

# Strong edge-colouring of graphs

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université  
de **BORDEAUX**



## Definition (Fouquet and Jolivet, 1983)

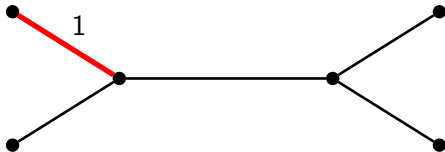
*Strong edge-colouring (distance 2 edge-colouring) is an assignment of colours to edges of graph such that:*

- *every two edges sharing a vertex have different colours (proper edge-colouring)*
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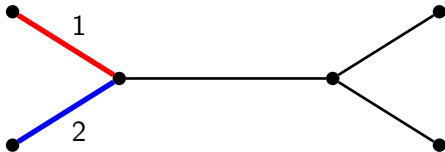
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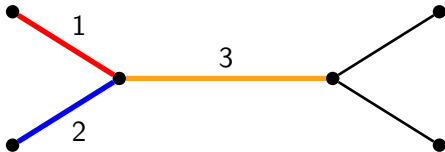
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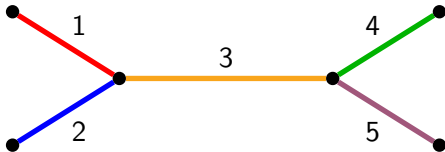
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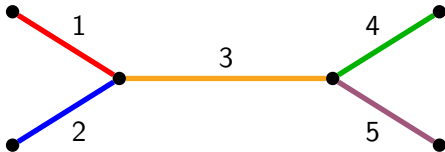
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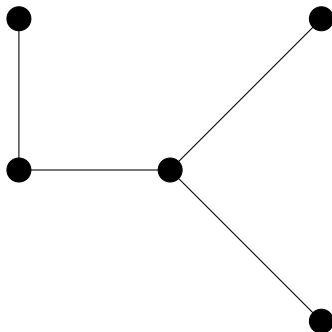


$\chi'_s(G)$  - minimum number of colours needed to obtain a strong edge-colouring of  $G$

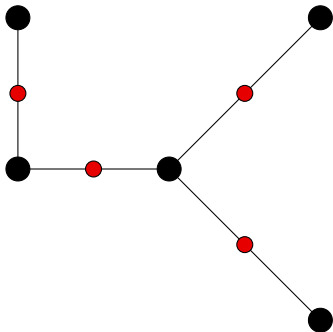


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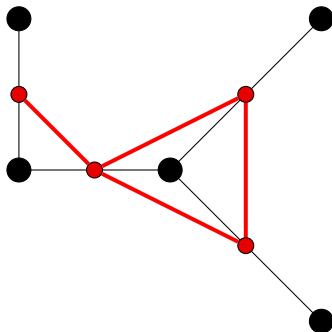
It can be seen as a proper vertex-colouring of the **square of the line graph**.



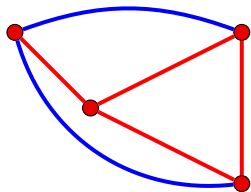
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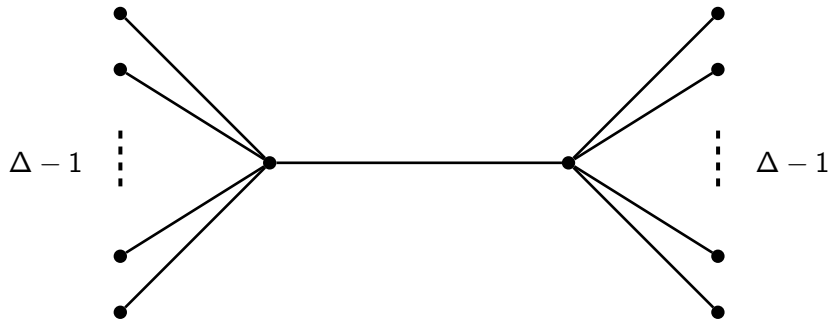
Vertex coloring of the **line graph**.



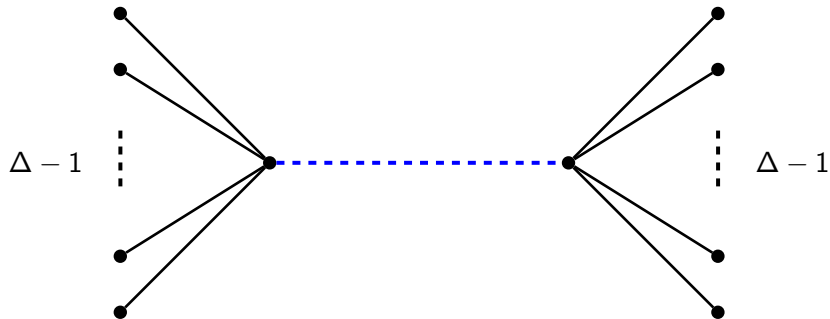
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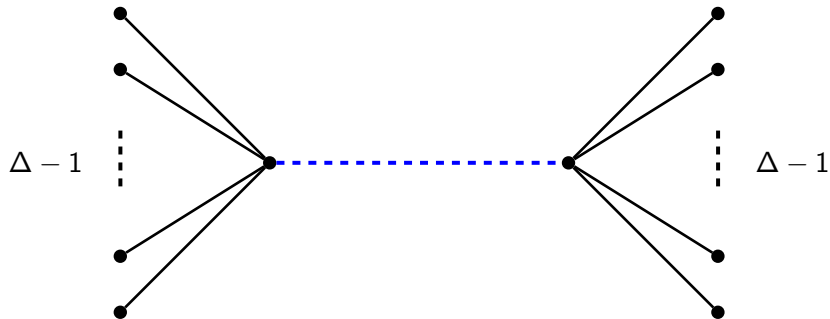
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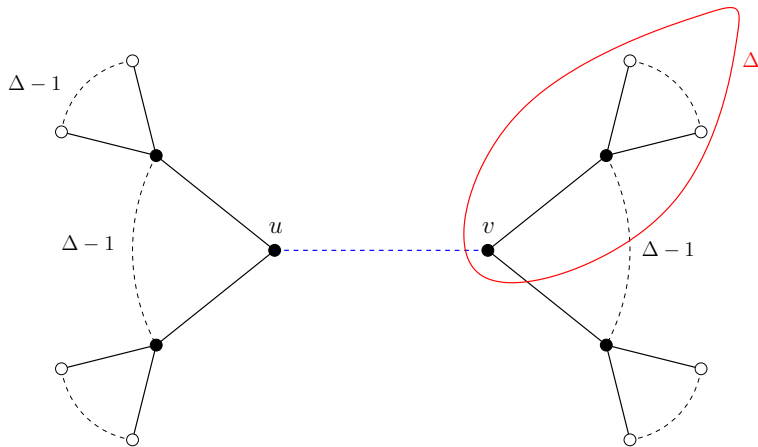


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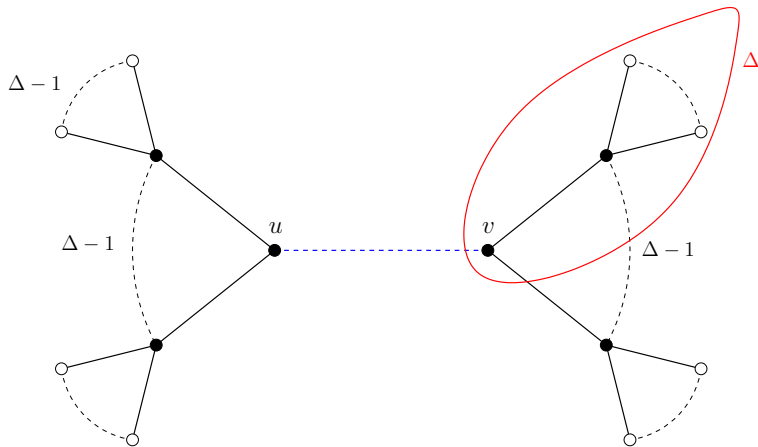
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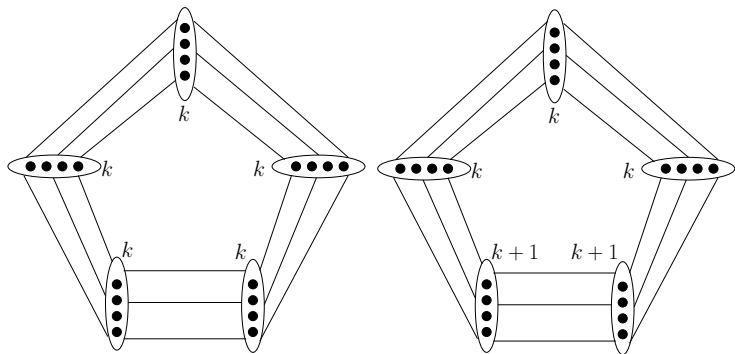
Conjecture [Erdős and Nešetřil, 1985]

For a graph  $G$ ,  $\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, \Delta \text{ even} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), \Delta \text{ odd} \end{cases}$

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$$\Delta = 2k, \chi'_s = 5k^2$$

$$\Delta = 2k + 1, \chi'_s = 5k^2 + 4k + 1$$

## To summarize

- Greedy algorithm :  $\chi'_s(G) \simeq 2\Delta^2$ .
- Erdős and Nešetřil Conjecture :  $\chi'_s(G) \simeq \frac{5}{4}\Delta^2$ .

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## Theorem [Molloy and Reed, 1997]

If  $\Delta$  is large enough, then  $\chi'_s(G) \leq 1.998\Delta^2$ .

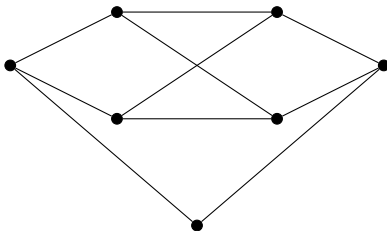
Theorem [Andersen, Horák *et al.*, 1992]

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$\chi'_s(G) \leq 20$  by Erdős and Nešetřil Conjecture.

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Conjecture [Brualdi and Quinn Massey, 1993]

If  $G$  is bipartite with parts  $X$  and  $Y$ , then  $\chi'_s(G) \leq \Delta(X)\Delta(Y)$ .

Theorem [Bensmail, Lagoutte and Valicov, 2014]

For every  $(3, \Delta)$ -bipartite graph  $G$ , we have  $\chi'_s(G) \leq 4\Delta$ .

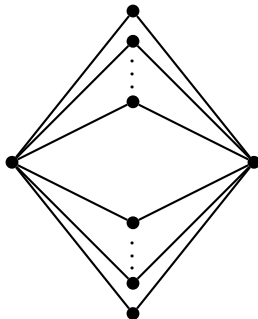
Theorem [Faudree *et al.*, 1990]

If  $G$  is a planar graph with maximum degree  $\Delta$ , then  
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# Other graph classes - planar graphs

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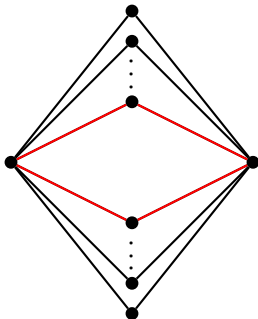
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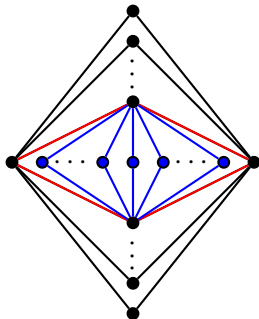
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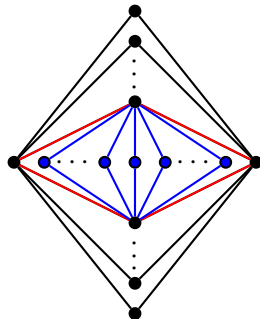




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$$\chi'_s(G) = 4\Delta - 4$$

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Theorem [Bensmail, Harutyunyan, H. and Valicov, 2013+]

If  $G$  is planar with  $g \geq 6$ , then  $\chi'_s(G) \leq 3\Delta + 1$ .

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proved for  $\Delta = 7$  (Sanders and Zhao, 2001)

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Question

If  $\Delta = g = 4$ , then can  $G$  be **Class 1**?

# Summary

	$\Delta \geq 7$	$\Delta \in \{5, 6\}$	$\Delta = 4$	$\Delta = 3$
no girth restriction	$4\Delta$	$4\Delta + 4$	$4\Delta + 4$	$3\Delta + 1$
$g \geq 4$	$4\Delta$	$4\Delta$	$4\Delta + 4$	$3\Delta + 1$
$g \geq 5$	$4\Delta$	$4\Delta$	$4\Delta$	$3\Delta + 1$
$g \geq 6$	<b><math>3\Delta + 1</math></b>	<b><math>3\Delta + 1</math></b>	<b><math>3\Delta + 1</math></b>	<b><math>3\Delta</math></b>
$g \geq 7$	$3\Delta$	$3\Delta$	$3\Delta$	$3\Delta$

## Other graph classes - planar graphs

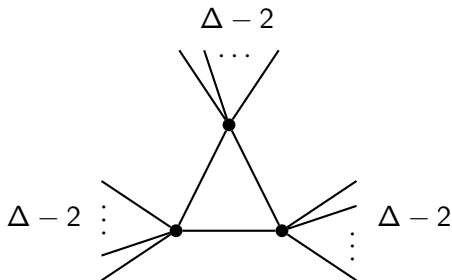
Theorem [H., Ochem and Valicov, 2011]

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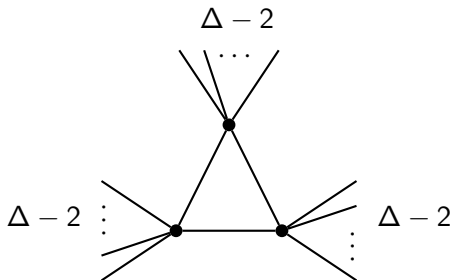
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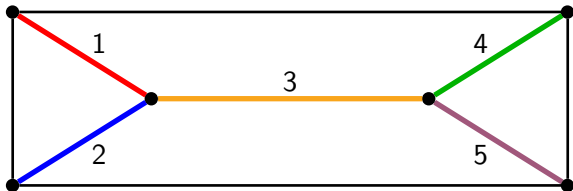
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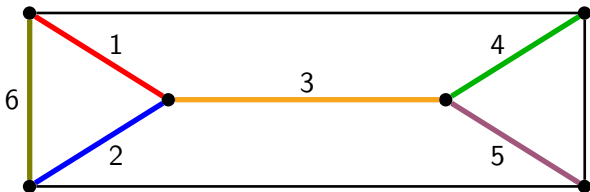
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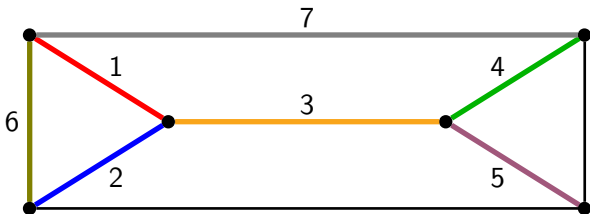
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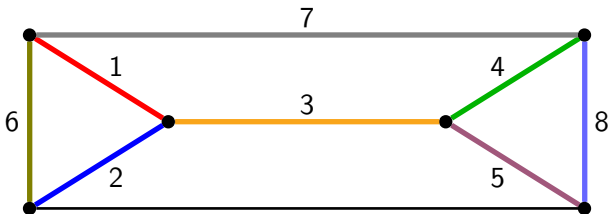
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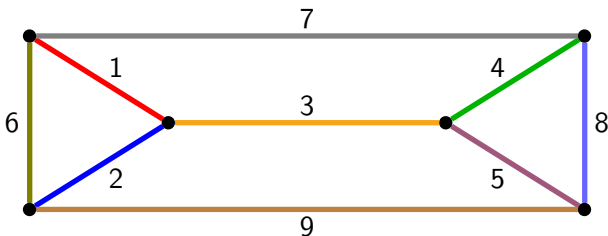
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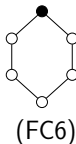
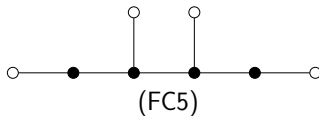
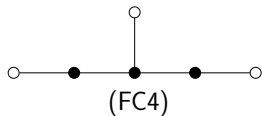
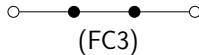
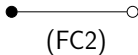
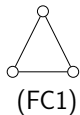
## Theorem

Let  $G$  be a planar subcubic graph with no induced cycles of length 4 nor 5. Then  $\chi'_s(G) \leq 9$ .

Proof by minimum counterexample and using planarity.

# Sketch of the proof

Let  $H$  be a smallest counterexample.  $H$  does not contain:





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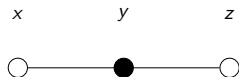
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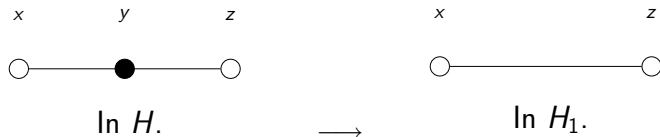
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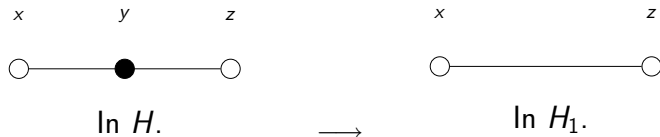


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Clearly,  $H_1$  is planar.

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$$\left. \begin{array}{l} H \text{ has no 1-vertices.} \\ H \text{ has no two adjacent 2-vertices.} \end{array} \right\}$$

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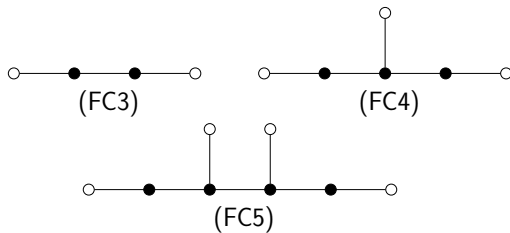
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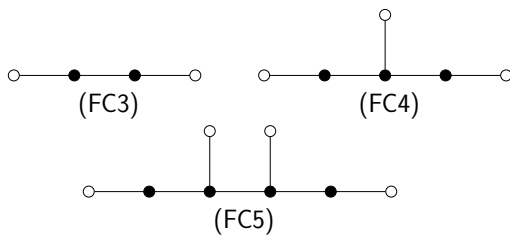
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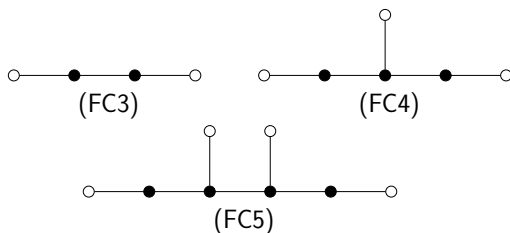
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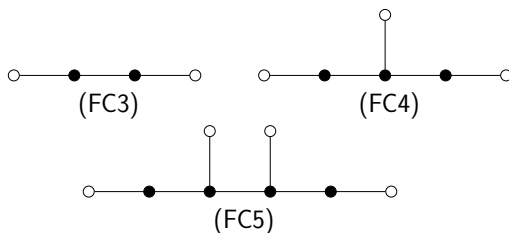
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## Definition

The **maximal average degree** of a graph  $G$ , denoted  $\text{mad}(G)$ , is the maximal of the average degrees of all the subgraphs of  $G$ :

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## Theorem

Let  $G$  be a subcubic graph:

- 1 If  $\text{mad}(G) < \frac{15}{7}$ , then  $\chi'_s(G) \leq 6$ .
- 2 If  $\text{mad}(G) < \frac{27}{11}$ , then  $\chi'_s(G) \leq 7$ .
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## Corollary

Let  $G$  be a planar subcubic graph with girth  $g$ :

- 1 If  $g \geq 30$ , then  $\chi'_s(G) \leq 6$  (can be improved to  $g \geq 16$ ).
- 2 If  $g \geq 11$ , then  $\chi'_s(G) \leq 7$ .
- 3 If  $g \geq 9$ , then  $\chi'_s(G) \leq 8$ .
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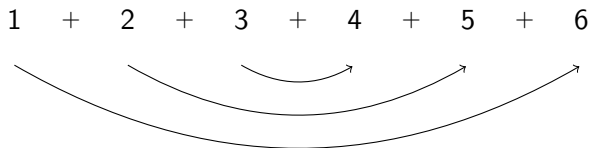
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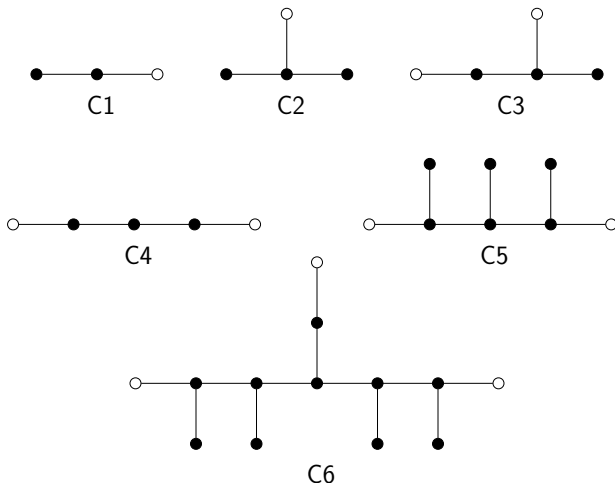
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**Therefore, the counterexample cannot exist.**

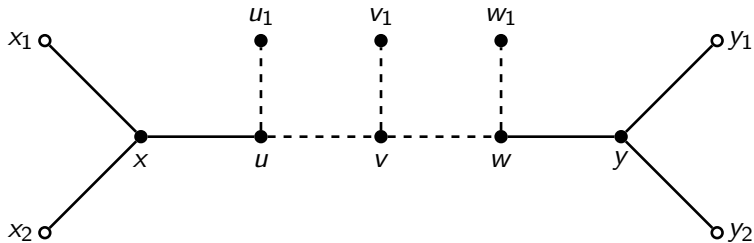
# Reducible configurations (Theorem 1.1)

Let  $H$  be a smallest counterexample.  $H$  does not contain:

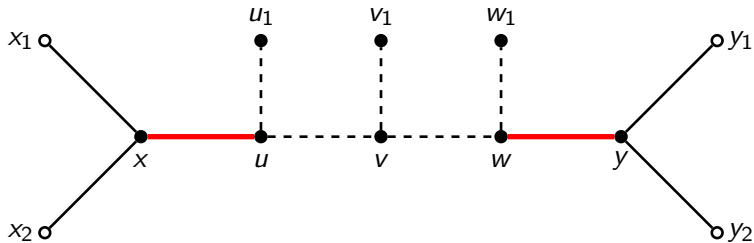




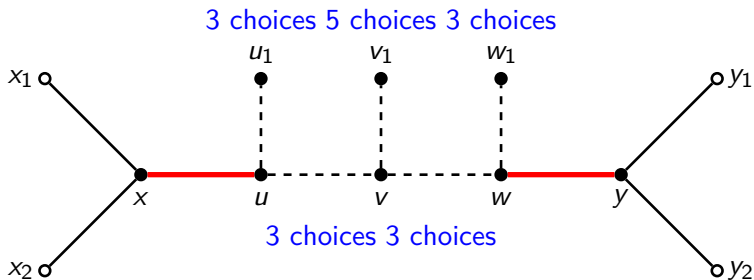
# Proof of configuration (C5)



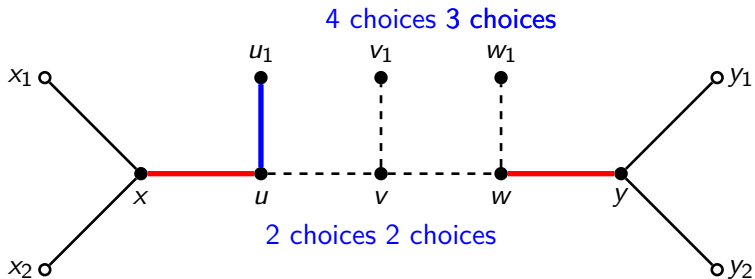
First case :  $c(xu) = c(wy)$



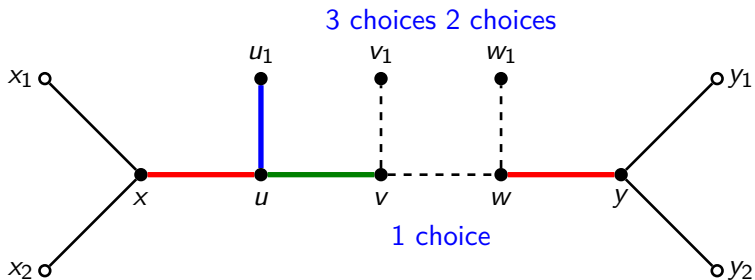
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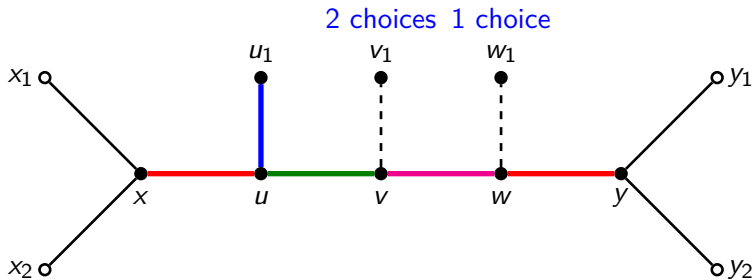
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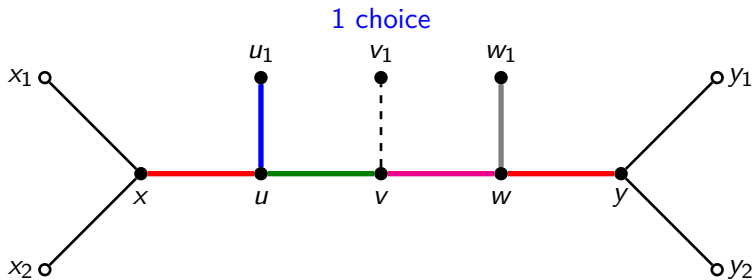
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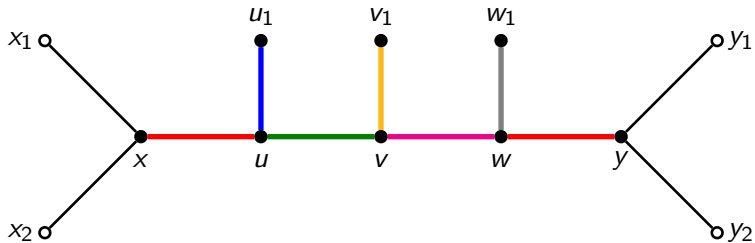
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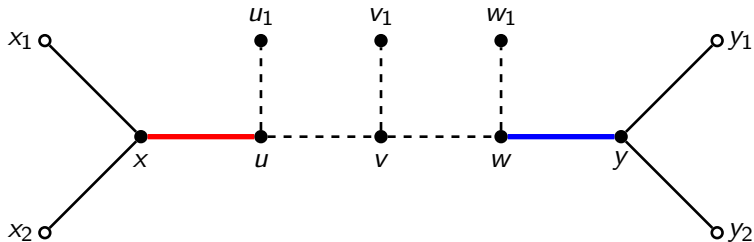


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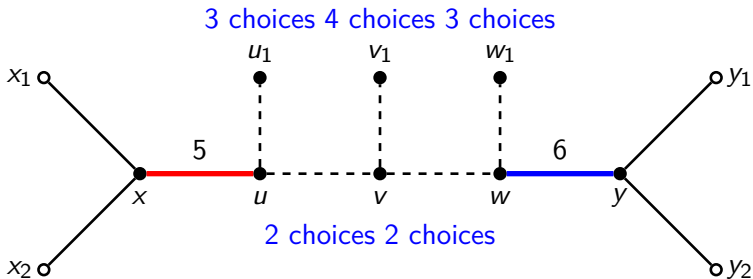




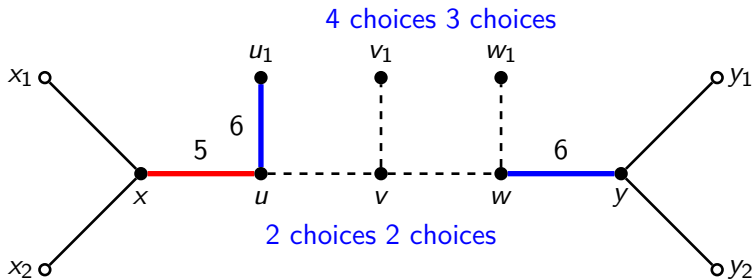
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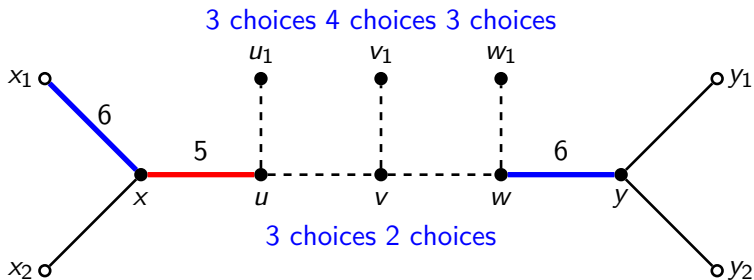
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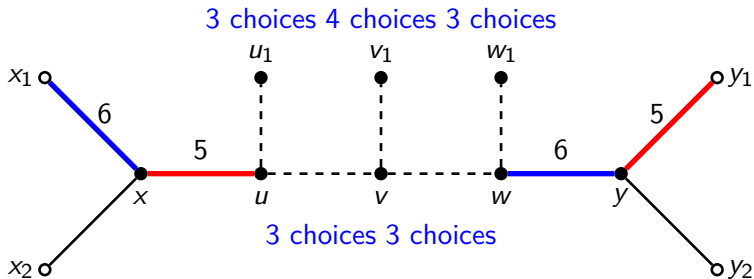
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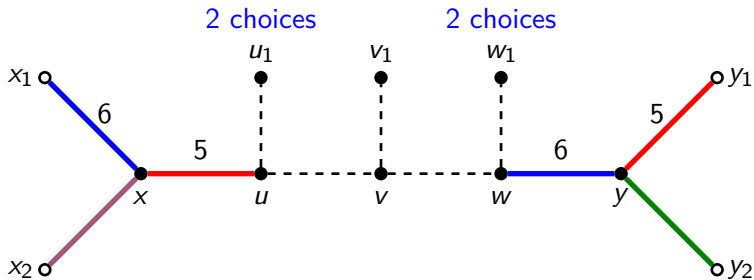
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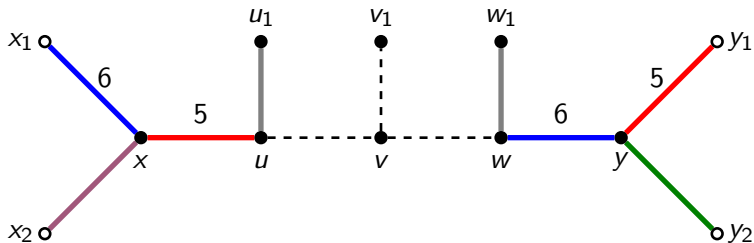
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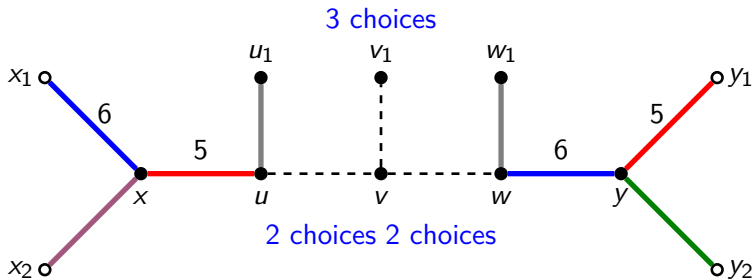
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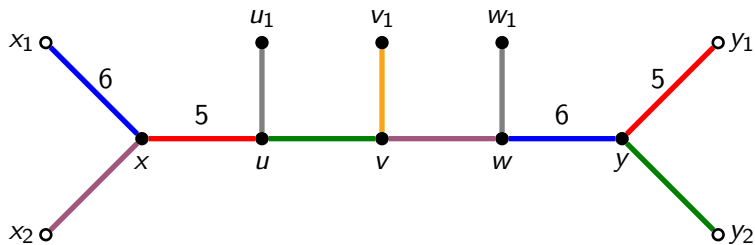


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# Discharging procedure

Recall that  $\text{mad}(H) < \frac{15}{7}$ !

We define a weight function  $\omega : V(H) \rightarrow \mathbb{R}$  by  
 $\omega(x) = d(x) - \frac{15}{7}$ .

$$\omega(u) = -\frac{8}{7} \quad \omega(v) = -\frac{1}{7} \quad \omega(w) = \frac{6}{7}$$

$$\begin{array}{c} \downarrow \\ d(u) = 1 \end{array}$$

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- 1 Every 3-vertex not adjacent to a 1-vertex gives  $\frac{2}{7}$  to each of its adjacent 3-vertices having a 1-vertex in its neighbourhood and  $\frac{1}{7}$  to each of its adjacent 2-vertices.
- 2 By (C3) every 3-vertex has at most one 1-vertex in its neighbourhood and it gives to this vertex  $\frac{8}{7}$ .

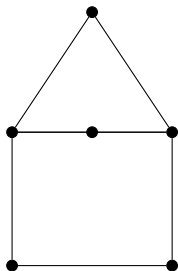
At the end of the discharging procedure, we have:

- 1 The total sum of weights did not change.
- 2 For every  $x \in V(H)$ ,  $\omega^*(x) \geq 0$ .

$$\text{Hence, } 0 \leq \sum_{x \in V(H)} \omega^*(x) = \sum_{x \in V(H)} \omega(x) < 0$$

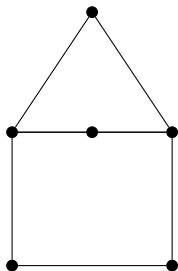
Contradiction!

# Tightness?



A graph  $G$  with  $\text{mad}(G) = \frac{7}{3}$  and  $\chi'_s(G) > 6$ .

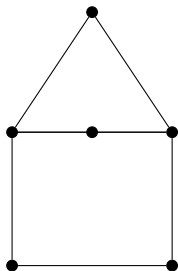
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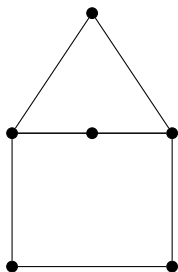
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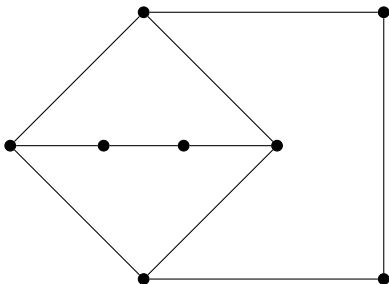


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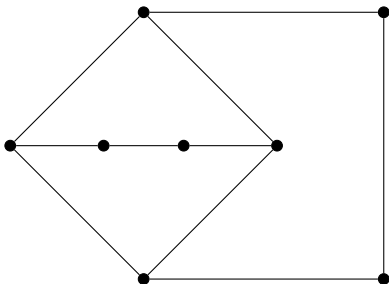


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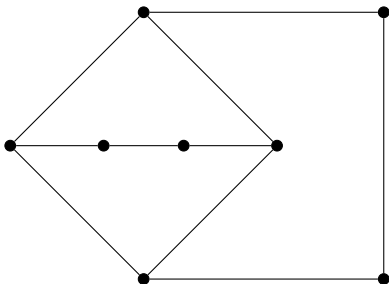
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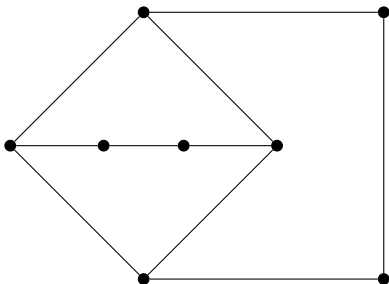
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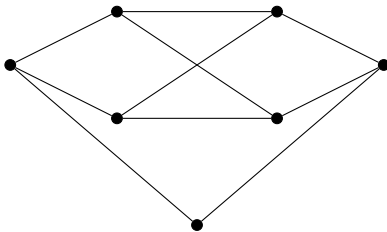
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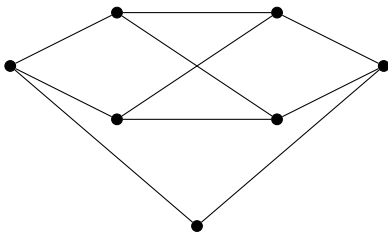
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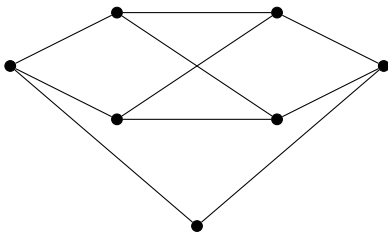
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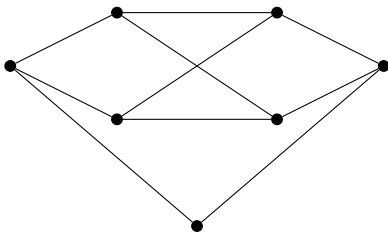
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# To summarize

Let  $f(n) = \inf\{\text{mad}(G) \mid \chi'_s(G) > n\}$ .

$$f(6) = \frac{7}{3}$$

$$f(7) = \frac{5}{2}$$

$$\frac{56}{21} = \frac{8}{3} < f(8) \leq \frac{20}{7} = \frac{60}{21}$$

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- The conjecture of Faudree *et al.*...

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- Could we reach the bound proposed by the conjecture of Erdős and Nešetřil for graphs with maximum degree 4?

- The conjecture of Faudree *et al.*...
- Does there exist a subcubic graph  $G$  with  $\text{mad}(G) < \frac{20}{7}$  and having  $\chi'_s(G) = 9$ ?
- Could we reach the bound proposed by the conjecture of Erdős and Nešetřil for graphs with maximum degree 4?
- Bigger challenge: the conjecture of Erdős and Nešetřil.

**Thank you for your attention**

**Dziękuję za uwagę**

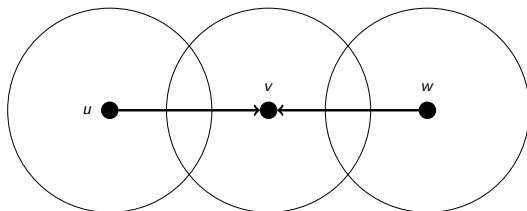
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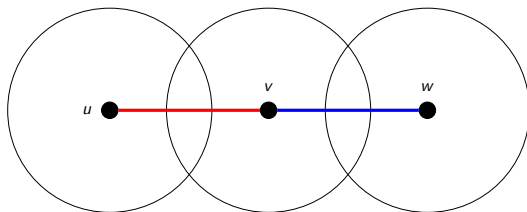
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Conflict at node  $v$

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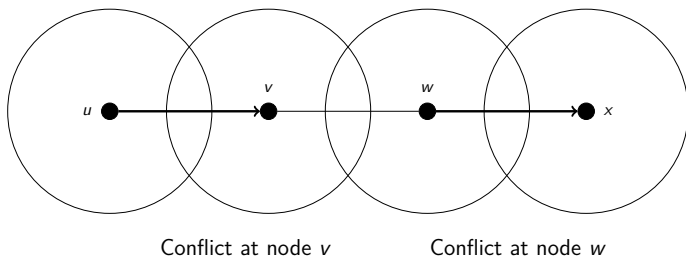
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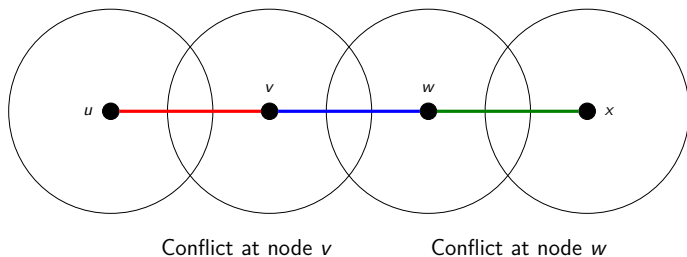
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- And what about planar graphs?

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3-COLOURING  $\leq_P$  4-SEC.

# What else?

By reducing 3-COLOURING of planar graphs of maximum degree 4 we also proved:

- 5-SEC is NP-complete for planar bipartite graphs of maximum degree 3 and girth  $g = 8$ .
- 6-SEC is NP-complete for planar graphs of maximum degree 3, with girth  $g = 4$ .