 1 to each linked  $3^*$ -face.
- (R2.1) Every  $3^*$ -vertex  $v$  gives 5 to its incident 3-face.
- (R2.2) Every 3-vertex  $v$ , which is not a  $3^*$ -vertex and is not linked to a  $3^*$ -face, gives 7 to its incident 3-face (if any).
- (R2.3) Every 3-vertex  $v$ , which is a  $3^*$ -vertex and is linked to a  $3^*$ -face, gives 1 to each linked  $3^*$ -face and gives 6 to its incident 3-face (if any).

In order to complete the proof, it suffices to prove that the new weight  $\omega^*(x)$  is non-negative for all  $x \in V(H) \cup F(H)$ .

Let  $v \in V(H)$  be a  $k$ -vertex. Then,  $k \geq 2$  by (C1).

- If  $k = 2$ , then  $\omega(v) = -4$  and  $v$  is adjacent to two  $\geq 3$ -vertices by (C3). By (R1.1),  $\omega^*(v) = -4 + 2 \cdot 2 = 0$ .
- If  $k = 3$ , then  $\omega(v) = 7$ . Since  $H$  contains no 4-cycles,  $v$  is incident to at most one 3-face. Assume first that  $v$  is not incident to a 3-face. Then by (R1.1) and (R2.3),  $v$  gives at most 3 times 2. Hence,  $\omega^*(v) \geq 7 - 3 \cdot 2 \geq 1$ . Assume now that  $v$  is incident to a 3-face. If  $v$  is a  $3^*$ -vertex, then  $\omega^*(v) = 7 - 5 - 2 = 0$  by (R1.1) and (R2.1). If  $v$  is linked to a  $3^*$ -face then  $\omega^*(v) \geq 7 - 6 - 1 = 0$  by (R2.3). If  $v$  is not adjacent to a 2-vertex and not linked to a  $3^*$ -face then  $\omega^*(v) = 7 - 7 = 0$  by (R2.2).
- If  $k \geq 4$ , then  $\omega(v) = 11k - 26$ . Observe by (C1), (C2) and definitions of  $n^*(v)$  and of linked vertices that:

$$m_3(v) \leq \left\lfloor \frac{k}{2} \right\rfloor \quad \text{and} \quad k - 2m_3(v) \geq n_2(v) + n^*(v)$$

$$k \geq 2m_3(v) + n_2(v) + n^*(v) \tag{2}$$

It follows by (R1.1), (R1.2) and Equation (2) that:

$$\begin{aligned}
\omega^*(v) &= 11k - 26 - 9m_3(v) - n^*(v) - 2n_2(v) \\
&\geq 11k - 26 - 9m_3(v) - \frac{9}{2}n^*(v) - \frac{9}{2}n_2(v) \\
&\geq 11k - 26 - \frac{9}{2}k \\
&\geq \frac{13}{2}k - 26 \\
&\geq 0
\end{aligned}$$

Suppose that  $f$  is a  $k$ -face. Then,  $k = 3$  or  $k \geq 13$  by the hypothesis.

- If  $k \geq 13$ , then  $\omega^*(f) = \omega(f) = 2k - 26 \geq 0$ .
- If  $k = 3$ , then  $\omega(f) = -20$ . Suppose  $f = [rst]$ . By (C2),  $f$  is not incident to a 2-vertex; hence,  $d(r) \geq 3, d(s) \geq 3, d(t) \geq 3$ . By (C4)  $f$  is incident to at most one  $3^*$ -vertex. Now, observe that if one of the vertices  $r, s, t$  is a  $\geq 4$ -vertex, then by (R1.2) (R2.1) (R2.2) (R2.3)  $\omega^*(f) \geq -20 + 9 + 5 + 6 = 0$ . So assume  $d(r) = d(s) = d(t) = 3$  and let  $r_0, s_0, t_0$  be the other neighbors of  $r, s, t$ , respectively. Suppose that  $f$  is a  $3^*$ -face and let  $r$  be its unique  $3^*$ -vertex. By (C6) none of  $s$  and  $t$  are linked to a  $3^*$ -face. Moreover  $s_0$  and  $t_0$  give 1 to  $f$  by (R1.2) and (R2.3). Hence  $\omega^*(f) = -20 + 5 + 2 \cdot 7 + 2 \cdot 1 = 1$ . Finally assume that  $f$  is not a  $3^*$ -face. By (C5) at most one of  $r, s, t$  is linked to a  $3^*$ -face. Hence  $\omega^*(f) \geq -20 + 6 + 2 \cdot 7 = 0$ , by (R1.2), (R2.2) and (R2.3).

We proved that, for all  $x \in V(H) \cup F(H)$ ,  $\omega^*(x) \geq 0$ . This completes the proof of Theorem 2.

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