

Acyclic coloring of graphs with given maximum degree

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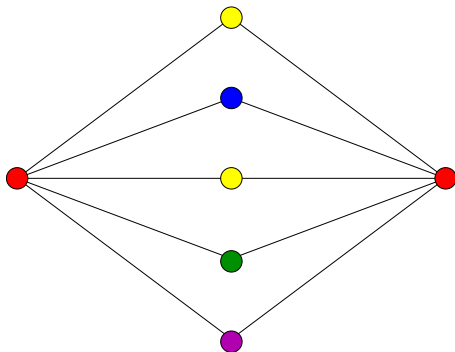
Proper k -coloring

A *proper k -coloring* of the vertices of a graph

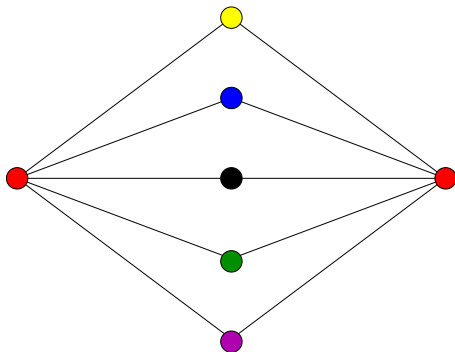
$G = (V(G), E(G))$ is a mapping $c : V(G) \rightarrow \{1, \dots, k\}$ such that $\forall uv \in E(G), c(u) \neq c(v)$.

Acyclic k -coloring

An *acyclic k -coloring* of a graph G is a proper k -coloring of G such that G contains no bicolored cycles ; in other words, the graph induced by every two color classes is a forest.



The coloring of G is not acyclic



The coloring of G is acyclic

Acyclic chromatic number

The *acyclic chromatic number* of a graph G , denoted by $\chi_a(G)$, is the smallest integer k such that G has an acyclic k -coloring.



B. Grünbaum.

Acyclic colorings of planar graphs.

Israel J. Math., (14):390-408, 1973.

Acyclic colorings were introduced by Grünbaum in 1973 who conjectured that 5 colors are sufficient to acyclically color any planar graph; this was confirmed by Borodin in 1979.

Theorem [O.V. Borodin '79]

Every planar graph is acyclically 5-colorable.



O.V. Borodin

On acyclic colorings of planar graphs.

Discrete Math., (25):211-236, 1979.

This bound is best possible since there exist 4-regular planar graphs which are not acyclically colorable with four colors.

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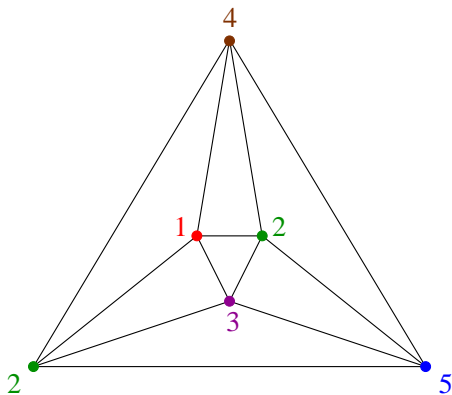
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Grünbaum's example



Graphs of bounded maximum degree

Motivation

As usual in the case of the coloring problems, one would like to bound $\chi_a(G)$ by a function of Δ ($f(\Delta)$).

Theorem [N. Alon, C. McDiarmid, B. Reed '91]

For large values of Δ , every graph is acyclically colorable with $O(\Delta^{4/3})$ colors. Moreover, there exist graphs with $\chi_a(G) = \Omega(\Delta^{4/3} / \log \Delta)$.

Both proofs use probabilistic method.

Graphs of bounded maximum degree

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Graphs of bounded maximum degree

Remark

$$\chi_a(K_n) = \chi(K_n) = n = \Delta + 1,$$

$$\chi_a(K_{2n} \setminus \mathcal{M}) = 2n - 1 = \Delta + 1 \gg \chi(K_{2n} \setminus \mathcal{M}) = n \simeq \frac{\Delta}{2}$$

with \mathcal{M} a perfect matching.

Question

Can we find a concrete example of graph with $\chi_a(G) \geq \Delta + 2$?

Graphs of bounded maximum degree

Observation

For $\Delta = 0, 1, 2 : f(\Delta) = 1, 2, 3$ (easy).

Theorem [Grünbaum '73]

Every graph with $\Delta \leq 3$ is acyclically 4-colorable, and the bound is tight (because of K_4).



B. Grünbaum.

Acyclic colorings of planar graphs.

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Graphs of bounded maximum degree

Theorem [M.I. Burstein '79]

Every graph with $\Delta \leq 4$ is acyclically 5-colorable, and the bound is tight.

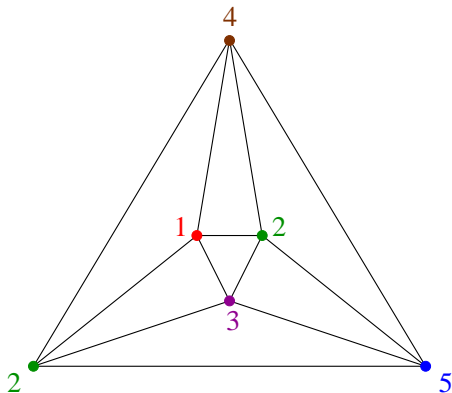


M.I. Burstein

Every 4-valent graph has an acyclic 5-coloring.

Soobšč. Akad. Nauk Gruzin SSR, (93):21-24, 1979 (in Russian).

Graphs of bounded maximum degree



$$K_6 \setminus \mathcal{M}$$

Graphs of bounded maximum degree

Theorem [G. Fertin, A. Raspaud '05]

Every graph, for any $\Delta \geq 6$, is acyclically $\left(\frac{\Delta(\Delta-1)}{2}\right)$ -colorable for even Δ and acyclically $\left(\frac{\Delta(\Delta-1)}{2} - 1\right)$ -colorable for odd Δ .



G. Fertin, A. Raspaud

Acyclic coloring of graphs of maximum degree Δ .

DMTCS proc. AE, 389-396, 2005.

Graphs of bounded maximum degree

Theorem [G. Fertin, A. Raspaud '05]

Every graph with $\Delta \leq 5$ is acyclically 9-colorable.



G. Fertin, A. Raspaud

Acyclic coloring of graphs of maximum degree five : nine colors are enough.

Inform. Process. Lett., (105):65-72, 2008.

Graphs of bounded maximum degree

Theorem [Yadav, Varagani, Kothapalli, Venkaiah '09]

Every graph with $\Delta \leq 5$ is acyclically 8-colorable.

Every graph with $\Delta \leq 6$ is acyclically 12-colorable.



K. Yadav, S. Varagani, K. Kothapalli, and V. Ch. Venkaiah
Acyclic Vertex Coloring of Graphs of Maximum Degree 5
International Conference on Graph Theory and its Applications, 2009.



K. Yadav, S. Varagani, K. Kothapalli, and V. Ch. Venkaiah
Acyclic Vertex Coloring of Graphs of Maximum Degree 6
V Latin-American Algorithms, Graphs, and Optimization Symposium (LAGOS), 2009.

Graphs of bounded maximum degree

Theorem [Yadav, Varagani, Kothapalli, Venkaiah '10]

Every graph is acyclically $\frac{3\Delta^2+4\Delta+8}{8}$ -colorable, for any $\Delta \geq 8$.



K. Yadav, S. Varagani, K. Kothapalli, and V. Ch. Venkaiah
Acyclic Vertex Coloring of Graphs of Maximum Degree Δ
Proc. of Indian Mathematical Society, 2009.

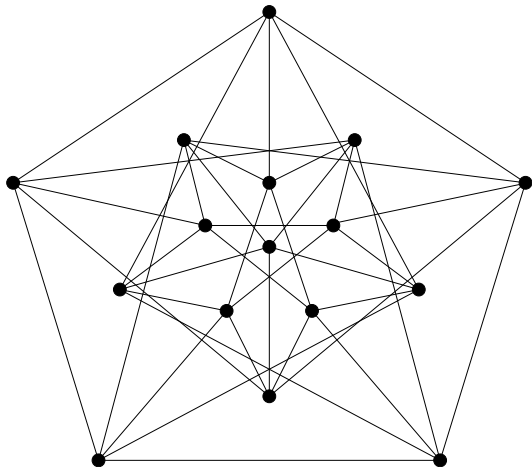
Graphs of bounded maximum degree

Theorem [Dieng, H., Naserasr '10]

$$\text{Let } f(\Delta) = \begin{cases} 8 & \text{if } \Delta = 5 \\ 11 & \text{if } \Delta = 6 \\ 17 & \text{if } \Delta = 7 \\ \frac{\Delta^2 - 5\Delta}{2} + 2 \times \lfloor \frac{\Delta - 1}{2} \rfloor + 3 & \text{if } \Delta \geq 8 \end{cases}$$

Every graph is acyclically $f(\Delta)$ -colorable, for any $\Delta \geq 5$.

Proof for $\Delta = 5$: the case of 5-regular graph

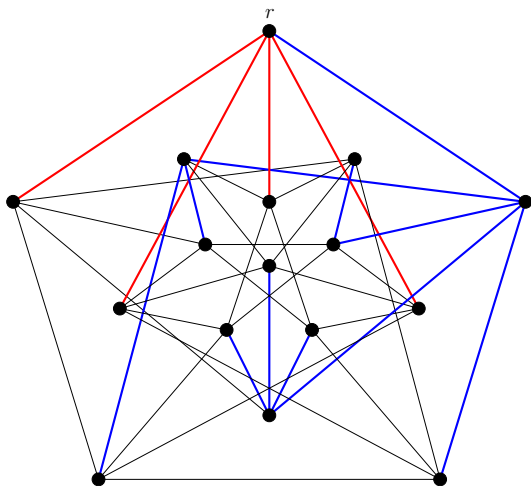


Good Spanning Trees

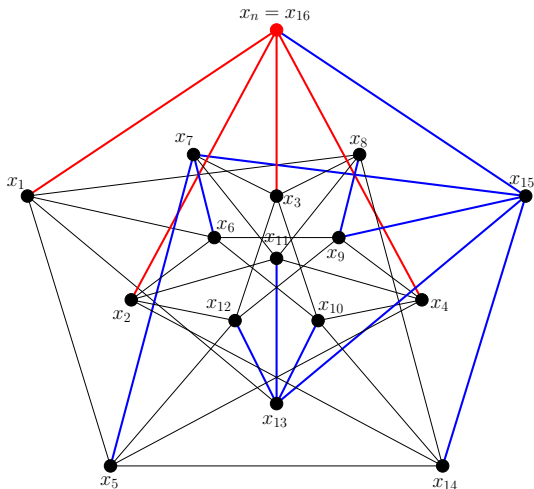
Theorem [Dieng, H., Naserasr '10]

Let G be a Δ -regular connected graph. There exists a spanning tree T of G having a vertex u with $\Delta - 1$ leaves in T .

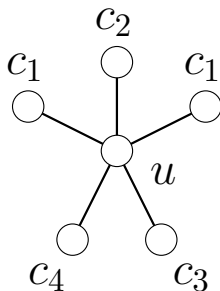
Good Spanning Trees



Post order walk

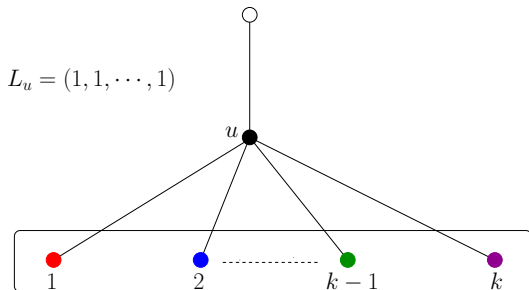


The color list of u

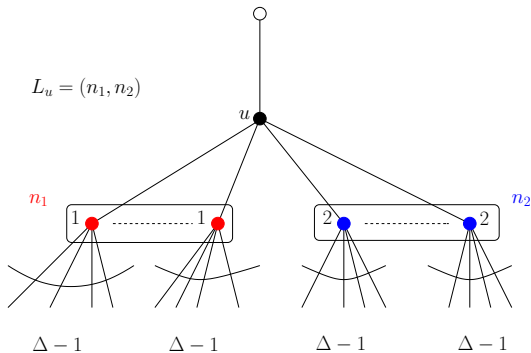


$$L_u = (2, 1, 1, 1)$$

Maximum number of dangerous colors to color u

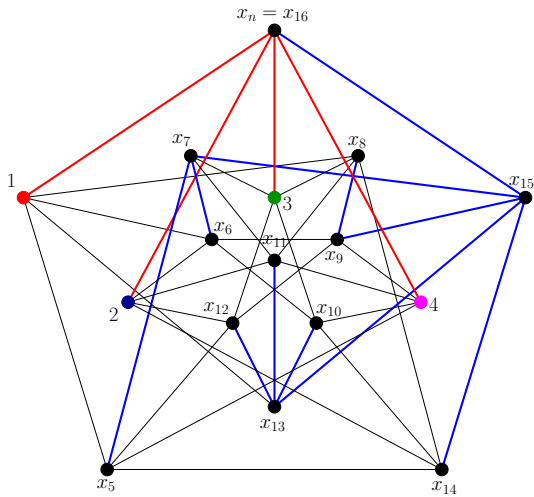


Maximum number of dangerous colors to color u



$$BC(u) = \lfloor \frac{n_1 \times (\Delta - 1)}{2} \rfloor + \lfloor \frac{n_2 \times (\Delta - 1)}{2} \rfloor = 2n_1 + 2n_2$$

Coloring the vertices



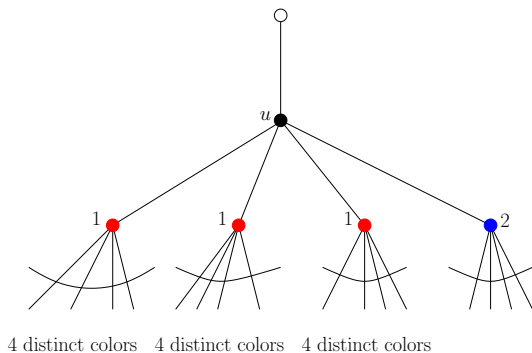
Simple cases

Based on the type of L_u we have several cases.

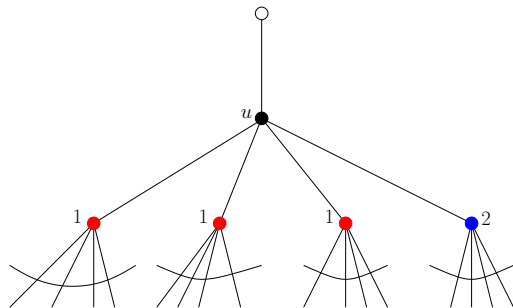
We assume $u \neq x_n$.

- $L_u = (1), (1, 1), (1, 1, 1), (1, 1, 1, 1)$ we color u properly, that is possible because there remains at least 4 colors.
- $L_u = (2), (2, 1), (2, 1, 1), (3)$ we can color u because there remains at least one color.

$$L_u = (3, 1)$$



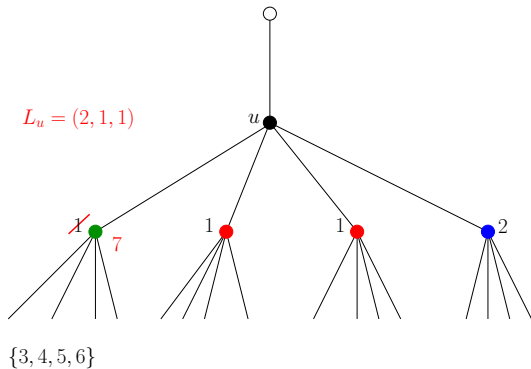
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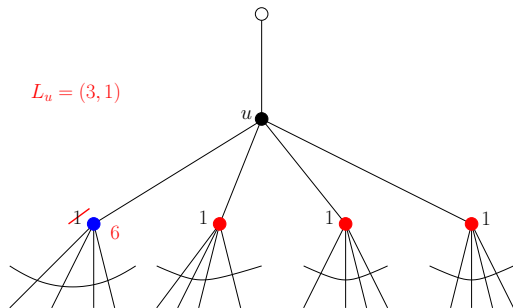
3 distinct colors 4 distinct colors 4 distinct colors

$$\lfloor \frac{3+2 \times 4}{2} \rfloor + 2 = 7$$

$$L_u = (3, 1)$$

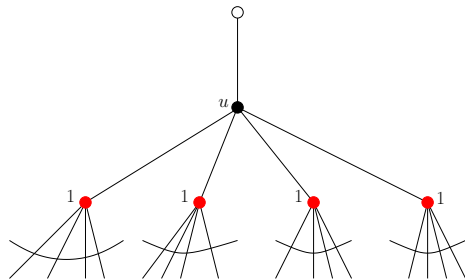


$$L_u = (4)$$



4 distinct colors
 $\{2, 3, 4, 5\}$

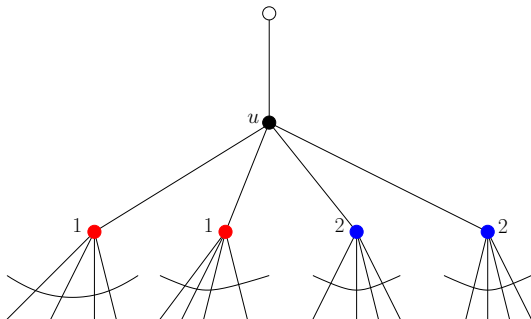
$$L_u = (4)$$



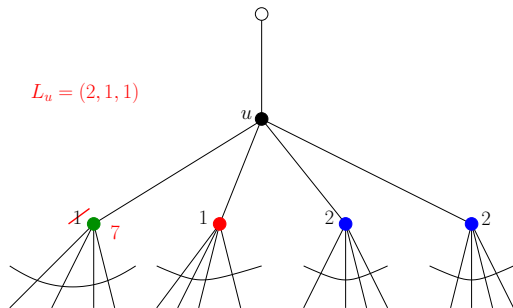
3^- distinct colors 3^- distinct colors 3^- distinct colors 3^- distinct colors

$$\lfloor \frac{4 \times 3}{2} \rfloor + 1 = 7$$

$$L_u = (2, 2)$$



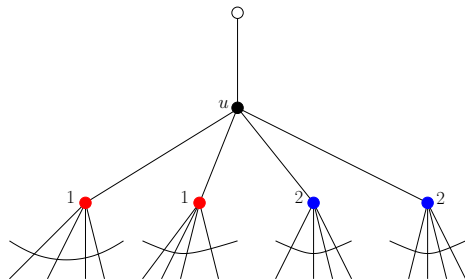
$$L_u = (2, 2)$$



$$L_u = (2, 1, 1)$$

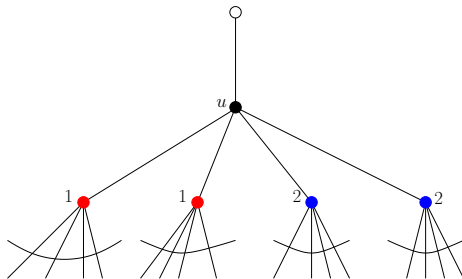
4 distinct colors
 $\{3, 4, 5, 6\}$

$$L_u = (2, 2)$$



3^- distinct colors 3^- distinct colors 3^- distinct colors 3^- distinct colors

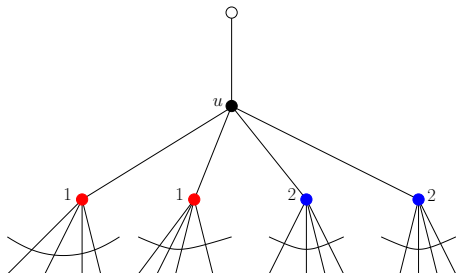
$$L_u = (2, 2)$$



2⁻ distinct colors 3⁻ distinct colors 3⁻ distinct colors 3⁻ distinct colors

$$\lfloor \frac{2+3}{2} \rfloor + \lfloor \frac{2 \times 3}{2} \rfloor + 2 = 7$$

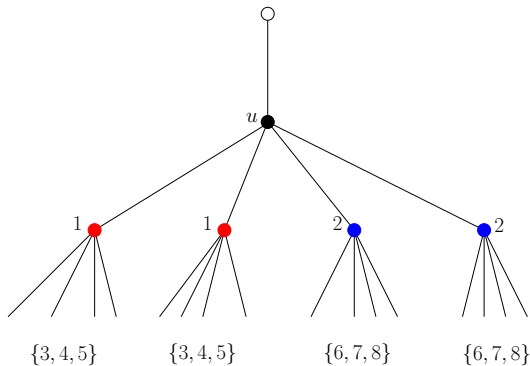
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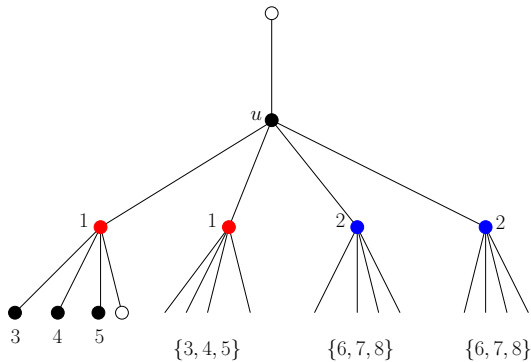
3 distinct colors 3 distinct colors 3 distinct colors 3 distinct colors

$$\lfloor \frac{2 \times 3}{2} \rfloor + \lfloor \frac{2 \times 3}{2} \rfloor + 2 = 8$$

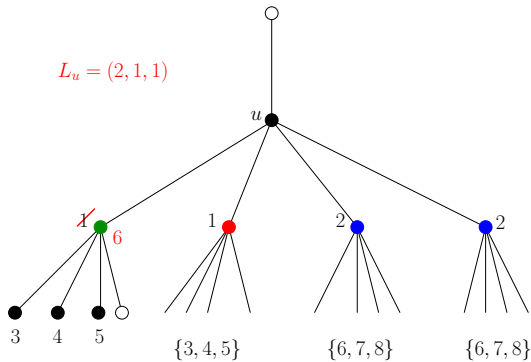
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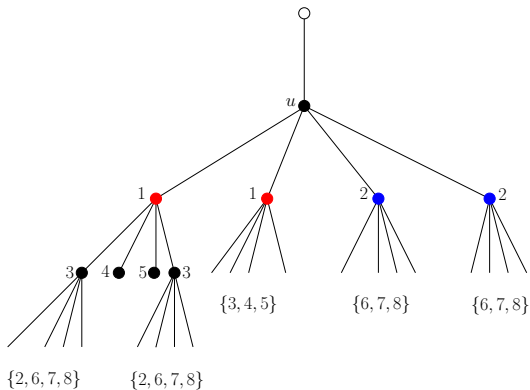
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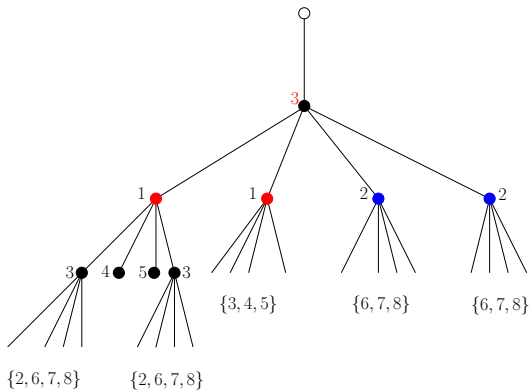
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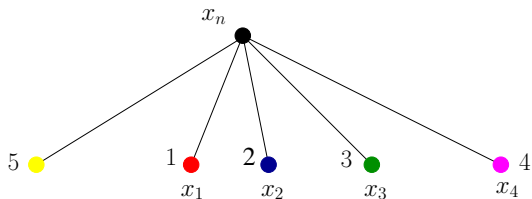
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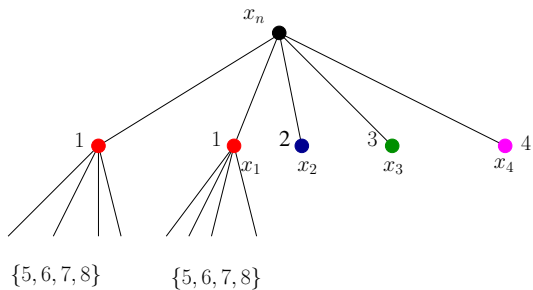
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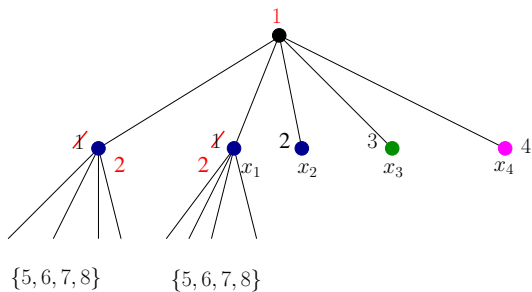
$$L_{x_n} = (1, 1, 1, 1, 1)$$



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$$L_{x_n} = (2, 1, 1, 1)$$



Proof for $\Delta \geq 6$

Remark

- For $\Delta = 6$, we use the same techniques but we make deeper analysis.
- For $\Delta \geq 7$, we do a similar analysis based on induction.

Questions

Is it true that looking at distance > 2 decreases the number of colors ?

There are graphs of maximum degree 5 that need 6 colors.
Can we find a graph G of maximum degree 5 with $\chi_a(G) > 6$?

For $\Delta \geq 5$, we suspect the bounds we found are not tight. Can we improve those bounds ?

What is the smallest value of Δ for which there is a graph with $\chi_a(G) \geq \Delta + 2$?

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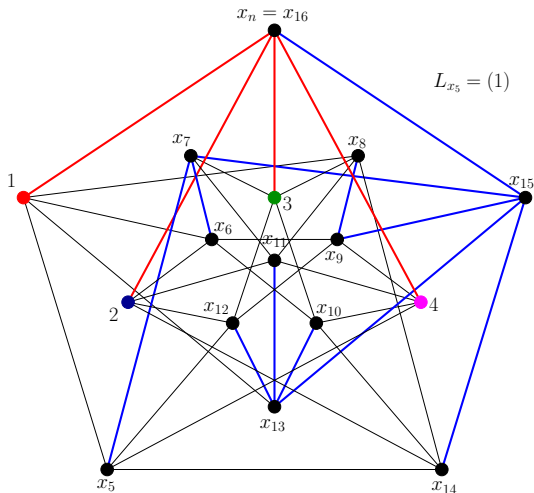
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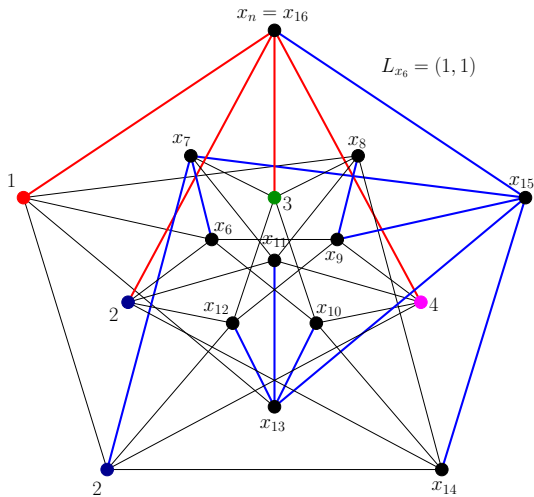
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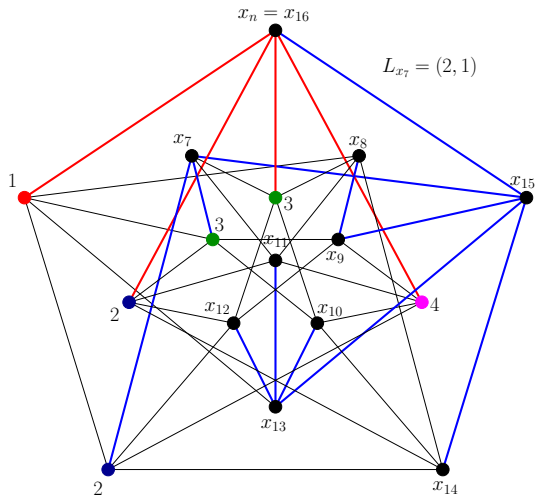
Proof for $\Delta = 5$



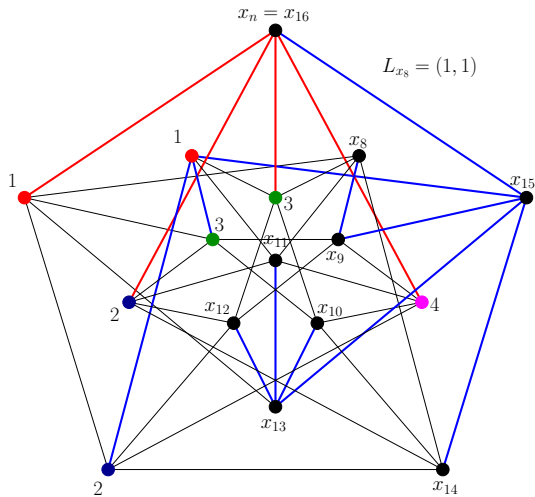
Proof for $\Delta = 5$



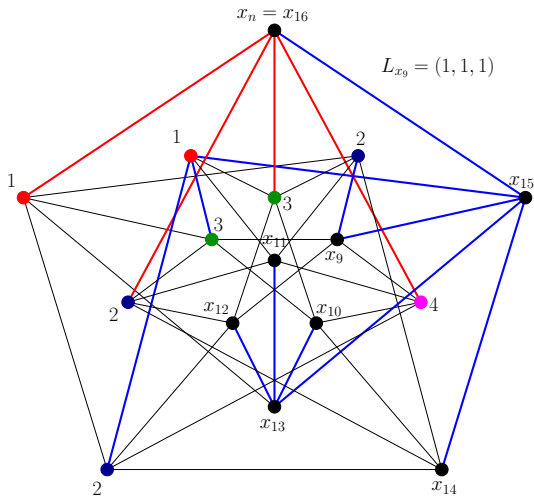
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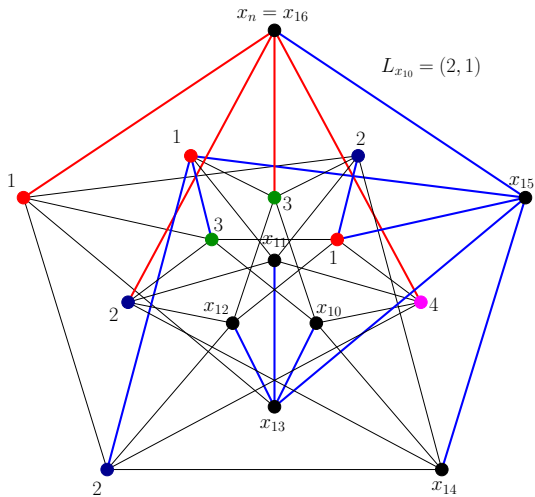
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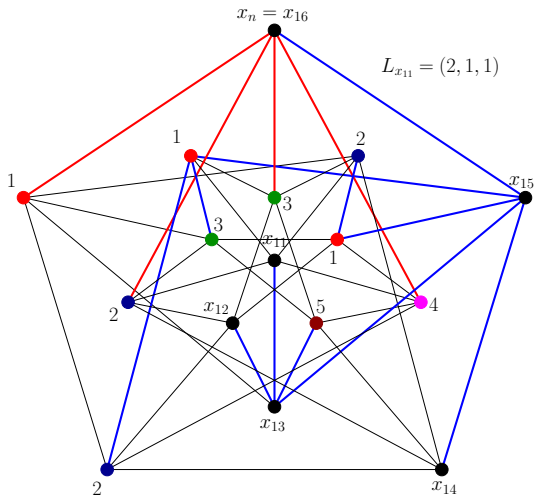
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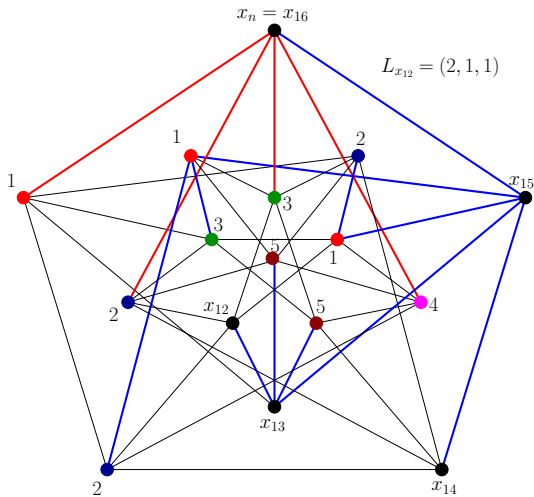
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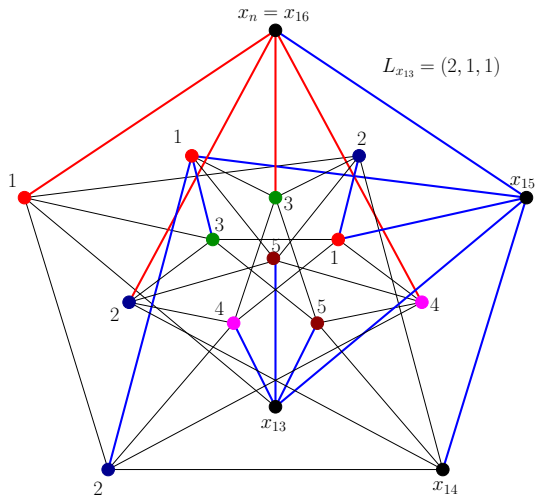
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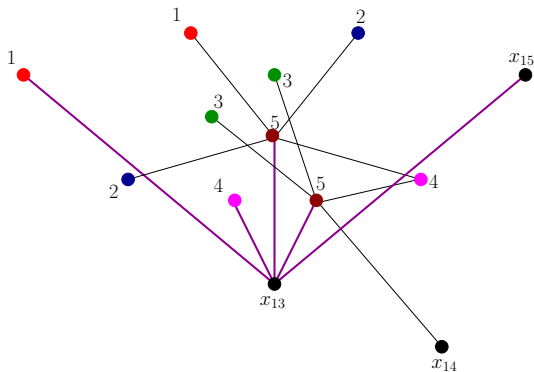


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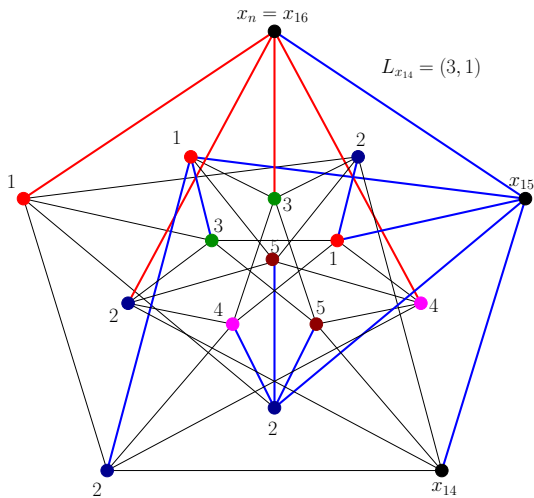


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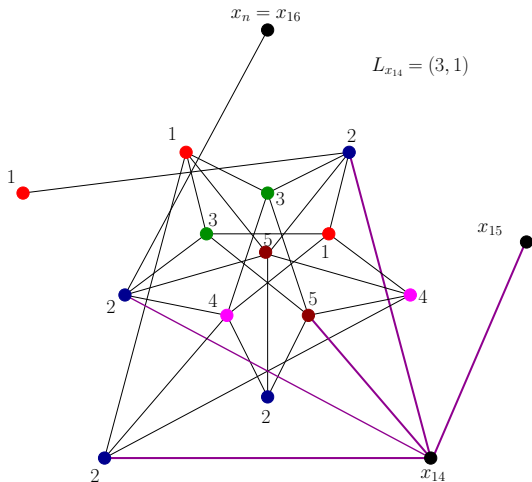
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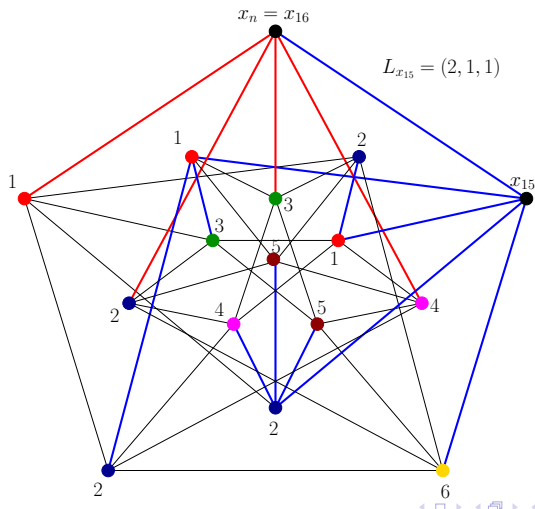
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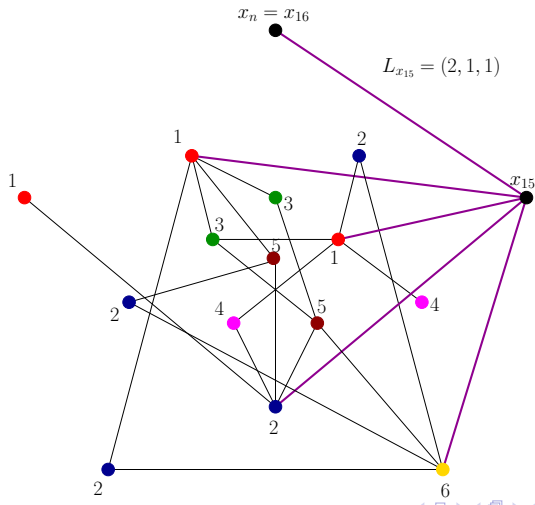
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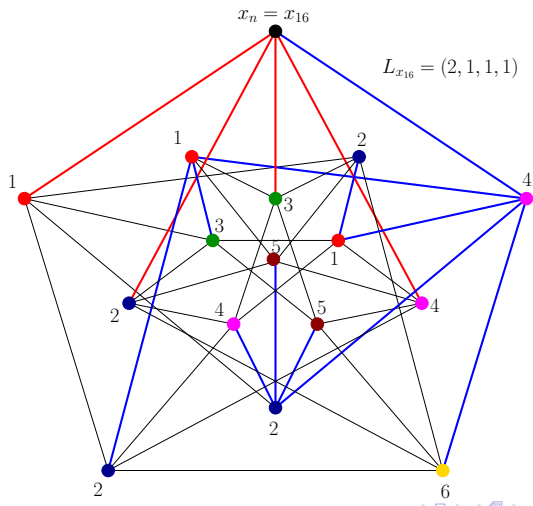
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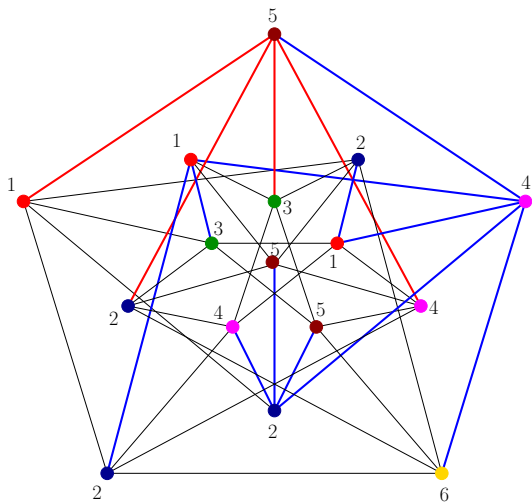
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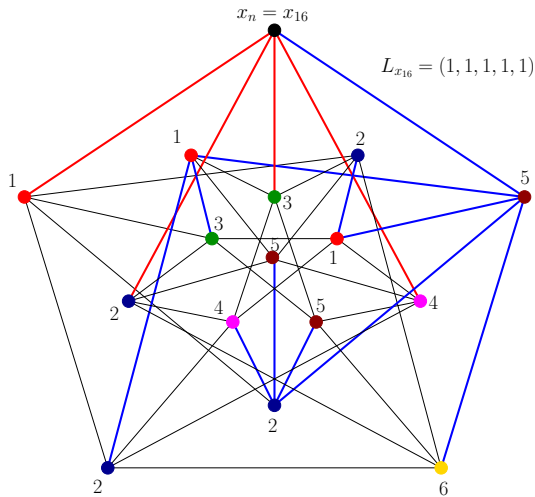
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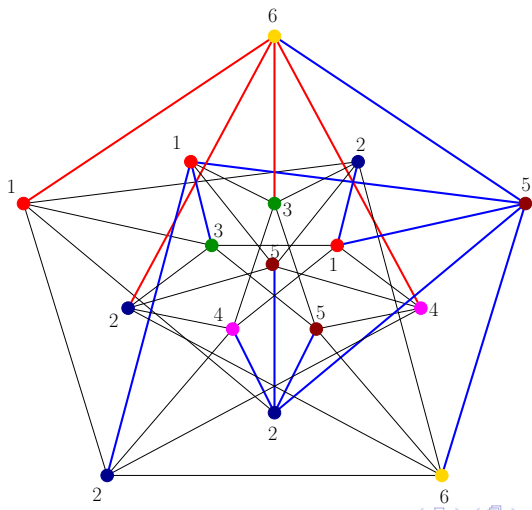
Proof for $\Delta = 5$



Proof for $\Delta = 5$



Proof for $\Delta = 5$



Proof for $\Delta \geq 8$

Lemma [Dieng, H., Naserasr '10]

Let G be a graph of maximum degree $\Delta \geq 8$ together with ordering of vertices defined above and let φ_i ($1 \leq i \leq \Delta - 2$) be a partial acyclic coloring which has colored vertices

x_1, x_2, \dots, x_i using at most $f(\Delta) = \frac{\Delta^2 - 5\Delta}{2} + 2 \times \lfloor \frac{\Delta - 1}{2} \rfloor + 3$ colors. Suppose $L_{x_{i+1}} = (1, 1, \dots, 1)$ (with $1 \leq l \leq \Delta - 1$ being the number 1's) or $L_{x_{i+1}} = (n_1, n_2, \dots, n_k, 1, \dots, 1)$ (with $n_1 \geq n_2 \geq \dots \geq n_k \geq 2$ and $1 \leq l \leq \Delta - 2$ being the number 1's). Then there is an acyclic coloring φ_{i+1} of $G[\{x_1, x_2, \dots, x_{i+1}\}]$ using at most $f(\Delta)$ colors.